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A SURVEY ON p-ADIC NEVANLINNA THEORY AND ITS APPLICATIONS TO DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we will give a brief survey on Nevanlinna theory of p-adic meromorphic functions and some of its applications. Also we will study p-adic meromorphic solutions of differential equations, and show that some differential equations have no admissible transcendental p-adic meromorphic solutions as in the complex-valued function cases.

1. Introduction

Recently, p-adic Nevanlinna theory has become one of active mathematical fields. For example, Khóai [19], Khóai-Quang [22], and Boutabaa [3] proved p-adic analogues of two "main theorems" and defect relations of classical Nevanlinna theory. Khóai [20] and Cherry-Ye [6] studied several-variable p-adic Nevanlinna theory, and proved the defect relation of hyperplanes in general position. Hu-Yang [12]-[14] proved p-adic analogues of the defect relation for moving targets, the second main theorem for differential polynomials and unique range sets with finite elements. Cherry-Yang [5] characterized some unique range sets with finite elements for p-adic entire functions. Bézivin-Boutabaa [2] studied decomposition of p-adic meromorphic functions. Also there are some results in several variables and hyperbolicity (see, e.g., [20], [21]). In [24], p-adic valued distributions in mathematical physics were studied.

There are a lot of results on meromorphic solutions, in particular, on Malmquist-type theorems, of algebraic differential equations. For example,

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see [7], [11], [15]-[17], and [25]-[37]. In this paper, we will give a brief survey on the p-adic Nevanlinna theory and the results mentioned in the previous paragraph, and will prove p-adic analogues related to Malmquist-type theorems, and show that some p-adic algebraic differential equations have no admissible transcendental meromorphic solutions.

2. Nevanlinna Theory of p-adic Meromorphic Functions

Let p be a prime number, let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\mathbb{C}_p$ be the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value $|\ |_p$ in $\mathbb{C}_p$ is normalized so that $|\ p\ |_p = p^{-1}$. We further use the notation ord_p for the additive valuation on $\mathbb{C}_p$.

Recall that in a complete metric space whose metric comes from a non-Archimedean norm, an infinite sum converges if and only if its general term approaches zero. Then expressions of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (a_n \in \mathbb{C}_p)$$

is well-defined whenever

$$|\ a_n z^n\ |_p \to 0.$$

Define the "radius ρ of convergence" by

$$\frac{1}{\rho} = \lim_{n\to\infty} \sup |\ a_n\ |_p^{\frac{1}{n}}\ .$$

Then the series converges if $|\ z\ |_p < \rho$ and diverges if $|\ z\ |_p > \rho$. Also the function $f(z)$ is said to be p-adic analytic on $B(\rho)$, where

$$B(\rho) = \{z \in \mathbb{C}_p \ \ |\ |\ z\ |_p < \rho\}.$$

If $\rho = \infty$, the function $f(z)$ is said to be p-adic entire on $\mathbb{C}_p$.

Let f be a nonconstant p-adic analytic function on $B(\rho)$ $(0 < \rho \leq \infty)$. The essence of the Wiman-Valiron method is the analysis of the behaviour of the function by means of the *maximum term*:

$$\mu(r, f) = \max_{n\geq 0} |\ a_n\ |_p\ r^n \quad (0 < r < \rho)$$

together with the *central index*:

$$\nu(r, f) = \max_{n\geq 0} \{n \ \ |\ |\ a_n\ |_p\ r^n = \mu(r, f)\}.$$

Define

$$\nu(0,f)=\lim_{r\to 0}\nu(r,f).$$

Further, we note that if h is another p-adic analytic function on $B(\rho)$, then

$$\mu(r,fh)=\mu(r,f)\mu(r,h). \tag{1}$$

Lemma 2.1 ([12]). *The central index $\nu(r,f)$ increases as $r\to\rho$, and satisfies the formula*:

$$\log\mu(r,f)=\log\mid a_{\nu(0,f)}\mid_p+\int_0^r\frac{\nu(t,f)-\nu(0,f)}{t}dt+\nu(0,f)\log r\quad(0<r<\rho).$$

The following technical lemma can be found in [6]:

Lemma 2.2 (Weierstrass Preparation Theorem). *There exists a unique monic polynomial P of degree $\nu(r,f)$ and a p-adic analytic function g on $B[r]$ such that $f=gP$, where*

$$B[r]=\{z\in\mathbb{C}_p\ \mid\ \mid z\mid_p\le r\}.$$

Furthermore, g does not have any zero inside $B[r]$, and P has exactly $\nu(r,f)$ zeros, counting multiplicity, on $B[r]$.

Let $n(r,\frac{1}{f})$ denote the number of zeros (counting multiplicity) of f with absolute value $\le r$ and define the *valence function* of f for 0 by

$$N\left(r,\frac{1}{f}\right)=\int_0^r\frac{n(t,\frac{1}{f})-n(0,\frac{1}{f})}{t}dt+n\left(0,\frac{1}{f}\right)\log r\quad(0<r<\rho).$$

Lemma 2.2 shows that

$$n\left(r,\frac{1}{f}\right)=\nu(r,f).$$

Then Lemma 2.1 implies the *Jensen formula*:

$$N\left(r,\frac{1}{f}\right)=\log\mu(r,f)-\log\mid a_{n(0,\frac{1}{f})}\mid_p. \tag{2}$$

We also denote the number of distinct zeros of f on $B[r]$ by $\bar{n}(r,\frac{1}{f})$ and define

$$\bar{N}\left(r,\frac{1}{f}\right)=\int_0^r\frac{\bar{n}(t,\frac{1}{f})-\bar{n}(0,\frac{1}{f})}{t}dt+\bar{n}\left(0,\frac{1}{f}\right)\log r\quad(0<r<\rho).$$

For each n we draw the graph $\gamma_n(t)$ which depicts $\mathrm{ord}_p(a_nz^n)$ as a function of $t=\mathrm{ord}_p(z)$. Then $\gamma_n(t)$ is a straight line with slope n. Let $\gamma(t,f)$ denote the boundary of the intersection of all of the half-planes lying under the lines $\gamma_n(t)$. This line is what we call the Newton polygon of the function $f(z)$ (see [22]). The points t at which $\gamma(t, f)$ has vertices are called the *critical points* of $f(z)$. A finite segment $[\alpha, \beta]$ contains only finitely many critical points. It is clear that if t is a critical point, then $\mathrm{ord}_p(a_n) + nt$ attains its minimum at least at two values of n. Obviously, we have

$$\mu(r, f) = p^{-\gamma(t,f)},$$

where $r = p^{-t}$. A basic property of the Newton polygon is that, if $t = \mathrm{ord}_p(z)$ is not a critical point, then

$$| f(z) |_p = p^{-\gamma(t,f)},$$

which implies

$$| f(z) |_p = \mu(r, f).$$

By a *meromorphic function* f on $B(\rho)$ we will mean the quotient $\frac{g}{h}$ of two p-adic analytic functions g and h such that g and h have no common factors in the ring of p-adic analytic functions on $B(\rho)$. Because the function μ satisfies (1) and because greatest common divisors of any two p-adic analytic functions exist, we can uniquely extend μ to a meromorphic function $f = \frac{g}{h}$ by defining

$$\mu(r, f) = \frac{\mu(r, g)}{\mu(r, h)}.$$

Also set

$$\gamma(t, f) = \gamma(t, g) - \gamma(t, h).$$

It is clear that, if $t = \mathrm{ord}_p(z)$ is not a critical point for $f(z)$, i.e., t is not a critical point for either $g(z)$ or $h(z)$, then

$$| f(z) |_p = p^{-\gamma(t,f)} = \mu(r, f).$$

Define

$$|\mathbb{C}_p| = \{|z|_p \mid z \in \mathbb{C}_p\}.$$

Note that $\{p^w \mid w \in \mathbb{Q}\} \subseteq |\mathbb{C}_p|$. Then $|\mathbb{C}_p|$ is dense in $\mathbb{R}[0, +\infty)$.

If $a : \mathbb{R}[0, +\infty) \longrightarrow \mathbb{R}$ and $b : \mathbb{C}_p \longrightarrow \mathbb{R}$ are real-valued functions, then

$$\| \quad a(r) \leq b(z)$$

means that for any finite positive number $0 < R < \rho$, there is a finite set E in $|\mathbb{C}_p| \cap [0, R]$ such that

$$a(r) \leq b(z), \quad r = |z|_p \in |\mathbb{C}_p| \cap [0, R] - E.$$

By using this notation, we have

$$\| \quad \mu(r,f) = |f(z)|_p$$

for a p-adic meromorphic function f on $B(\rho)$.

Define the *counting function* $n(r,f)$ and the *valence function* $N(r,f)$ of f for poles respectively by

$$n(r,f) = n\left(r, \frac{1}{h}\right), \quad N(r,f) = N\left(r, \frac{1}{h}\right).$$

Then applying (2) for g and h, we obtain the *Jensen formula*:

$$N\left(r, \frac{1}{f}\right) - N(r,f) = \log \mu(r,f) - C_f, \tag{3}$$

where C_f is a constant depending only on f. Define

$$m(r,f) = \log^+ \mu(r,f) = \max\{0, \log \mu(r,f)\}.$$

Finally, we define the *characteristic function*:

$$T(r,f) = m(r,f) + N(r,f).$$

Here we exhibit some basic facts which will be used in the following sections.

Lemma 2.3 (First Main Theorem; cf. [3], [22]). *Let f be a nonconstant meromorphic function in $B(\rho)$. Then for every $a \in \mathbb{C}_p$ we have*

$$m\left(r, \frac{1}{f-a}\right) + N\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1) \quad (r \to \rho).$$

Lemma 2.4 (Lemma of Logarithmic Derivative; cf. [3], [6], [22]). *Let f be a nonconstant meromorphic function in $B(\rho)$. For any positive integer n,*

$$m\left(r, \frac{f^{(n)}}{f}\right) = O(1) \quad (r \to \rho).$$

Lemma 2.5 (Second Main Theorem; cf. [3], [6], [22]). *Let f be a nonconstant meromorphic function in $B(\rho)$ and let $a_1, \ldots, a_q$ be distinct numbers in $\mathbb{C}_p$. Then*

$$(q-1)T(r,f) \leq N(r,f) + \sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right) - N_1(r,f) - \log r + O(1),$$

where

$$N_1(r,f) = 2N(r,f) - N(r,f') + N\left(r, \frac{1}{f'}\right).$$

Furthermore, we have

$$N(r,f)+\sum_{j=1}^{q} N\left(r, \frac{1}{f-a_j}\right)-N_1(r,f) \le \bar{N}(r,f)+\sum_{j=1}^{q} \bar{N}\left(r, \frac{1}{f-a_j}\right)-N_0\left(r, \frac{1}{f'}\right),$$

$$\sum_{a\in\mathbb{C}_p\cup\{\infty\}} \Theta_f(a) \le 2,$$

where $N_0(r, \frac{1}{f'})$ *is the valence function of the zeros of* f' *where* f *does not take one of the values* $a_1, \dots, a_q$*, and where*

$$\Theta_f(a) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, \frac{1}{f-a})}{T(r,f)}.$$

3. A Defect Relation for Moving Targets

Let $\mathbb{P}^n(\mathbb{C}_p)$ denote the projective n-space over $\mathbb{C}_p$. By a *holomorphic curve*

$$f : \mathbb{C}_p \longrightarrow \mathbb{P}^n(\mathbb{C}_p),$$

we mean an equivalence class of $(n+1)$-tuples of p-adic entire functions

$$\tilde{f} = (f_0, \dots, f_n) : \mathbb{C}_p \longrightarrow \mathbb{C}_p^{n+1}$$

such that $f_0, \dots, f_n$ have no common factors in the ring of p-adic entire functions on $\mathbb{C}_p$ and such that not all of the f_j are identically zero. Here $\tilde{f}$ is also called a reduced representation of f. Write

$$\| \tilde{f}(z) \| = \max_k | f_k(z) |_p .$$

Then the *characteristic function*

$$T(r,f) = \log \| \tilde{f}(z) \| \quad (| z |_p = r)$$

is well-defined up to $O(1)$.

Let $g : \mathbb{C}_p \longrightarrow \mathbb{P}^n(\mathbb{C}_p)$ be another holomorphic curve with a reduced representation $\tilde{g} = (g_0, ..., g_n)$. The pair (f, g) is said to be *free* if

$$\langle \tilde{f}, \tilde{g} \rangle = g_0 f_0 + \cdots + g_n f_n \not\equiv 0.$$

Assume that the pair (f, g) is free and put

$$N_f(r, g) = N\left(r, \frac{1}{\langle \tilde{f}, \tilde{g} \rangle}\right), \quad m_f(r, g) = -\log \frac{\mu(r, \langle \tilde{f}, \tilde{g} \rangle)}{\| \tilde{f}(z) \| \| \tilde{g}(z) \|},$$

where $| z |_p = r$. Then the Jensen formula implies the *first main theorem*:

$$N_f(r, g) + m_f(r, g) = T(r, f) + T(r, g) + O(1).$$

The *defect* of f for g is defined by

$$\delta_f(g) = 1 - \lim_{r\to\infty} \sup \frac{N_f(r, g)}{T(r, f) + T(r, g)}.$$

For $q \geq n$, let

$$g_j : \mathbb{C}_p \longrightarrow \mathbb{P}^n(\mathbb{C}_p), \quad j = 0, \ldots, q,$$

be $q + 1$ holomorphic curves with reduced representations

$$\tilde{g} = (g_{j0}, \ldots, g_{jn}) : \mathbb{C}_p \longrightarrow \mathbb{C}_p^{n+1}.$$

The family $\{g_j\}$ is said to be in general position if $\det(g_{j_k l}) \not\equiv 0$ for any $j_0, \ldots, j_n$ with $0 \leq j_0 < \cdots < j_n \leq q$. If so, we may assume that

$$g_{j0} \not\equiv 0, \quad j = 0, \ldots, q,$$

by changing the homogeneous coordinate system of $\mathbb{P}^n(\mathbb{C}_p)$ if necessary. Then put

$$\zeta_{jk} = \frac{g_{jk}}{g_{j0}}$$

with $\zeta_{j0} = 1$. Let $\mathcal{G}$ be the smallest subfield containing

$$\{\zeta_{jk} \mid 0 \leq j \leq q, 0 \leq k \leq n\} \cup \mathbb{C}_p$$

of the meromorphic function field on $\mathbb{C}_p$. The holomorphic curve f is said to be *non-degenerate* over $\mathcal{G}$ if $f_0, ..., f_n$ are linearly independent over $\mathcal{G}$. We have the following *defect relation*:

Theorem 3.1 (Hu-Yang [12]). *Given holomorphic curves*

$$f, g_j : \mathbb{C}_p \longrightarrow \mathbb{P}^n(\mathbb{C}_p), \quad j = 0, ..., q.$$

with $q \geq n$. If the family $\{g_j\}$ is in general position such that

$$T(r, g_j) = o(T(r, f)), \quad r \to \infty, \quad j = 0, ..., q,$$

and if f is non-degenerate over $\mathcal{G}$, then

$$\sum_{j=0}^{q} \delta_f(g_j) \leq n+1.$$

Theorem 3.2 (Hu-Yang [14]). *Let V be a vector space of dimension $n+1$ over $\mathbb{C}_p$. Let $\mathcal{G} = \{g_j\}_{j=0}^{q}$ be a finite family of p-adic holomorphic curves $g_j : \mathbb{C}_p \longrightarrow \mathbb{P}(V^*)$ in general position with $q \geq n$. Take an integer k with $1 \leq k \leq n$. Let $f : \mathbb{C}_p \longrightarrow \mathbb{P}(V)$ be a p-adic holomorphic curve which is k-flat over $\mathcal{R}$ such that each pair (f, g_j) is free for $j = 0, ..., q$. Assume that g_j grows slower than f for each j. Then we have*

$$\sum_{j=0}^{q} \delta_f(g_j) \leq 2n - k + 1.$$

For the case of constant targets, see Khóai-Tu [23], Cherry-Ye [6].

4. Uniqueness of p-adic Meromorphic Functions

For a nonconstant meromorphic function f on $\mathbb{C}$ and a set $S \subset \mathbb{C} \cup \{\infty\}$ we define

$$E_f(S) = \bigcup_{a \in S} \{mz \ \mid \ f(z) = a \text{ with multiplicity } m\},$$

and

$$\bar{E}_f(S) = \bigcup_{a \in S} \{z \ \mid \ f(z) = a \text{ ignoring multiplicities }\}.$$

A set $S \subset \mathbb{C} \cup \{\infty\}$ is called a *unique range set for meromorphic functions* (URSM) if for any pair of nonconstant meromorphic functions f and g on $\mathbb{C}$, the condition $E_f(S) = E_g(S)$ implies $f = g$. A set $S \subset \mathbb{C} \cup \{\infty\}$ is called a *unique range set for entire functions* (URSE) if for any pair of non-constant entire functions f and g on $\mathbb{C}$, the condition $E_f(S) = E_g(S)$ implies $f = g$. Classical theorems of Nevanlinna show that $f = g$ if $\bar{E}_f(a_j) = \bar{E}_g(a_j)$ for distinct values $a_1, ..., a_5$, and that f is a Möbius transformation of g if $E_f(a_j) = E_g(a_j)$ for distinct values $a_1, ..., a_4$. Gross and Yang [10] showed that the set

$$S = \{z \in \mathbb{C} \ \mid \ z + e^z = 0\}$$

is a URSE. Recently, URSE and also URSM with finitely many elements have been found by Yi ([38], [39]), Li-Yang ([27], [28]), Mues-Reinders [31], and Frank-Reinders [9]. Li-Yang [27] introduced the notation

$$\begin{aligned} \lambda_M &= \inf\{\#S \ \mid \ S \text{ is a URSM }\}, \\ \lambda_E &= \inf\{\#S \ \mid \ S \text{ is a URSE }\}, \end{aligned}$$

where $\#S$ is the cardinality of the set S. The best lower and upper bounds known so far are

$$5 \leq \lambda_E \leq 7, \quad 6 \leq \lambda_M \leq 11.$$

For a p-adic meromorphic (or entire) function f on $\mathbb{C}_p$, we can similarly define $E_f(S)$ and $\bar{E}_f(S)$ for a set $S \subset \mathbb{C}_p \cup \{\infty\}$, and introduce the notation λ_M and λ_E. We recall the following useful fact (see [6]):

Lemma 4.1. *If f is a p-adic entire function on $\mathbb{C}_p$ that is never zero, then f is constant.*

Theorem 4.1 (Hu-Yang [12]). *Let f, g be nonconstant p-adic meromorphic functions on $\mathbb{C}_p$. Let a_1, a_2, a_3, a_4 be four different points in $\mathbb{C}_p \cup \{\infty\}$. Assume*

$$\bar{E}_f(a_j) = \bar{E}_g(a_j), \quad j = 1, ..., 4.$$

Then $f \equiv g$.

Theorem 4.2 (Hu-Yang [12]). *Assume that f, g are nonconstant p-adic meromorphic functions on $\mathbb{C}_p$ for which there exist three distinct values $a_1, a_2, a_3 \in \mathbb{C}_p \cup \{\infty\}$ such that*

$$E_f(a_j) = E_g(a_j), \quad j = 1, 2, 3.$$

Then $f \equiv g$.

Adams-Strauss [1] showed that if f and g are two nonconstant p-adic entire functions on $\mathbb{C}_p$ for which there exist two distinct values $a_1, a_2 \in \mathbb{C}_p$ such that

$$E_f(a_j) = E_g(a_j), \quad j = 1, 2,$$

then $f \equiv g$.

Theorem 4.3 (Hu-Yang [12]). *If f is a nonconstant p-adic analytic function on $\mathbb{C}_p$, then there is no $a \in \mathbb{C}_p$ such that $E_f(a) = E_{f'}(a)$.*

Theorem 4.4 (Hu-Yang [12]). *Take an integer $n \geq 4$ and choose $a, b \in \mathbb{C}_p - \{0\}$ such that the set*

$$S = \{z \in \mathbb{C}_p \ \mid \ z^n + az^{n-1} + b = 0\}$$

contains n distinct elements. If f and g are nonconstant p-adic analytic functions on $\mathbb{C}_p$ such that $E_f(S) = E_g(S)$, then $f \equiv g$.

Theorem 4.5 (Hu-Yang [12]). *Take an integer $n \geq 12$ and choose $a, b \in \mathbb{C}_p - \{0\}$ such that the set*

$$S = \{z \in \mathbb{C}_p \mid z^n + az^{n-2} + b = 0\}$$

contains n distinct elements. If f and g are nonconstant p-adic meromorphic functions on $\mathbb{C}_p$ such that $E_f(S) = E_g(S)$, then $f \equiv g$.

Of course, a URSE must have at least three points, because given two points a, b, there does exist an affine function of the form $h(z) = cz + d$ with $c \neq 0$ such that $h(a) = b, h(b) = a$, and therefore, putting $S = \{a, b\}$, it is easily seen that for every entire function f, we have $E_f(S) = E_{h\circ f}(S)$.

In the same way, a URSM must have at least 4 points, because given 3 points a, b, c, there does exist a bilinear function h that permutes the set $S = \{a, b, c\}$ (in a nontrivial way) and therefore, for every meromorphic function f, we have $E_f(S) = E_{h\circ f}(S)$.

Boutabaa, Escassut and Haddad [4] announced recently that they have characterized the URS's for polynomials in any algebraically closed field, and proved that in non-Archimedean analysis, there exist URS's of n elements for entire functions for any $n \geq 3$. When $n = 3$, they characterized the sets of three elements that are URSE.

Theorem 4.6 (Hu-Yang [13]). *Take an integer $n \geq 10$ and let $b \in \mathbb{C}_p - \{0, -1\}$. Then the polynomial $P(z)$ defined by*

$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} + b$$

has only simple zeros, and if f and g are nonconstant p-adic meromorphic functions on $\mathbb{C}_p$ such that $E_f(S) = E_g(S)$, then $f \equiv g$, where

$$S = \{z \in \mathbb{C}_p \mid P(z) = 0\}.$$

5. Growth Estimates of p-adic Meromorphic Functions

Let $\mathcal{M}(\mathbb{C}_p)$ be the space of p-adic meromorphic functions on $\mathbb{C}_p$. Define

$$A(z, w) = \sum_{j=0}^{k} a_j(z) w^j, \tag{4}$$

where $a_j \in \mathcal{M}(\mathbb{C}_p)$ with $a_k \not\equiv 0$.

Lemma 5.1. *If $w \in \mathcal{M}(\mathbb{C}_p)$, then*

$$N(r, A) = kN(r, w) + O\left(\sum_{j=0}^{k}\left(N(r, a_j) + N\left(r, \frac{1}{a_j}\right)\right)\right). \tag{5}$$

Proof. For $a \in \mathbb{C}_p \cup \{\infty\}$, let $\mu_w^a(z)$ denote the a-valued multiplicity of w at z. Obviously, we have

$$\mu_A^\infty \leq k\mu_w^\infty + \sum_{j=0}^{k} \mu_{a_j}^\infty,$$

and hence

$$N(r, A) \leq kN(r, w) + \sum_{j=0}^{k} N(r, a_j). \tag{6}$$

Now we prove the following inequality

$$\mu_A^\infty \geq k\mu_w^\infty - k\sum_{j=0}^{k}(\mu_{a_j}^\infty + \mu_{a_j}^0). \tag{7}$$

Define

$$b_j(z) = a_j(z)w(z)^j, \quad j = 0, ..., k.$$

Now fix $z \in \mathbb{C}_p$. If $\mu_w^\infty(z) = 0$, it clearly holds. Assume that $\mu_w^\infty(z) > 0$. If

$$\mu_{b_j}^\infty(z) < \mu_{b_k}^\infty(z) \quad (j < k),$$

then

$$\mu_A^\infty(z) = \mu_{b_k}^\infty(z) \geq k\mu_w^\infty(z) + \mu_{a_k}^\infty(z) - \mu_{a_k}^0(z) \geq k\mu_w^\infty(z) - \mu_{a_k}^0(z).$$

If there exists $l < k$ such that

$$\mu_{b_j}^\infty(z) < \mu_{b_l}^\infty(z) \quad (j \neq l),$$

then for $j = k$ we have

$$k\mu_w^\infty(z) + \mu_{a_k}^\infty(z) - \mu_{a_k}^0(z) < l\mu_w^\infty(z) + \mu_{a_l}^\infty(z) - \mu_{a_l}^0(z),$$

which implies

$$\mu_w^\infty(z) \leq (k - l)\mu_w^\infty(z) \leq \mu_{a_l}^\infty(z) + \mu_{a_k}^0(z).$$

If $\mu_{b_j}^\infty(z) = \mu_{b_l}^\infty(z)$ for some $j > l$, then

$$\mu_w^\infty(z) \leq (j - l)\mu_w^\infty(z) \leq \mu_{a_l}^\infty(z) + \mu_{a_j}^0(z).$$

Hence (7) follows, and consequently

$$N(r,A) \geq kN(r,w) - k\sum_{j=0}^{k}\left(N(r,a_j) + N\left(r,\frac{1}{a_j}\right)\right). \tag{8}$$

Now clearly (5) follows from (6) and (8). ∎

Lemma 5.2. *If $w \in \mathcal{M}(\mathbb{C}_p)$, then*

$$m(r,A) = km(r,w) + O\left(\sum_{j=0}^{k}\left(m(r,a_j) + m\left(r,\frac{1}{a_j}\right)\right)\right). \tag{9}$$

Proof. Note that

$$\begin{aligned} \| \ \mu(r,A) &= |A(z,w(z))|_p \leq \max_{0\leq j\leq k}\{|a_j(z)|_p|w(z)|_p^j\} \\ &= \max_{0\leq j\leq k}\{\mu(r,a_j)\mu(r,w)^j\}, \end{aligned}$$

and consequently

$$\mu(r,A) \leq \max_{0\leq j\leq k}\{\mu(r,a_j)\mu(r,w)^j\}$$

holds for all $r > 0$ by continuity of the μ functions. Thus we obtain

$$m(r,A) \leq km(r,w) + \max_{0\leq j\leq k} m(r,a_j). \tag{10}$$

Take $z \in \mathbb{C}_p$ with

$$w(z) \neq 0,\infty; \quad a_j(z) \neq 0,\infty \quad (0 \leq j \leq k),$$

and define

$$\mathcal{A}(z) = \max_{0\leq j<k}\left\{1, \left(\frac{|a_j(z)|_p}{|a_k(z)|_p}\right)^{\frac{1}{k-j}}\right\}.$$

If $|w(z)|_p > \mathcal{A}(z)$, we see

$$|a_j(z)|_p|w(z)|_p^j \leq |a_k(z)|_p\mathcal{A}(z)^{k-j}|w(z)|_p^j < |a_k(z)|_p|w(z)|_p^k.$$

Hence

$$|A(z,w(z))|_p = |a_k(z)|_p|w(z)|_p^k.$$

Setting $r = |z|_p$, we obtain

$$\mu(r,w)^k = \frac{\mu(r,A)}{\mu(r,a_k)}.$$

If $|w(z)|_p \leq \mathcal{A}(z)$, we have

$$\mu(r,w)^k \leq \max_{0\leq j<k} \left\{1, \left(\frac{\mu(r,a_j)}{\mu(r,a_k)}\right)^{\frac{k}{k-j}}\right\}.$$

Therefore we obtain

$$\| \ \mu(r,w)^k \leq \max_{0\leq j<k} \left\{1, \frac{\mu(r,A)}{\mu(r,a_k)}, \left(\frac{\mu(r,a_j)}{\mu(r,a_k)}\right)^{\frac{k}{k-j}}\right\}.$$

Thus by continuity of the μ functions, we have

$$km(r,w) \leq m(r,A) + km\left(r, \frac{1}{a_k}\right) + k \max_{0\leq j<k} m(r,a_j). \tag{11}$$

Thus (9) follows from (10) and (11). ∎

Now Lemmas 5.1 and 5.2 yield the following result:

Theorem 5.1. *If $w \in \mathcal{M}(\mathbb{C}_p)$, then*

$$T(r,A) = kT(r,w) + O\left(\sum_{j=0}^{k} T(r,a_j)\right). \tag{12}$$

Take $\{b_0, ..., b_q\} \subset \mathcal{M}(\mathbb{C}_p)$ with $b_q \not\equiv 0$ and define

$$B(z,w) = \sum_{j=0}^{q} b_j(z)w^j. \tag{13}$$

Assume that $A(z,w)$ and $B(z,w)$ are coprime polynomials in w. Define

$$R(z,w) = \frac{A(z,w)}{B(z,w)}. \tag{14}$$

Theorem 5.2. *If $w \in \mathcal{M}(\mathbb{C}_p)$, then*

$$T(r,R) = \max\{k,q\}T(r,w) + O\left(\sum_{j=0}^{k} T(r,a_j) + \sum_{j=0}^{q} T(r,b_j)\right). \tag{15}$$

Proof. W.l.o.g, we may assume $\deg(A) = k \geq q = \deg(B)$. By using the algorithm of division, we have

$$\begin{aligned} A(z,w) &= P_1(z,w)B(z,w) + Q_1(z,w), \\ &\quad \deg(P_1) = k-q, \quad \deg(Q_1) = t_1 < q, \end{aligned}$$

$$\begin{aligned} B(z,w) &= P_2(z,w)Q_1(z,w)+Q_2(z,w),\\ &\quad \deg(P_2)=q-t_1, \quad \deg(Q_2)=t_2<t_1,\\ &\quad \dots\ \dots\\ Q_{m-2}(z,w) &= P_m(z,w)Q_{m-1}(z,w)+Q_m(z),\\ &\quad \deg(P_m)=t_{m-2}-t_{m-1}, \quad \deg(Q_m)=t_m=0. \end{aligned}$$

Since $A(z,w)$ and $B(z,w)$ are coprime, then $Q_m(z)\not\equiv 0$, and

$$A(z,w)Q(z,w)+B(z,w)P(z,w)=1, \tag{16}$$

where $P(z,w)$ and $Q(z,w)$ are polynomials in w such that

$$\deg(P)\le k-1, \quad \deg(Q)\le q-1,$$

and such that coefficients are rational functions of $\{a_j(z)\}$ and $\{b_j(z)\}$. Note that

$$k+\deg(Q)=q+\deg(P), \quad k\ge q.$$

We also have $\deg(Q)\le\deg(P)$. By Theorem 5.1 and the first main theorem, we see

$$\begin{aligned} T(r,R) &= T\left(r,\frac{A}{B}\right)\le T(r,P_1)+T\left(r,\frac{Q_1}{B}\right)\\ &= T(r,P_1)+T\left(r,\frac{B}{Q_1}\right)+O(1)\\ &\le T(r,P_1)+\cdots+T(r,P_m)+T\left(r,\frac{Q_{m-1}}{Q_m}\right)+O(1)\\ &= (k-q)T(r,w)+(q-t_1)T(r,w)+\cdots(t_{m-1}-t_m)T(r,w) \qquad (17)\\ &\quad +O\left(\sum_{j=0}^{k}T(r,a_j)+\sum_{j=0}^{q}T(r,b_j)\right)\\ &= kT(r,w)+O\left(\sum_{j=0}^{k}T(r,a_j)+\sum_{j=0}^{q}T(r,b_j)\right). \end{aligned}$$

Now we use induction. If $q=0$, Theorem 5.2 follows from Theorem 5.1. Assume Theorem 5.2 holds for rational functions of w with the degree of denominators $\le q-1$. By (16), we have

$$\begin{aligned} T\left(r,\frac{Q}{P}+\frac{B}{A}\right) &= T\left(r,\frac{1}{AP}\right)=T(r,AP)+O(1)\\ &= (k+\deg(P))T(r,w)+O\left(\sum_{j=0}^{k}T(r,a_j)+\sum_{j=0}^{q}T(r,b_j)\right). \end{aligned}$$

By the assumption of the induction, we see

$$\begin{aligned} T\left(r, \frac{Q}{P} + \frac{B}{A}\right) &\leq T\left(r, \frac{P}{Q}\right) + T(r, R) + O(1) \\ &\leq \deg(P)T(r, w) + T(r, R) \\ &\quad + O\left(\sum_{j=0}^{k} T(r, a_j) + \sum_{j=0}^{q} T(r, b_j)\right). \end{aligned}$$

Therefore we obtain

$$kT(r, w) \leq T(r, R) + O\left(\sum_{j=0}^{k} T(r, a_j) + \sum_{j=0}^{q} T(r, b_j)\right),$$

which combines (17) to imply (15). ∎

For meromorphic functions on the complex plane $\mathbb{C}$, Theorem 5.2 was proved by Gackstatter and Laine [7], and Mokhon'ko [30]. Also see He-Xiao [8]. For several variables, see Hu-Yang [16], [17].

Theorem 5.3. *If w is a nonconstant p-adic entire function and if $f \in \mathcal{M}(\mathbb{C}_p) - \mathbb{C}_p(z)$, then*

$$\lim_{r\to\infty} \frac{T(r, f\circ w)}{T(r, w)} = +\infty.$$

Proof. Since $f \in \mathcal{M}(\mathbb{C}_p) - \mathbb{C}_p(z)$, there exists some $c \in \mathbb{C}_p$ such that $f - c = 0$ has infinitely many zero points $a_1, a_2, ...$ with $|a_j - a_l|_p > 1$ $(j \neq l)$. Set

$$f(z) - c = (z - a_j)g_j(z), \quad j = 1, 2, ... \ .$$

Then for any positive integer ν, there exist positive constants K and $\delta(< \frac{1}{2})$ such that

$$|g_j(z)|_p \leq K, \quad |z - a_j|_p \leq \delta, \quad j = 1, ..., \nu.$$

Hence we have

$$\log^+ \frac{1}{|f(z) - c|_p} \geq \sum_{j=1}^{\nu} \log^+ \frac{\delta}{|z - a_j|_p} - \log^+(\delta K), \quad z \in \mathbb{C}_p,$$

which yields

$$m\left(r, \frac{1}{f\circ w - c}\right) \geq \sum_{j=1}^{\nu} m\left(r, \frac{1}{w - a_j}\right) - \nu \log^+ \frac{1}{\delta} - \log^+(\delta K).$$

Note that
$$\sum_{j=1}^{\nu} N\left(r, \frac{1}{w-a_j}\right) \le N\left(r, \frac{1}{f\circ w - c}\right).$$
Adding up the two inequalities above and using the first main theorem, we have
$$\nu T(r,w) \le T(r, f\circ w) + O(1).$$
Since $T(r,w)\to\infty$ as $r\to\infty$, the theorem follows. ∎

Corollary 5.1. *A p-adic meromorphic function f on $\mathbb{C}_p$ is a rational function of degree d if and only if, for any nonconstant p-adic entire function w on $\mathbb{C}_p$, we have*
$$\lim_{r\to\infty} \frac{T(r, f\circ w)}{T(r,w)} = d.$$

Corollary 5.2. *A p-adic meromorphic function f on $\mathbb{C}_p$ is a rational function of degree d if and only if*
$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = d.$$

Traditionally, p-adic meromorphic functions in $\mathcal{M}(\mathbb{C}_p) - \mathbb{C}_p(z)$ are called transcendental. Obviously, a p-adic meromorphic function f on $\mathbb{C}_p$ is transcendental if and only if
$$\lim_{r\to\infty} \frac{T(r,f)}{\log r} = +\infty.$$

6. Malmquist-type Theorems (I)

We talk of a p-adic algebraic differential equation if it is of the form
$$\Omega(z,w,w',...,w^{(n)}) = R(z,w), \tag{18}$$
where
$$\Omega(z,w,w',...,w^{(n)}) = \sum_{i\in I} c_i w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n} \tag{19}$$
and $i=(i_0,i_1,...,i_n)$ are nonnegative integer indices, I is a finite set, $c_i\in\mathcal{M}(\mathbb{C}_p)$, and $R(z,w)$ is a p-adic meromorphic function on $\mathbb{C}_p^2$. Define
$$\deg(\Omega) = \max_{i\in I}\left\{\sum_{\alpha=0}^{n} i_\alpha\right\},\ \Gamma(\Omega) = \max_{i\in I}\left\{\sum_{\alpha=0}^{n}(\alpha+1)i_\alpha\right\},\ \gamma(\Omega) = \max_{i\in I}\left\{\sum_{\alpha=1}^{n}\alpha i_\alpha\right\}.$$

We first give some properties of the differential operator Ω. For $w \in \mathcal{M}(\mathbb{C}_p)$, we abbreviate

$$\Omega(z) = \Omega(z, w(z), w'(z), ..., w^{(n)}(z)).$$

Note that

$$N(r, w^{(\alpha)}) = N(r, w) + \alpha \bar{N}(r, w) \leq (\alpha + 1) N(r, w).$$

We have

$$N(r, \Omega) \leq \deg(\Omega) N(r, w) + \gamma(\Omega) \bar{N}(r, w) + \sum_{i \in I} N(r, c_i), \tag{20}$$

and

$$N(r, \Omega) \leq \Gamma(\Omega) N(r, w) + \sum_{i \in I} N(r, c_i). \tag{21}$$

Obviously, we have

$$m(r, \Omega) \leq \deg(\Omega) m(r, w) + \max_{i \in I} \left\{ m(r, c_i) + \sum_{\alpha=1}^{n} i_\alpha m\left(r, \frac{w^{(\alpha)}}{w}\right) \right\}. \tag{22}$$

Thus we obtain from the Logarithmic Derivative Lemma

$$T(r, \Omega) \leq \deg(\Omega) T(r, w) + \gamma(\Omega) \bar{N}(r, w) + \sum_{i \in I} T(r, c_i) + O(1) \tag{23}$$

and

$$T(r, \Omega) \leq \Gamma(\Omega) T(r, w) + \sum_{i \in I} T(r, c_i) + O(1). \tag{24}$$

Next, we will keep the notations in §5 and consider the equation (18) with $R(z, w)$ defined by (14). The following Clunie-type theorem will play an important role in the proof of Malmquist-type theorems.

Lemma 6.1. *Let $w \in \mathcal{M}(\mathbb{C}_p)$ be a solution of* (18). *If $q \geq k$, then*

$$m(r, \Omega) \leq \sum_{i \in I} m(r, c_i) + \sum_{j=0}^{k} m(r, a_j) + O\left(m\left(r, \frac{1}{b_q}\right) + \sum_{j=0}^{q} m(r, b_j) \right), \tag{25}$$

$$N(r, \Omega) \leq \sum_{i \in I} N(r, c_i) + \sum_{j=0}^{k} N(r, a_j) + O\left(\sum_{j=0}^{q} N\left(r, \frac{1}{b_j}\right) \right). \tag{26}$$

Proof. Take $z \in \mathbb{C}_p$ with

$$w(z) \neq 0, \infty; \quad a_j(z) \neq 0, \infty \quad (0 \leq j \leq k);$$

$$c_i(z) \neq 0, \infty \quad (i \in I); \quad b_j(z) \neq 0, \infty \quad (0 \leq j \leq q),$$

and define

$$\mathcal{B}(z) = \max_{0 \leq j < q} \left\{ 1, \left(\frac{|b_j(z)|_p}{|b_q(z)|_p} \right)^{\frac{1}{q-j}} \right\}.$$

If $|w(z)|_p > \mathcal{B}(z)$, we see

$$|b_j(z)|_p |w(z)|_p^j \leq |b_q(z)|_p \mathcal{B}(z)^{q-j} |w(z)|_p^j < |b_q(z)|_p |w(z)|_p^q.$$

Hence

$$|B(z, w(z))|_p = |b_q(z)|_p |w(z)|_p^q.$$

Then

$$|\Omega(z)|_p = \frac{|A(z, w(z))|_p}{|B(z, w(z))|_p} \leq \frac{1}{|b_q(z)|_p} \max_{0 \leq j \leq k} |a_j(z)|_p.$$

If $|w(z)|_p \leq \mathcal{B}(z)$, then

$$|\Omega(z)|_p \leq \mathcal{B}(z)^{\deg(\Omega)} \max_{i \in I} |c_i(z)|_p \left| \frac{w'(z)}{w(z)} \right|_p^{i_1} \cdots \left| \frac{w^{(n)}(z)}{w(z)} \right|_p^{i_n}.$$

Therefore

$$\begin{aligned} \| \mu(r, \Omega) \leq \max_{0 \leq j \leq k, i \in I} \Bigg\{ & \frac{\mu(r, a_j)}{\mu(r, b_q)}, \mu(r, c_i) \mu \left(r, \frac{w'}{w} \right)^{i_1} \cdots \\ & \mu \left(r, \frac{w^{(n)}}{w} \right)^{i_n} \max_{0 \leq j < q} \left\{ 1, \mu \left(r, \frac{b_j}{b_q} \right)^{\frac{\deg(\Omega)}{q-j}} \right\} \Bigg\}, \end{aligned}$$

which also holds for all $r > 0$ by continuity of the μ functions. Thus (25) follows from this inequality and the lemma of logarithmic derivative.

Now we prove (26). Take a point $z_0 \in \mathbb{C}_p$ with $w(z_0) = \infty$. Then

$$\mu_A^\infty(z_0) \leq k \mu_w^\infty(z_0) + \sum_{j=0}^{k} \mu_{a_j}^\infty(z_0),$$

$$\mu_B^\infty(z_0) \geq q \mu_w^\infty(z_0) - \sum_{j=0}^{q} \mu_{b_j}^0(z_0).$$

If $q\mu_w^\infty(z_0) - \sum_{j=0}^q \mu_{b_j}^0(z_0) > 0$, then

$$\mu_\Omega^\infty(z_0) \le \mu_A^\infty(z_0) - \mu_B^\infty(z_0) \le \sum_{j=0}^k \mu_{a_j}^\infty(z_0) + \sum_{j=0}^q \mu_{b_j}^0(z_0),$$

since $q \ge k$. If $q\mu_w^\infty(z_0) - \sum_{j=0}^q \mu_{b_j}^0(z_0) \le 0$, i.e.,

$$\mu_w^\infty(z_0) \le \frac{1}{q}\sum_{j=0}^q \mu_{b_j}^0(z_0),$$

then

$$\mu_\Omega^\infty(z_0) \le \Gamma(\Omega)\mu_w^\infty(z_0) + \sum_{i\in I}\mu_{c_i}^\infty(z_0) \le \frac{\Gamma(\Omega)}{q}\sum_{j=0}^q \mu_{b_j}^0(z_0) + \sum_{i\in I}\mu_{c_i}^\infty(z_0).$$

Therefore,

$$\mu_\Omega^\infty \le \sum_{j=0}^k \mu_{a_j}^\infty + \max\left\{1, \frac{\Gamma(\Omega)}{q}\right\}\sum_{j=0}^q \mu_{b_j}^0 + \sum_{i\in I}\mu_{c_i}^\infty.$$

Hence (26) follows. ∎

Definition 6.1. A solution w of (18) with $R(z,w)$ defined by (14) is said to be admissible if $w \in \mathcal{M}(\mathbb{C}_p)$ satisfies (18) with

$$\sum_{i\in I} T(r, c_i) + \sum_{j=0}^k T(r, a_j) + \sum_{j=0}^q T(r, b_j) = o(T(r,w)).$$

Theorem 6.1. *If R is of the form* (14) *and if* (18) *has an admissible solution w, then*

$$q = 0, \quad k \le \min\{\Gamma(\Omega), \deg(\Omega) + \gamma(\Omega)(1 - \Theta_w(\infty))\}.$$

Proof. The equation (18) can be rewritten as follows

$$\Omega(z, w, w', ..., w^{(n)}) = A_1(z,w) + \frac{A_2(z,w)}{B(z,w)},$$

where $\deg(A_1) = k - q$ if $k \ge q$, and $\deg(A_2) = k_2 < q$. By Lemma 6.1, we have

$$T(r, \Omega - A_1) = o(T(r,w)).$$

Theorem 5.2 implies

$$T(r, \Omega - A_1) = T\left(r, \frac{A_2}{B}\right) = qT(r,w) + o(T(r,w)).$$

Thus we obtain $q = 0$, and (18) becomes

$$\Omega(z, w, w', ..., w^{(n)}) = A(z, w).$$

By using Theorem 5.1, we have

$$T(r, \Omega) = T(r, A) = kT(r, w) + o(T(r, w)).$$

Our result follows from this, (23) and (24). ■

For meromorphic functions on the complex plane $\mathbb{C}$, this theorem is well-known, called Malmquist-type theorem; see Malmquist [29], Gackstatter-Laine [7], Laine [26], Toda [34] and Yosida [37]. Also see He-Xiao [8]. For several variables, see Hu-Yang [16] and [17]. Theorem 6.1 implies the following result of Boutabaa [3].

Corollary 6.1. *Let $\Omega(z, w, w', ..., w^{(n)})$ be a differential polynomial with coefficients in $\mathbb{C}_p(z)$ and let $R(z, w) \in \mathbb{C}_p(z, w)$. If (18) has a p-adic meromorphic solution $w = w(z) \notin \mathbb{C}_p(z)$, then $R(z, w)$ is a polynomial in w of degree $\leq \Gamma(\Omega)$.*

Finally, we study (18) for more general $R(z, w)$.

Theorem 6.2. *Take $R \in \mathcal{M}(\mathbb{C}_p)$. If the following differential equation*

$$\Omega(z, w, w', ..., w^{(n)}) = R(w)$$

has a nonconstant solution $w \in \mathcal{M}(\mathbb{C}_p)$ satisfying

$$\sum_{i \in I} T(r, c_i) = o(T(r, w)),$$

then R is a polynomial with

$$\deg(R) \leq \min\{\Gamma(\Omega), \deg(\Omega) + \gamma(\Omega)(1 - \Theta_w(\infty))\}.$$

Proof. Note that

$$\lim_{r \to \infty} \frac{T(r, R \circ w)}{T(r, w)} = \lim_{r \to \infty} \frac{T(r, \Omega)}{T(r, w)} \leq \Gamma(\Omega).$$

Hence R is a rational function. Now the theorem follows from Theorem 6.1. ■

For meromorphic functions on the complex plane $\mathbb{C}$, this theorem is also well-known; see Rellich [32], Wittich [36], Laine [25], or He-Xiao [8]. For several variables, see Hu-Yang [15] and [16].

Theorem 6.3. *Let $a_1, a_2, \ldots$ be a sequence of distinct p-adic numbers which tends to a finite limit value a, and let $R(z,w)$ be a p-adic meromorphic function on $\mathbb{C}_p^2$. If (18) has a p-adic meromorphic solution w satisfying*

$$\sum_{i\in I} T(r,c_i) + T(r,R_j) = o(T(r,w)), \quad j = 1,2,\ldots,$$

where $R_j(z) = R(z,a_j)$, then $R(z,w)$ is a polynomial in w with

$$\deg_w(R) \le \min\{\Gamma(\Omega), \deg(\Omega) + \gamma(\Omega)(1-\Theta_w(\infty))\}.$$

Proof. Define

$$\varphi[a_1] = \frac{\Omega - R_1}{w - a_1},$$

$$\begin{aligned}\varphi[a_1,a_2] &= \frac{\varphi[a_1]-\varphi[a_2]}{a_1-a_2} = \frac{\Omega}{(w-a_1)(w-a_2)} \\ &\quad - \frac{R_1}{(a_1-a_2)(w-a_1)} + \frac{R_2}{(a_1-a_2)(w-a_2)},\end{aligned}$$

and inductively define

$$\begin{aligned}\varphi[a_1,\ldots,a_l] &= \frac{\varphi[a_1,\ldots,a_{l-1}] - \varphi[a_1,\ldots,a_{l-2},a_l]}{a_{l-1}-a_l} \\ &= \frac{\Omega}{(w-a_1)\cdots(w-a_l)} + \sum_{j=1}^{l} \frac{\hat{a}_{lj}R_j}{w-a_j} \\ &= \frac{\Omega - Q_l(z,w)}{(w-a_1)\cdots(w-a_l)} \quad (l \ge 3),\end{aligned}$$

where $\hat{a}_{lj}$ are constants depending on $\{a_1,\ldots,a_l\}$, and $Q_l(z,w)$ is a polynomial in w of degree $\le l-1$ with coefficients being linear combinations in $R_j (1 \le j \le l)$. Here we write

$$\nu = \Gamma(\Omega), \quad \varphi_l = \varphi[a_{l(\nu+1)+1},\ldots,a_{(l+1)(\nu+1)}], \quad l = 0,1,\ldots\ .$$

We claim that $\varphi_l \equiv 0$ for some $l \ge 0$.

Assume to the contrary that $\varphi_l \not\equiv 0$ for all $l \ge 0$. Then

$$\begin{aligned}T(r,w) &= T(r,w-a_{\nu+1}) + O(1) \\ &\le T(r,(w-a_{\nu+1})\varphi_0) + T(r,\varphi_0) + O(1).\end{aligned} \tag{27}$$

Note that

$$\left|\frac{w(z)}{w(z)-a_j}\right|_p \le \hat{a}\max\left\{1, \frac{1}{|w(z)-a_j|_p}\right\}, \quad j=1,...,\nu+1,$$

where $\hat{a}=\max_{1\le j\le \nu+1}\{1+|a_j|_p\}$. By using the lemma of logarithmic derivative, we have

$$\begin{aligned} m(r,\varphi_0) &\le m\left(r, \frac{\Omega}{(w-a_1)\cdots(w-a_{\nu+1})}\right) + m\left(r, \sum_{j=1}^{\nu+1}\frac{\hat{a}_{\nu+1,j}R_j}{w-a_j}\right) \\ &\le 2\sum_{j=1}^{\nu+1} m\left(r, \frac{1}{w-a_j}\right) + \sum_{i\in I} m(r,c_i) + \sum_{j=1}^{\nu+1} m(r,R_j) + O(1), \end{aligned} \tag{28}$$

$$\begin{aligned} m(r,(w-a_{\nu+1})\varphi_0) \le & 2\sum_{j=1}^{\nu} m\left(r, \frac{1}{w-a_j}\right) + \sum_{i\in I} m(r,c_i) \\ & + \sum_{j=1}^{\nu+1} m(r,R_j) + O(1). \end{aligned} \tag{29}$$

Now we consider the poles of φ_0. Fix $z_0 \in \mathbb{C}_p$. Since w is the solution of (18), we have

$$\mathrm{supp}\mu_w^{a_1} \subset \mathrm{supp}\mu_{\Omega-R_1}^0.$$

Thus if $\mu_w^{a_1}(z_0) > 0$, then

$$\mu_{\varphi[a_1]}^{\infty}(z_0) \le \mu_w^{a_1}(z_0) - 1.$$

By induction, if $\mu_w^{a_j}(z_0) > 0$ for some j with $1 \le j \le \nu+1$, but $c_i(z_0) \ne \infty (i \in I)$, $R_l(z_0) \ne \infty (1 \le l \le \nu+1)$, then we have

$$\mu_{\varphi_0}^{\infty}(z_0) \le \mu_w^{a_j}(z_0) - 1.$$

If $\mu_w^{\infty}(z_0) > 0$, then

$$\begin{aligned} \mu_{\varphi_0}^{\infty}(z_0) &\le \max\{0, \max\{\mu_{\Omega}^{\infty}(z_0), \mu_{Q_{\nu+1}}^{\infty}(z_0)\} - (\nu+1)\mu_w^{\infty}(z_0)\} \\ &\le \sum_{i\in I}\mu_{c_i}^{\infty}(z_0) + \sum_{j=1}^{\nu+1}\mu_{R_j}^{\infty}(z_0). \end{aligned}$$

Therefore,

$$\begin{aligned} N(r,\varphi_0) \le & \sum_{j=1}^{\nu+1}\left\{N\left(r, \frac{1}{w-a_j}\right) - \bar{N}\left(r, \frac{1}{w-a_j}\right)\right\} \\ & + \sum_{i\in I} N(r,c_i) + \sum_{j=1}^{\nu+1} N(r,R_j). \end{aligned} \tag{30}$$

Similarly, we have

$$N(r,(w-a_{\nu+1})\varphi_0) \le \sum_{j=1}^{\nu+1}\left\{N\left(r,\frac{1}{w-a_j}\right)-\bar{N}\left(r,\frac{1}{w-a_j}\right)\right\} + \sum_{i\in I}N(r,c_i)+\sum_{j=1}^{\nu+1}N(r,R_j). \tag{31}$$

Therefore, by (27)-(31), we obtain

$$T(r,w)\le 4\sum_{j=1}^{\nu+1}m(r,\frac{1}{w-a_j})+2\sum_{j=1}^{\nu+1}\{N(r,\frac{1}{w-a_j})-\bar{N}(r,\frac{1}{w-a_j})\}+o(T(r,w)).$$

In a similar fashion, we have, for $l\ge 0$,

$$T(r,w)\le 2\sum_{j=l(\nu+1)+1}^{(l+1)(\nu+1)}\{2m(r,\frac{1}{w-a_j})+N(r,\frac{1}{w-a_j})-\bar{N}(r,\frac{1}{w-a_j})\}+o(T(r,w)).$$

Therefore,

$$lT(r,w)\le 2\sum_{j=1}^{l(\nu+1)}\{2m(r,\frac{1}{w-a_j})+N(r,\frac{1}{w-a_j})-\bar{N}(r,\frac{1}{w-a_j})\}+o(T(r,w)).$$

By using the second main theorem, we obtain

$$(l-8)T(r,w)\le o(T(r,w)).$$

This is impossible if $l>8$.

Hence $\varphi_l\equiv 0$ for some $l\ge 0$, say, $l=0$. It follows that w satisfies the following equation

$$\Omega(z,w,w',...,w^{(n)})=Q_{\nu+1}(z,w).$$

Define

$$H(z,w)=R(z,w)-Q_{\nu+1}(z,w),\quad H_j(z)=H(z,a_j).$$

If $H_j\not\equiv 0$, then

$$\bar{N}\left(r,\frac{1}{w-a_j}\right)\le N\left(r,\frac{1}{H_j}\right)\le T(r,H_j)+O(1) \le T(r,R_j)+\sum_{l=1}^{\nu+1}T(r,R_l)+O(1)=o(T(r,w)).$$

By the second main theorem, there are at most two values of a_j such that the above inequality holds. Hence $H_j \equiv 0$, or

$$R(z, a_j) = Q_{\nu+1}(z, a_j), \quad z \in \mathbb{C}_p,$$

holds except for these two values of a_j. Then $R(z, w) = Q_{\nu+1}(z, w)$ by the identity theorem which can be proved according to the standard method in complex analysis. The rest of the theorem follows from Theorem 6.1. ■

For meromorphic functions on the complex plane $\mathbb{C}$, this theorem is proved by Steinmetz [33], or see He-Xiao [8]. For several variables, see Hu-Yang [16].

7. Malmquist-type Theorems (II)

In this section, we consider the following differential equation:

$$\Omega(z, w, w', ..., w^{(n)}) = R(z, w)\Phi(z, w, w', ..., w^{(n)}), \tag{32}$$

where

$$\Phi(z, w, w', ..., w^{(n)}) = \sum_{i\in J} d_i w^{i_0}(w')^{i_1} \cdots (w^{(n)})^{i_n} \quad (\#J < \infty, d_i \in \mathcal{M}(\mathbb{C}_p)). \tag{33}$$

The following Clunie-type result will be needed.

Lemma 7.1. *Let $R(z, w)$ be defined by (14) and let w be a solution of (32). If $q \geq k$, then*

$$T\left(r, \frac{\Omega}{\Phi}\right) \leq T(r, \Phi) + \sum_{i\in I} T(r, c_i) + \sum_{j=0}^{k} T(r, a_j) + O\left(\sum_{j=0}^{q} T(r, b_j)\right).$$

If $q \geq k + \deg(\Phi)$, then

$$m(r, \Omega) \leq \sum_{i\in I} m(r, c_i) + \sum_{i\in J} m(r, d_i) + \sum_{j=0}^{k} m(r, a_j) + O\left(\sum_{j=0}^{q} m(r, b_j) + m(r, \frac{1}{b_q})\right).$$

Proof. Following the proof of Lemma 6.1, we can prove

$$\begin{aligned} m\left(r, \frac{\Omega}{\Phi}\right) \leq & m\left(r, \frac{1}{\Phi}\right) + \sum_{i\in I} m(r, c_i) + \sum_{j=0}^{k} m(r, a_j) \\ & + O\left(\sum_{j=0}^{q} m(r, b_j) + m\left(r, \frac{1}{b_q}\right)\right), \end{aligned} \tag{34}$$

$$(35)\ N\left(r, \frac{\Omega}{\Phi}\right) \leq N\left(r, \frac{1}{\Phi}\right) + \sum_{i\in I} N(r, c_i) + \sum_{j=0}^{k} N(r, a_j) + O\left(\sum_{j=0}^{q} N(r, \frac{1}{b_j})\right),$$

if $q \geq k$. Thus the first inequlity follows from (34) and (35). Similarly, we can prove the second inequality. ■

Theorem 7.1. *If there exists a solution w of* (32) *with $R(z, w)$ defined by* (14) *such that*

$$\sum_{i\in I} T(r, c_i) + \sum_{i\in J} T(r, d_i) + \sum_{j=0}^{k} T(r, a_j) + \sum_{j=0}^{q} T(r, b_j) = o(T(r, w)),$$

then

$$q \leq \min\{\Gamma(\Phi), \deg(\Phi) + \gamma(\Phi)(1 - \Theta_w(\infty))\},$$
$$k \leq \min\{\Gamma(\Omega), \deg(\Omega) + \gamma(\Omega)(1 - \Theta_w(\infty))\}.$$

Proof. The equation (32) can be rewritten as follows

$$\Omega(z, w, w', ..., w^{(n)}) = \left(A_1(z, w) + \frac{A_2(z, w)}{B(z, w)}\right) \Phi(z, w, w', ..., w^{(n)}),$$

where $\deg(A_1) = k - q$ if $k \geq q$, and $\deg(A_2) = k_2 < q$. By Lemma 7.1, we have

$$T\left(r, \frac{\Omega - A_1\Phi}{\Phi}\right) \leq T(r, \Phi) + o(T(r, w)).$$

Theorem 5.2 implies

$$T\left(r, \frac{\Omega - A_1\Phi}{\Phi}\right) = T\left(r, \frac{A_2}{B}\right) = qT(r, w) + o(T(r, w)).$$

Thus we obtain

$$qT(r, w) \leq T(r, \Phi) + o(T(r, w)).$$

By combining this with (23) and (24), we obtain the upper bound for q.

Rewriting (32) as follows

$$\frac{\Phi}{\Omega} = \frac{1}{R} = \frac{B}{A},$$

by the conclusion above, one can obtain the upper bound for k. ■

For meromorphic functions on the complex plane $\mathbb{C}$, this theorem is proved by Tu [35]. For several variables, see Hu-Yang [16]. Similarly, we can prove

Theorem 7.2. *Take $R \in \mathcal{M}(\mathbb{C}_p)$. If the following differential equation*

$$\Omega(z, w, w', ..., w^{(n)}) = R(w)\Phi(z, w, w', ..., w^{(n)})$$

has a nonconstant solution $w \in \mathcal{M}(\mathbb{C}_p)$ satisfying

$$\sum_{i \in I} T(r, c_i) + \sum_{i \in J} T(r, d_i) = o(T(r, w)),$$

then $R = \frac{A}{B}$ is a rational function with

$$\deg(B) \leq \min\{\Gamma(\Phi), \deg(\Phi) + \gamma(\Phi)(1 - \Theta_w(\infty))\},$$
$$\deg(A) \leq \min\{\Gamma(\Omega), \deg(\Omega) + \gamma(\Omega)(1 - \Theta_w(\infty))\}.$$

For the complex case, also see Hu-Yang [16].

8. Admissible Solutions of Some Differential Equations

In this section, we discuss the following differential equations

$$\Omega(z, w, w', ..., w^{(n)}) = \sum_{j=0}^{k} a_j(z) w^j \tag{36}$$

for some special forms of Ω. For $w \in \mathcal{M}(\mathbb{C}_p)$, here and in the sequel $\Omega(z, w, w', ..., w^{(n)})$ is called a differential polynomial of w if

$$T(r, c_i) = o(T(r, w)) \quad (i \in I).$$

Lemma 8.1. *If $w_0, w_1 \in \mathcal{M}(\mathbb{C}_p)$ are linearly independent, then*

$$\begin{aligned} T(r, w_0) \leq & m(r, w_0 + w_1) + N(r, w_0) + \bar{N}(r, w_0) \\ & + \bar{N}(r, \tfrac{1}{w_0}) + \bar{N}(r, w_1) + \bar{N}(r, \tfrac{1}{w_1}) + O(1). \end{aligned}$$

Proof. Setting $w = w_0 + w_1$, we have

$$w' = \frac{w_0'}{w_0} w_0 + \frac{w_1'}{w_1} w_1,$$

which implies

$$w_0 = w \left(\frac{w_1'}{w_1} - \frac{w'}{w} \right) \left(\frac{w_1'}{w_1} - \frac{w_0'}{w_0} \right)^{-1}.$$

Hence by using the first main theorem and the lemma of logarithmic derivative, we have

$$\begin{aligned} m(r,w_0) \leq & m(r,w) + m\left(r, \frac{w_1'}{w_1} - \frac{w'}{w}\right) + m\left(r, \left(\frac{w_1'}{w_1} - \frac{w_0'}{w_0}\right)^{-1}\right) \\ \leq & m(r,w) + m\left(r, \frac{w_1'}{w_1}\right) + m\left(r, \frac{w'}{w}\right) \\ & + m\left(r, \frac{w_1'}{w_1} - \frac{w_0'}{w_0}\right) + N\left(r, \frac{w_1'}{w_1} - \frac{w_0'}{w_0}\right) + O(1) \\ \leq & m(r,w) + \bar{N}(r,w_0) + \bar{N}(r,w_1) \\ & + \bar{N}(r, \frac{1}{w_0}) + \bar{N}(r, \frac{1}{w_1}) + O(1). \end{aligned}$$

Thus the lemma follows. ■

Theorem 8.1. *Assume*

$$\Omega(z,w,w',...,w^{(n)}) = (P(z,w,w',...,w^{(n)}) + Q(z,w,w',...,w^{(n)}))^l, \tag{37}$$

where P is a differential monomial of w and Q is a differential polynomial of w with

$$\deg(P) \geq \deg(Q), \quad \gamma(P) > \gamma(Q).$$

If $k < l$, and if (36) *has an admissible transcendental meromorphic solution, then* (36) *assumes the following form*

$$\Omega(z,w,w',...,w^{(n)}) = a_k(z)(w+b(z))^k, \quad b(z) = \frac{a_{k-1}(z)}{ka_k(z)}. \tag{38}$$

Proof. The case $k = 1$ is obvious. Assume $1 < k < l$ and that, to the contrary,

$$\sum_{j=0}^{k} a_j w^j - a_k(w+b)^k = \sum_{j=0}^{m} A_j w^j \not\equiv 0, \quad b = \frac{a_{k-1}}{ka_k},$$

where $0 \leq m \leq k-2, A_m \not\equiv 0$ and A_j are rational functions of $\{a_j\}$. Then $w_0 = -a_k(w+b)^k$ and $w_1 = \Omega = (P+Q)^l$ are linearly independent. In fact, suppose that

$$\alpha w_0 + \beta w_1 \equiv 0, \quad \{\alpha, \beta\} \subset \mathbb{C}_p - \{0\}.$$

Then

$$\sum_{j=0}^{k} \beta a_j w^j = \alpha a_k(w+b)^k = \alpha a_k w^k + \alpha a_{k-1} w^{k-1} + \cdots + \alpha a_k b^k.$$

Let $\mathcal{M}_w(\mathbb{C}_p)$ be the set of p-adic meromorphic functions f satisfying $T(r,f) = o(T(r,w))$. Then $1, w, w^2, ..., w^k$ are linearly independent over $\mathcal{M}_w(\mathbb{C}_p)$. Thus $\alpha = \beta$. Note that

$$w_0 + w_1 = \sum_{j=0}^{m} A_j w^j \not\equiv 0.$$

Hence $\alpha = \beta = 0$.

Under the conditions of the theorem, we have

$$\begin{aligned} N(r,\Omega) &= lN(r, P+Q) \\ &= l\{\deg(P)N(r,w) + \gamma(P)\bar{N}(r,w) + o(T(r,w))\} \\ &= N(r, \sum_{j=0}^{k} a_j w^j) = kN(r,w) + o(T(r,w)). \end{aligned}$$

Since $\deg(P) > 0, l > k$, we obtain

$$N(r,w) = o(T(r,w)).$$

Similarly, we can prove

$$T(r, P+Q) = \frac{k}{l}T(r,w) + o(T(r,w)).$$

Note that

$$T(r,w_0) = kT(r,w) + o(T(r,w)),$$

$$T(r,w_1) = lT(r,P+Q) = kT(r,w) + o(T(r,w)),$$

$$N(r,w_0) = o(T(r,w)), \quad N(r,w_1) = o(T(r,w)),$$

$$\bar{N}\left(r, \frac{1}{w_0}\right) = \bar{N}\left(r, \frac{1}{w+b}\right) + o(T(r,w)) \le T(r,w) + o(T(r,w)),$$

$$\begin{aligned} \bar{N}\left(r, \frac{1}{w_1}\right) &= \bar{N}\left(r, \frac{1}{P+Q}\right) \le T(r, P+Q) \\ &= \frac{k}{l}T(r,w) + o(T(r,w)), \end{aligned}$$

$$m(r, w_0 + w_1) = m\ m(r,w) + o(T(r,w)) \le mT(r,w) + o(T(r,w)).$$

By Lemma 8.1, we obtain

$$kT(r,w) \le mT(r,w) + T(r,w) + \frac{k}{l}T(r,w) + o(T(r,w)),$$

which is impossible since $k > m + 1 + \frac{k}{l}$. ■

Corollary 8.1. *If*

$$(w')^n = \sum_{j=0}^{k} a_j(z) w^j \quad (k < n) \tag{39}$$

has an admissible transcendental meromorphic solution, then (39) *assumes the following form*

$$(w')^n = a_k(z)(w + b(z))^k, \quad b(z) = \frac{a_{k-1}(z)}{k a_k(z)}. \tag{40}$$

Corollary 8.2. *If $n > k$ and if $n - k$ is not a factor of n, then* (39) *with constant coefficients a_j has no admissible transcendental meromorphic solution.*

The result can be proved easily by Corollary 8.1 and by comparing the multiplicity of poles and $(-b)$-valued points of w by using (40).

Conjecture 8.1. *The equation* (39) *has no admissible transcendental meromorphic solution.*

For the complex case, this is the conjecture of Gackstatter-Laine [7].

Lemma 8.2. *Assume*

$$\begin{aligned} \Omega(z, w, w', ..., w^{(n)}) = & B(z, w) P(z, w, w', ..., w^{(n)}) \\ & + Q(z, w, w', ..., w^{(n)}) \not\equiv 0, \end{aligned} \tag{41}$$

where $P(\not\equiv 0)$ and $Q(\not\equiv 0)$ are differential polynomials of w, and where $B(z, w)$ is defined by (13)*. If*

$$q = \deg(B) > \min\{\Gamma(Q), \deg(Q) + \gamma(Q)(1 - \Theta_w(\infty))\},$$

then

$$\begin{aligned} (q - \deg(Q)) T(r, w) \leq & (\Gamma(Q) - \deg(Q) + 1) \bar{N}(r, w) \\ & + \bar{N}\left(r, \frac{1}{\Omega}\right) + \bar{N}\left(r, \frac{1}{B}\right) + o(T(r, w)). \end{aligned}$$

Proof. Theorem 7.1 implies that $\frac{BP}{Q}$ is not constant, and hence $\frac{\Omega}{Q}$ is not constant, i.e., Ω, Q are linearly independent. Thus

$$Q^* = \left(\frac{Q'}{Q} - \frac{\Omega'}{\Omega}\right) Q \not\equiv 0.$$

Note that

$$\frac{\Omega'}{\Omega}BP + \frac{\Omega'}{\Omega}Q = \Omega' = B'P + BP' + Q'.$$

Then $BP^* = Q^*$, where

$$P^* = \left(\frac{\Omega'}{\Omega} - \frac{B'}{B} - \frac{P'}{P}\right)P.$$

By using Lemma 7.1, we see

$$m(r, P^*) = o(T(r, w)).$$

Also we have the estimate

$$m(r, Q^*) \leq \deg(Q)m(r, w) + o(T(r, w)).$$

By the first main theorem, we obtain

$$\begin{aligned} m\left(r, \frac{1}{P^*}\right) &= m(r, P^*) + N(r, P^*) - N\left(r, \frac{1}{P^*}\right) + O(1) \\ &= N(r, P^*) - N\left(r, \frac{1}{P^*}\right) + o(T(r, w)). \end{aligned}$$

Hence

$$\begin{aligned} qm(r, w) &= m(r, B) + o(T(r, w)) \\ &\leq m(r, Q^*) + m\left(r, \frac{1}{P^*}\right) + o(T(r, w)) \\ &\leq \deg(Q)m(r, w) + N(r, P^*) - N\left(r, \frac{1}{P^*}\right) + o(T(r, w)). \end{aligned}$$

Fix $z_0 \in \mathbb{C}_p$. If $\mu_B^0(z_0) > 0$ but z_0 is not a pole or a zero of the coefficients of B, P and Q, then $w(z_0) \neq \infty$, and

$$\mu_{P^*}^{\infty}(z_0) \leq 1.$$

If $\mu_w^{\infty}(z_0) > 0$ but z_0 is not a pole or a zero of the coefficients of B, P and Q, then

$$\mu_{Q^*}^{\infty}(z_0) \leq \deg(Q)\mu_w^{\infty}(z_0) + \Gamma(Q) - \deg(Q) + 1,$$

and further, if $\mu_{P^*}^{\infty}(z_0) > 0$, then

$$\mu_{P^*}^{\infty}(z_0) = \mu_{Q^*/B}^{\infty}(z_0) \leq \deg(Q)\mu_w^{\infty}(z_0) + \Gamma(Q) - \deg(Q) + 1 - q\mu_w^{\infty}(z_0),$$

otherwise if $\mu_{P^*}^{\infty}(z_0) = 0$, then

$$\mu_{1/P^*}^{\infty}(z_0) = \mu_{B/Q^*}^{\infty}(z_0) \geq q\mu_w^{\infty}(z_0) - \{\deg(Q)\mu_w^{\infty}(z_0) + \Gamma(Q) - \deg(Q) + 1\}.$$

Therefore, the lemma follows from

$$\begin{aligned} N(r,P^*) - N\left(r,\frac{1}{P^*}\right) \leq & \bar{N}\left(r,\frac{1}{\Omega}\right) + \bar{N}\left(r,\frac{1}{B}\right) - (q-\deg(Q))N(r,w) \\ & + (\Gamma(Q) - \deg(Q) + 1)\bar{N}(r,w) + o(T(r,w)). \end{aligned}$$

■

Theorem 8.2. *Assume*

(42) $\Omega(z,w,w',...,w^{(n)}) = (w^q P(z,w,w',...,w^{(n)}) + Q(z,w,w',...,w^{(n)}))^N$,

where $P(\not\equiv 0)$ and $Q(\not\equiv 0)$ are differential polynomials of w with

$$q > \max\{\deg(Q)+2, \Gamma(Q)\}.$$

If $k < N$, then (36) *has no admissible transcendental meromorphic solutions.*

Proof. Assume, to the contrary, that (36) has an admissible transcendental meromorphic solution w. Then

$$N(r,\Omega) = N\left(r,\sum_{j=0}^{k} a_j w^j\right) = kN(r,w) + o(T(r,w)),$$

$$N(r,\Omega) = N\ N(r,w^qP+Q) \geq NqN(r,w) + o(T(r,w)),$$

and hence

$$N(r,w) = o(T(r,w)).$$

According to the proof of Theorem 8.1, we can prove that (36) assumes the form (38). Thus Lemma 8.2 implies

$$\begin{aligned} (q-\deg(Q))T(r,w) &\leq \bar{N}\left(r,\frac{1}{w^qP+Q}\right) + \bar{N}\left(r,\frac{1}{w}\right) + o(T(r,w)) \\ &= \bar{N}\left(r,\frac{1}{w+b}\right) + \bar{N}\left(r,\frac{1}{w}\right) + o(T(r,w)) \\ &\leq 2T(r,w) + o(T(r,w)), \end{aligned}$$

which is impossible since $q - \deg(Q) > 2$. ■

Similarly, we can prove

Theorem 8.3. *Assume that Ω is defined by* (42) *with $q > \Gamma(Q)+3$. Then* (38) *has no admissible transcendental meromorphic solutions for any positive integers k and N.*

We end this paper by the following problem:

Conjecture 8.2. *The following p-adic differential equation*

$$w^{(n)} + a_n(z)w^{(n-1)} + \cdots + a_2(z)w' + a_1(z)w + a_0(z) = 0 \tag{43}$$

with $a_j \in \mathcal{M}(\mathbb{C}_p)$ *has no admissible transcendental meromorphic solutions.*

For related topics of this section in the complex-variable case, see Hu [11], and Hu-Yang [18].

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