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ALGORITHMIC ASPECTS OF LINEAR k-ARBORICITY

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Abstract. For a fixed positive integer k, the linear k-arboricity $\mathrm{la}_k(G)$ of a graph G is the minimum number ℓ such that the edge set $E(G)$ can be partitioned into ℓ disjoint sets, each induces a subgraph whose components are paths of lengths at most k. This paper examines linear k-arboricity from an algorithmic point of view. In particular, we present a linear-time algorithm for determining whether a tree T has $\mathrm{la}_2(T) \leq m$. We also give a characterization for a tree T with maximum degree $2m$ having $\mathrm{la}_2(T) = m$.

1. Introduction

All graphs in this paper are simple, i.e., finite, undirected, loopless, and without multiple edges. A *linear k-forest* is a graph whose components are paths of lengths at most k. A *linear k-forest partition* of G is a partition of the edge set $E(G)$ into linear k-forests. The *linear k-arboricity* of G, denoted by $\mathrm{la}_k(G)$, is the minimum size of a linear k-forest partition of G.

The notion of linear k-arboricity was introduced by Habib and Peroche [18]. It is a natural refinement of the linear arboricity introduced by Harary [20], which is the same as linear k-arboricity except that the paths have no length constraints. Suppose $\chi'(G)$ is the chromatic index of G and $\mathrm{la}(G)$ the linear arboricity. Let $\Delta(G)$ denote the maximum degree of a vertex in G. The following propositions are easy to verify.

Proposition 1. *If G is a subgraph of H, then* $\mathrm{la}_k(G) \leq \mathrm{la}_k(H)$ *for* $k \geq 1$.

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Proposition 2. *If G is a graph with n vertices, then*

$$\mathrm{la}(G) = \mathrm{la}_{n-1}(G) \leq \mathrm{la}_{n-2}(G) \leq \cdots \leq \mathrm{la}_2(G) \leq \mathrm{la}_1(G) = \chi'(G).$$

Proposition 3. *If G is a graph with n vertices and m edges, then*

$$\mathrm{la}_k(G) \geq \max\left\{ \left\lceil \frac{\Delta(G)}{2} \right\rceil, \left\lceil \frac{m}{\lfloor \frac{kn}{k+1} \rfloor} \right\rceil \right\}.$$

On the other hand, Habib and Peroche [18] made the following conjecture.

Conjecture 4. ([18]) *If G is a graph with n vertices and $k \geq 2$, then $\mathrm{la}_k(G) \leq \lceil \frac{\Delta(G)n+\alpha}{2\lfloor \frac{kn}{k+1} \rfloor} \rceil$, where $\alpha = 1$ when $\Delta(G) < n-1$ and $\alpha = 0$ when $\Delta(G) = n-1$.*

This conjecture subsumes Akiyama's conjecture [2] as follows.

Conjecture 5. ([2]) $\mathrm{la}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.

Considerable work has been done in determining exact values and bounds for linear k-arboricity, aimed at these conjectures (see the references at the end of this paper).

We examine linear k-arboricity from an algorithmic point of view in this paper. Habib and Peroche [19] showed the first results along this line. They gave an algorithm for proving that if T is a tree with exactly one vertex of maximum degree $2m$, then $\mathrm{la}_2(T) \leq m$. Using this as a basis for induction, they then characterized a tree T with maximum degree $2m$ as having $\mathrm{la}_2(T) = m$. However, their characterization has a flaw as shown in Section 2. Holyer [21] proved that determining $\mathrm{la}_1(G)$ is NP-complete, Peroche [24] that determining $\mathrm{la}(G)$ is NP-complete, and Bermond *et al.* [9] that determining whether $\mathrm{la}_3(G) = 2$ is NP-complete for cubic graphs of $4m$ vertices. Bermond *et al.* [9] conjectured that determining $\mathrm{la}_k(G)$ for any fixed k is NP-complete.

In this paper, we present a linear-time algorithm for determining whether a tree T has $\mathrm{la}_2(T) \leq m$. We employ a "local message passing" approach that reduces the problem on T to the problem on another tree T' obtained from T by a local modification. We then give a characterization for a tree with maximum degree $2m$ having $\mathrm{la}_2(T) = m$. This makes up a gap of Habib and Peroche's result [19].

2. Linear 2-arboricity on Trees

A *leaf* is a vertex of degree one. A *penultimate vertex* is a vertex that is not a leaf and all of whose neighbors are leaves, with the possible exception of one. Note that a penultimate vertex of a connected graph is always adjacent to a non-leaf, unless the graph is a star. It is well-known that a non-trivial tree has at least two leaves, and a tree with at least three vertices has at least one penultimate vertex.

Theorem 6. *For any tree T, $\mathrm{la}_2(T) \le \lceil \frac{\Delta(T)+1}{2} \rceil$.*

Proof. We prove this theorem by induction on the number of vertices of T. The theorem is trivial when T is a star. We thus consider a general tree T that is not a star. Choose a penultimate vertex x that is adjacent to a non-leaf y and $r \ge 1$ leaves $x_1, x_2, \ldots, x_r$. Let $T' = T - \{x_1, x_2, \ldots, x_r\}$. According to the induction hypothesis, $\mathrm{la}_2(T') \le \lceil \frac{\Delta(T')+1}{2} \rceil \le \lceil \frac{\Delta(T)+1}{2} \rceil$. The edge set $E(T')$ can then be partitioned to at most $\lceil \frac{\Delta(T)+1}{2} \rceil$ linear 2-forests. Since $r + 1 = \deg_T(x) \le \Delta(T)$, we have $\lceil \frac{r}{2} \rceil \le \lceil \frac{\Delta(T)+1}{2} \rceil - 1$. Note that the star with center x and leaves $x_1, x_2, \ldots, x_r$ can be partitioned into $\lceil \frac{r}{2} \rceil$ paths of length at most 2. We then add these paths to $\lceil \frac{r}{2} \rceil$ different linear 2-forests of T' that do not contain the edge xy to form a linear 2-forest partition of T. This proves that $\mathrm{la}_2(T) \le \lceil \frac{\Delta(T)+1}{2} \rceil$. ■

Theorem 6 and Propositions 2 and 3 yield the following two consequences. Corollar 7 was also obtained by Habib and Peroche [19] from a different approach.

Corollary 7. *If T is a tree with $\Delta(T) = 2m - 1$, then $\mathrm{la}_k(T) = m$ for $k \ge 2$.*

Corollary 8. *If T is a tree with $\Delta(T) = 2m$, then $m \le \mathrm{la}_k(T) \le m + 1$ for $k \ge 2$.*

So, it remains to determine whether $\mathrm{la}_k(T)$ is m or $m+1$ when $\Delta(T) = 2m$. Habib and Peroche [19] gave the following characterization of the case in which $k = 2$. In a tree T, a vertex x_0 is called an *m-center* if there exist p vertices $x_1, x_2, \ldots, x_p$ of degree $2m$ such that x_0 is of degree $p + q$ with $p + \lceil \frac{q}{2} \rceil > m$, and all internal vertices of the x_0-x_i path in T are of degree $2m - 1$ for $1 \le i \le p$. Habib and Peroche proved that for a tree T of maximum degree $2m$, $\mathrm{la}_2(T) = m$ if and only if T has no m-center. This results has a flaw as Figure 1 shows a tree T of maximum degree 6 such that $\mathrm{la}_2(T) = 4$ but T has no 3-center.

FIG. 1. A tree T with $\Delta(T) = 6$ and $\mathrm{la}_2(T) = 4$, but without any 3-center.

This paper is intended to give a linear-time algorithm for determining whether a tree T has $\mathrm{la}_2(T) \leq m$. We shall then give a characterization for a tree T of maximum degree $2m$ having $\mathrm{la}_2(T) = m$. The algorithm is based on the following theorem.

Theorem 9. *Suppose T is a tree in which x is a penultimate vertex adjacent to a vertex y and $r \geq 1$ leaves $x_1, x_2, \ldots, x_r$. Let $T' = T - \{x_1, x_2, \ldots, x_r\}$ when $r \leq 2m - 2$, and T' be obtained from $T - \{x_1, x_2, \ldots, x_r\}$ by adding a new leaf x' adjacent to y when $r = 2m - 1$. Then $\mathrm{la}_2(T) \leq m$ if and only if $r \leq 2m - 1$ and $\mathrm{la}_2(T') \leq m$.*

Proof. ($\Rightarrow$) Suppose $\mathrm{la}_2(T) \leq m$. According to Proposition 3, it is necessary that $r \leq 2m - 1$. For the case in which $r \leq 2m - 2$, according to Proposition 1, $\mathrm{la}_2(T') \leq \mathrm{la}_2(T) \leq m$. For the case in which $r = 2m - 1$, since x has degree $r + 1 = 2m$ and $\mathrm{la}_2(T) \leq m$, there exists some $1 \leq i \leq r$ such that the path (y, x, x_i) is a component of a linear 2-forest in an optimal linear 2-forest partition of T. Deleting all edges $xx_1, xx_2, \ldots, xx_r$ and replacing the path (y, x, x_i) with (x, y, x') in the optimal linear 2-forest partition of T, we obtain a linear 2-forest partition of T' of size at most $\mathrm{la}_2(T) \leq m$. This proves that $\mathrm{la}_2(T') \leq m$ for $r = 2m - 1$.

($\Leftarrow$) On the other hand, suppose $r \leq 2m - 1$ and $\mathrm{la}_2(T') \leq m$. We first consider the case in which $r \leq 2m - 2$. The edge set $E(T')$ can be partitioned into m linear 2-forests. Since $r \leq 2m - 2$, we have $\lceil \frac{r}{2} \rceil \leq m - 1$. Note that the star with center x and leaves $x_1, x_2, \ldots, x_r$ can be partitioned into $\lceil \frac{r}{2} \rceil$ paths of length at most 2. We then add these paths to $\lceil \frac{r}{2} \rceil$ linear 2-forests of T' different from the one containing the edge xy to form a linear 2-forest partition of T. This proves that $\mathrm{la}_2(T) \leq m$.

Next, consider the case in which $r = 2m - 1$. The edge set $E(T')$ can be partitioned into m linear 2-forests. Since x and x' are two leaves in T', we may assume that the path (x, y, x') is in a linear 2-forest of an optimal linear 2-forest partition $\mathcal{P}'$ of T'. Otherwise, suppose (x, y, z) and (x', y, z') are paths in different linear 2-forests F_1 and F_2 of $\mathcal{P}'$. Let T'_z be the subtree of $T' - y$ that contains z. We can then replace the path (x, y, z) of F_1 with (x, y, x'), replace the path (x', y, z') of F_2 with (z, y, z'), and interchange the roles of $F_1 \cap T'_z$ and $F_2 \cap T'_z$ to obtain a new optimal linear 2-forest partition of T' with the desired property. Similar (but even simpler) arguments work for the case in which (x, y, z) in F_1 and (x', y) in F_2, or in which (x, y) in F_1 and (x', y, z') in F_2, or in which (x, y) in F_1 and (x', y) in F_2. So, we may assume that (x, y, x') is in a linear 2-forest F of $\mathcal{P}'$. Then replace (x, y, x') in F with (y, x, x_1) and add (x_{2i}, x, x_{2i+1}) for $1 \le i \le m-1$ to $m-1$ different linear 2-forests other than F. This results in a linear 2-forest partition of T of size m, and so, $\mathrm{la}_2(T) \le m$. ■

Theorem 9 leads to the following algorithm.

Algorithm Tree. Test whether $\mathrm{la}_2(T) \le m$ for a tree T.
Input. A tree T and a positive integer m.
Output. "Yes" if $\mathrm{la}_2(T) \le m$ and "no" otherwise.
Method.

```
if (m = 1 and T has more than two edges)
then output "no" and stop;
while (T is not an edge) do
        choose a penultimate vertex x adjacent to
        a vertex y and r ≥ 1 leaves x1, x2, ..., xr;
        case 1. r ≤ 2m − 2:
                delete x1, x2, ..., xr from T;
        case 2. r = 2m − 1:
                delete x1, x2, ..., xr from T and
                add a new leaf x' adjacent to y;
        case 3. r ≥ 2m:
                output "no" and stop;
end while;
output "yes".
```

Note that in Theorem 9, when $m = 1$ and T has exactly three vertices, T' is isomorphic to T. This is the reason we have the first line in the above algorithm to treat this special case.

To implement the algorithm efficiently, we do not really delete and add vertices from and to T. Instead, we choose a vertex v^* and order the vertices of T into $v_1, v_2, \dots, v_n$ such that

$$d_T(v_1, v^*) \geq d_T(v_2, v^*) \geq \dots \geq d_T(v_n, v^*).$$

It is then clear that the first vertex v_i that is not a leaf is a penultimate vertex. We also use two arrays deleted[1..n] and degree[1..n]. Initially, deleted[i] $= 0$ and degree[i] is the degree of v_i in T for $1 \leq i \leq n$. To delete a leaf v_i from T we simply make deleted[i] $= 1$ and decrease degree[j] by one, where v_j is the vertex adjacent to v_i still in T. To add a new leaf v' adjacent to some vertex v_j, we increase degree[j] by one without creating a new vertex v' in the tree. In a general step, we choose the first v_i with deleted[i] $= 0$ and degree[i] ≥ 2 as the penultimate vertex x. The deletion and addition of vertices in cases 1 and 2 of the algorithm are performed by updating the arrays "deleted" and "degree" described above. The running time of the algorithm is clearly linear.

Theorem 10. *Algorithm* **Tree** *determines whether a tree T has* $\mathrm{la}_2(T) \leq m$ *in linear time.*

Next, we shall give a characterization for a tree T of maximum degree $2m$ having $\mathrm{la}_2(T) = m$.

For a tree T, $D(T)$ denotes the graph obtained from T by deleting all leaves of T. Note that if T has at least 3 vertices, $D(T)$ remains a tree. A tree is called *m-critical* if it has at least 3 vertices and $\deg_T(v) + \deg_{D(T)}(v) = 2m+1$ for any vertex v in $D(T)$. The tree T in Figure 1 is 3-critical, where the black vertices induce $D(T)$.

Lemma 11. *If T is an m-critical tree, then* $\mathrm{la}_2(T) = m + 1$.

Proof. According to the definition of an m-critical tree, $\deg_T(v) \leq 2m + 1$ for all vertices v in T. By Theorem 6, $\mathrm{la}_2(T) \leq \lceil \frac{\Delta(T)+1}{2} \rceil \leq \lceil \frac{2m+1+1}{2} \rceil = m+1$.

Next, we prove by induction on the number n of vertices of $D(T)$ that $\mathrm{la}_2(T) \geq m + 1$. For $n = 1$, T is a star with exactly $2m + 2$ vertices. Thus, $\mathrm{la}_2(T) \geq m + 1$. Suppose T is an m-critical graph with $n \geq 2$. Choose a penultimate vertex x adjacent to a non-leaf y and $2m-1$ leaves. Let T' be the tree as defined in Theorem 9. Note that x is in $D(T)$ but not in $D(T')$. Then, $\deg_{T'}(v) = \deg_T(v)$ and $\deg_{D(T')}(v) = \deg_{D(T)}(v)$ for all vertices v in $D(T')$ except $\deg_{T'}(y) = \deg_T(y) + 1$ and $\deg_{D(T')}(y) = \deg_{D(T)}(y) - 1$. Therefore, $\deg_{T'}(v) + \deg_{D(T')}(v) = \deg_T(v) + \deg_{D(T)}(v) = 2m + 1$ for all vertices v in $D(T')$. This shows that T' is an m-critical tree with $n' = n - 1$. By the induction hypothesis, $\mathrm{la}_2(T') \geq m + 1$. Then, by Theorem 9, $\mathrm{la}_2(T) \geq m + 1$. ■

Theorem 12. *For any tree T of maximum degree $2m$,* $\mathrm{la}_2(T) = m$ *if and only if T contains no m-critical tree as an induced subgraph.*

Proof. ($\Rightarrow$) Suppose $\mathrm{la}_2(T) = m$. If T contains an m-critical tree G, then $\mathrm{la}_2(T) \geq \mathrm{la}_2(G) = m + 1$ by Proposition 1 and Lemma 11, a contradiction. So, T contains no m-critical tree.

($\Leftarrow$) On the other hand, suppose T contains no m-critical tree but $\mathrm{la}_2(T) = m + 1$. Choose a minimal subtree G of T with $\mathrm{la}_2(G) = m + 1$. Since $\Delta(G) \leq \Delta(T) \leq 2m$, $D(G)$ has at least two vertices.

Suppose $D(G)$ has a vertex x such that $\deg_G(x) + \deg_{D(G)}(x) \leq 2m$. Let x have r neighbors $x_1, x_2, \ldots, x_r$ in G, where $x_1, x_2, \ldots, x_s$ are in $D(G)$ with $1 \leq s \leq r$ and $r+s \leq 2m$. For each $1 \leq i \leq r$, consider the subtree G_i obtained from the component of $G - x$ containing x_i by adding the edge xx_i. By the minimality of G, $\mathrm{la}_2(G_i) \leq m$. Let $\mathcal{P}_i = \{F_{i,1}, F_{i,2}, \ldots, F_{i,m}\}$ be a 2-forest partition of G_i such that xx_i is in $F_{i,i}$ for $1 \leq i \leq s$. Since $\lceil \frac{r-s}{2} \rceil \leq m - s$, the $r - s$ edges $xx_{s+1}, xx_{s+2}, \ldots, xx_r$ can be partitioned into $m - s$ paths P_j of lengths at most two, where $s + 1 \leq j \leq m$. Then, $\mathcal{P} = \{F_1, F_2, \ldots, F_m\}$ is clearly a 2-forest partition of G, where $F_j = \cup_{1 \leq i \leq r} F_{i,j}$ for $1 \leq j \leq s$ and $F_j = \cup_{1 \leq i \leq r} F_{i,j} \cup \{P_j\}$ for $s + 1 \leq j \leq m$. So, $\mathrm{la}_2(G) \leq m$, a contradiction. Therefore, $\deg_G(v) + \deg_{D(G)}(v) \geq 2m + 1$ for all vertices v in $D(G)$.

Choose a minimal subtree H of G such that $\deg_H(x) + \deg_{D(H)}(x) \geq 2m + 1$ for all vertices in $D(H)$. Suppose $D(H)$ has a vertex x such that $\deg_H(x) + \deg_{D(H)}(x) \geq 2m + 2$. Choose any vertex y adjacent to x in H. Delete y from H when y is a leaf of H; otherwise, delete all vertices of $H - y$ not in the component containing x from H. This results a smaller tree H' such that $\deg_{H'}(v) + \deg_{D(H')}(v) \geq 2m + 1$ for all vertices v in $D(H')$, a contradiction to the choice of H. Hence H is an m-critical tree in T, a contradiction. Thus, $\mathrm{la}_2(T) = m$. ■

An efficient algorithm for determining whether $\mathrm{la}_k(T) \leq m$ for a general k is desirable.

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References

1. H. Ait-Djafter, Linear arboricity for graphs with maximum degree six or seven and edge multiplicity two, *Ars Combin.* **20** (1985), 5-16.

2. J. Akiyama, *Three Developing Topics in Graph Theory*, Doctoral Dissertation, Univ. of Tokyo, 1980.

3. J. Akiyama, A status on the linear arboricity, *Lecture Notes in Comput. Sci.* **108** (1981), 38-44.

4. J. Akiyama and V. Chvátal, A short proof of the linear arboricity for cubic graphs, *Bull. Liber. Arts and Sci., NMS* **2** (1981), 1-3.

5. J. Akiyama, G. Exoo and F. Harary, Covering and packing in graphs III, cyclic and acyclic invariants, *Math. Slovaca* **30** (1980), 405-417.

6. J. Akiyama, G. Exoo and F. Harary, Covering and packing in graphs IV, linear arboricity, *Networks* **11** (1981), 69-72.

7. J. Akiyama and M. Kano, Path factors of a graph, in: *Graph Theory and its Applications*, Wiley and Sons, New York, 1984.

8. N. Alon, The linear arboricity of graphs, *Israel J. Math.* **62** (1988), 311-325.

9. J. C. Bermond, J. L. Fouquet, M. Habib and B. Peroche, On linear k-arboricity, *Discrete Math.* **52** (1984), 123-132.

10. B. L. Chen, H. L. Fu and K. C. Huang, Decomposing graphs into forests of paths with size less than three, *Australas. J. Combin.* **3** (1991), 55-73.

11. B. L. Chen and K. C. Huang, On the linear k-arboricity of K_n and $K_{n,n}$, manuscript (1996).

12. B. L. Chen and K. C. Huang, Some results on linear k-arboricity, manuscript (1996).

13. H. Enomoto, The linear arboricity of 5-regular graphs, Technical Report, Dept. of Information Sci., Univ. of Tokyo (1981).

14. H. Enomoto and B. Peroche, The linear arboricity of some regular graphs, *J. Graph Theory* **8** (1984), 309-324.

15. H. L. Fu and K. C. Huang, The linear 2-arboricity of complete bipartite graphs, *Ars Combin.* **38** (1994), 309-318.

16. F. Guldan, The linear arboricity of 10-regular graphs, *Math. Slovaca* **36** (1986), 225-228.

17. F. Guldan, Some results on linear arboricity, *J. Graph Theory* **10** (1986), 505-509.

18. M. Habib and P. Peroche, Some problems about linear arboricity, *Discrete Math.* **41** (1982), 219-220.

19. M. Habib and P. Peroche, La k-arboricité linéaire des arbres, in: *Combinatorial Math., Ann. Discrete Math.* **17**, North-Holland, Amsterdam, 1983.

20. F. Harary, Covering and packing in graphs I, *Ann. New York Acad. Sci.* **175** (1970), 198-205.

21. I. Holyer, The NP-completeness of edge colorings, *SIAM J. Comput.* **10** (1981), 718-720.

22. F. Jaeger, *Etude de quelques invariants et problèmes d'existence en théorie des graphes*, Thèse d'Etat, IMAG Grenbole, 1976.

23. A. Nakayama and B. Peroche, Linear arboricity of digraphs, *Networks* **17** (1987), 39-53.

24. B. Peroche, Complexité de l'arboricité lináire d'un graphe, *RAIRO* **16** (1982).

25. P. Tomasta, Note on linear arboricity, *Math. Slovaca* **32** (1982), 239-242.

26. T. W. Yeh, *Linear Arboricities of Complete r-Partite Graphs*, Master Thesis, Dept. Applied Math., National Chiao Tung Univ., Taiwan, June 1997.

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