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ON THE COLLECTIVE COMPACTNESS OF STRONGLY CONTINUOUS SEMIGROUPS AND COSINE FUNCTIONS OF OPERATORS

Hernán R. Henríquez*

Abstract. Let X be a complex Banach spcce, and denote by T a strongly continuous semigroup of linear operators defined on X and by C a cosine function of operators with associated sine function S defined on X. In this note we characterize in terms of spectral properties of the infinitesimal generator those semigroups T and cosine functions C such that $\{T(t) - I : t \geq 0\}$, $\{C(t) - I : t \in \mathbb{R}\}$ and $\{S(t) : t \in \mathbb{R}\}$ are collectively compact sets of bounded linear operators.

1. Introduction

The object of this note is the study of the compactness of semigroups and cosine functions of operators.

Throughout this work we will denote by X a complex Banach space endowed with a norm $\|\cdot\|$ and by $\mathcal{B}(X)$ the Banach algebra of bounded linear operators defined on X. If A is a linear operator with domain $D(A)$ and range $\mathcal{R}(A)$ in X, then $\sigma(A)$ (resp. $\sigma_p(A)$) denotes the spectrum (resp. point spectrum) of A. If λ belongs to the resolvent set of A, then $R(\lambda, A)$ denotes the resolvent operator $(\lambda I - A)^{-1}$. Some additional notations that we will use are $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ := (0, \infty)$, $\mathbb{R}_0^+ := [0, \infty)$, $\mathbb{R}^- := (-\infty, 0)$ and $\mathbb{R}_0^- := (-\infty, 0]$. For the necessary concepts in the theories of semigroups and cosine functions of linear operators we refer to Nagel [14] and Fattorini [6], respectively.

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The compactness properties of strongly continuous semigroups and cosine operator functions have been considered by several authors. Cuthbert [5] studies those strongly continuous semigroups $T(\cdot)$ such that $T(t) - I$ is compact for some $t > 0$ and Henríquez [8] characterizes those semigroups such that $(T(t) - I)^n$, $n \in \mathbb{N}$, is compact for all $t > 0$. Likewise, if C denotes a cosine operator function with associated sine function S the compactness of operators $C(t)$ and $S(t)$ was studied in [18] while the compactness of operators $(C(t) - I)^n$, $n \in \mathbb{N}$, and $(S(t) - tI)^n$, $n \in \mathbb{N}$, for all $t > 0$ was studied in [7, 8, 9]. Finally, Lizama [12] considered similar problems for resolvent families of operators.

In this note we will present a characterization for the collective compactness of sets $\{T(t) - I : t \geq 0\}$, $\{C(t) - I : t \in \mathbb{R}\}$ and $\{S(t) : t \in \mathbb{R}\}$ in terms of the infinitesimal generator A of T and C, respectively. These problems are related with the existence of mild solutions with relatively compact range of the abstract Cauchy problem

$$x'(t) = Ax(t) + f(t), \;\; x(0) \in X, \;\; t \geq 0,$$

when A generates a semigroup, and

$$x''(t) = Ax(t) + f(t), \;\; x(0), \; x'(0) \in X, \;\; t \geq 0,$$

in the case A generates a cosine operator function.

In order to prove our results we need certain properties of almost periodic functions. Next, for completeness, we present some preliminaries including those properties that we will use extensively. We refer the reader to [4, 19] for most of the basic aspects which are used afterward. First we shall give Bohr's definition of almost periodicity. We denote by J either the real line or the half-line $\mathbb{R}_0^+$.

Definition 1.1. *A continuous function $f : J \to X$ is called almost periodic (a.p.) if for every $\varepsilon > 0$ there exists a set P_ε relatively dense in J such that*

$$\|f(t + \tau) - f(t)\| \leq \varepsilon$$

for every $t \in J$ and every $\tau \in P_\varepsilon$.

In the sequel we denote by $C_b(J; X)$ the Banach space of bounded continuous functions from J into X endowed with the uniform convergence norm and we indicate by $C_0(\mathbb{R}_0^+; X)$ the space of all continuous functions from $\mathbb{R}_0^+$ into X which vanish at infinity. Moreover, the set of all a. p. functions from J into X will be denoted by $AP(J; X)$. We define the translation H_t on $C_b(J; X)$ by

$$(H_t f)(s) := f(t + s), \;\; s, \; t \in J.$$

Bochner's characterization of almost periodicity asserts that a function $f \in C_b(\mathbb{R}; X)$ is a.p. if and only if $\{H_t f : t \in \mathbb{R}\}$ is a relatively compact subset of $C_b(\mathbb{R}; X)$.

Definition 1.2. *A continuous function $f : \mathbb{R}_0^+ \to X$ is called asymptotically almost periodic (a.a.p.) if there are functions $g \in AP(\mathbb{R}; X)$ and $q \in C_0(\mathbb{R}_0^+; X)$ such that $f(t) = g(t) + q(t)$ for every $t \geq 0$.*

Asymptotic almost periodicity has been studied by Ruess and Summer (see [16] and the references therein). In particular, similar to Bochner's compactness criterion, a function $f : \mathbb{R}_0^+ \to X$ is a.a.p. if and only if the set $\{H_t f : t \geq 0\}$ is relatively compact in $C_b(\mathbb{R}_0^+; X)$.

For operator-valued functions it is convenient to consider some weaker forms of these definitions. A strongly continuous operator-valued function $F : J \to \mathcal{B}(X)$ is called a.p. (resp. a.a.p.) if for every $x \in X$ the function $t \to F(t)x$ is a.p. (resp. a.a.p.). We will use this concept with semigroups and cosine functions of operators. The a.p. semigroups were studied by Bart and Goldberg [1], while the a.p. cosine functions were considered in [2]. Afterward, Ruess and Summers [16] have characterized the a.a.p. semigroups and Henríquez [10] studied the a.a.p. cosine operator functions . The essential result is in [1]:

Proposition 1.1. *A strongly continuous semigroup T with infinitesimal generator A is almost periodic if and only if the following conditions hold:*

(a) *The semigroup T is uniformly bounded;*

(b) $\sigma(A) \subseteq i\mathbb{R}$;

(c) *The set of eigenvectors of A is total in X.*

Moreover, if T is a.p. and ir is an isolated point of $\sigma(A)$ then ir is an eigenvalue of A and a simple pole of $R(z, A)$.

We also need the following consequence of almost periodicity.

Lemma 1.1. *Let T be an a.p. semigroup with infinitesimal generator A. Then*

$$Px := \lim_{t\to\infty} \frac{1}{t} \int_0^t T(s)x\, ds, \quad x \in X,$$

exists and the following properties are satisfied:

(a) *P is a linear bounded projection;*

(b) *For all $t > 0$, P commutes with $T(t)$;*

(c) $\mathcal{R}(P) = \mathrm{Ker}(A)$;

(d) $\mathrm{Ker}(P) = \overline{\mathcal{R}(A)}$.

Consequently, $X = \mathrm{Ker}(P) \oplus \overline{\mathcal{R}(A)}$.

The a.a.p. semigroups have been characterized in [16]:

Proposition 1.2. *A strongly continuous semigroup T is asymptotically almost periodic if and only if $X = X_0 \oplus X_1$, where $X_0 := \{x \in X : \lim_{t\to\infty} T(t)x = 0\}$ and $X_1 := \{x \in X : T(t)x \text{ is a.p.}\}$.*

It follows from this result that every a.a.p. semigroup T can be decomposed as $T(t) = T_0(t) \oplus T_1(t)$, where $T_0(t) := T(t)|_{X_0}$ is a strongly stable C_0-semigroup and $T_1(t) := T(t)|_{X_1}$ is an a.p. C_0-semigroup.

Related to cosine functions, the a.p. strongly continuous cosine operator functions were characterized by Cioranescu [2]:

Proposition 1.3. *A strongly continuous cosine operator function C with infinitesimal generator A is almost periodic if and only if $C(t)$ is uniformly bounded on $\mathbb{R}$ and the set of eigenvectors of A is total in X.*

Finally, Henríquez [10] proved that an a.a.p. strongly continuous cosine operator function is also a.p.

To complete this introduction we remind ([15]) that an operator-valued map $F : J \to \mathcal{B}(X)$ is called collectively compact if $\bigcup_{t\in J} F(t)(B)$ is relatively compact for every bounded set $B \subseteq X$.

2. Results for Semigroups

In this section we denote by T a strongly continuous semigroup of linear operators on the complex Banach space X with infinitesimal generator A.

Lemma 2.1. *If $T(t) - I$ is collectively compact, then T is a.a.p.*

Proof. Let $x \in X$ and define the function $f(t) := T(t)x$. We will prove that $H(f) = \{H_t f : t \geq 0\}$ is relatively compact in $C_b(\mathbb{R}_0^+; X)$. Let $(h_n)_n$ be a sequence of positive real numbers. Since there exists a compact set K such that $T(t)x - x \in K$ for all $t \geq 0$, T is uniformly bounded and there is a subsequence $(s_k)_k$ such that $T(s_k)x$ is convergent, as $k \to \infty$, to some element $y \in X$. Therefore,

$$\|H_{s_k} f - T(\cdot)y\|_\infty = \sup_{t\geq 0} \|T(t)\,(T(s_k)x - y)\,\| \to 0, \quad k \to \infty,$$

which implies the assertion. ■

Theorem 2.1. *The operator-valued function $T(t) - I$ is collectively compact if and only if the following conditions hold:*

(a) *The semigroup T is uniformly bounded and its infinitesimal generator is compact;*

(b) *The spectrum of A is a finite set of eigenvalues included in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq 0\}$;*

(c) *The generalized eigenvectors of A span the space X.*

Proof. We begin the proof by assuming that $T(t) - I$ is collectively compact. It is immediate that T is uniformly bounded and the compactness of A follows from [5]. Consequently,

$$\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \leq 0\}$$

and

$$\sigma(A) \subseteq \sigma_p(A) \cup \{0\}.$$

If X is a finite-dimensional space, the previous inclusions prove conditions (b) and (c).

We consider now that X is an infinite-dimensional space. To prove condition (b), we will show that 0 is an isolated point of $\sigma(A)$. This statement is verified by the following considerations. From Lemma 2.1 and Proposition 1.2 it follows that we can decompose X as $X = X_0 \oplus X_1$ so that $T_0(t) := T(t)|_{X_0}$ is a strongly stable semigroup with infinitesimal generator $A_0 := A|_{X_0}$ and $T_1(t) := T(t)|_{X_1}$ is an a.p. semigroup with infinitesimal generator $A_1 := A|_{X_1}$. It is clear that $T_0(t) - I$ is collectively compact, that is, there exists a compact set $K_0 \subseteq X_0$ such that $T_0(t)x - x \in K_0$ for all $t \geq 0$ and all $x \in X_0$, $\|x\| \leq 1$. Taking limit as $t \to \infty$ we obtain that $x \in K_0$ which implies that X_0 has finite dimension and $\sigma(A_0)$ is a finite set. From this we conclude that the set $\sigma_p^-(A) := \{\lambda \in \sigma_p(A) : \mathrm{Re}(\lambda) < 0\}$ is finite. In fact, if $\lambda \in \sigma_p^-(A)$, then there is $x \in X$, $x \neq 0$, such that $Ax = \lambda x$. Since $T(t)x = e^{\lambda t}x \to 0$ as $t \to \infty$, it follows that $x \in X_0$ and $\lambda \in \sigma(A_0)$. On the other hand, since A is compact, we have that $0 \in \sigma(A)$. If we assume that 0 is a cluster point of $\sigma(A)$, then, since $\sigma_p(A_0)$ is finite, there exists a sequence $(\lambda_n)_n$ in $\sigma_p(A_1)$ such that $\lambda_n \to 0$ as $n \to \infty$. By the property we have just established we can choose $\lambda_n = i\beta_n$ with $0 \neq \beta_n \in \mathbb{R}$. Let $x_n \in X_1$, $\|x_n\| = 1$, be the eigenvector corresponding to λ_n. From the equality

$$T_1(t)x_n - x_n = (e^{i\beta_n t} - 1)x_n, \quad t \geq 0$$

and selecting $t := t_n$ so that $e^{i\beta_n t_n} = -1$ we infer that the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact. Hence, by passing to subsequences if needed, we may assume that $(x_n)_n$ converges to some element $y \in X_1$. From this it follows that $0 = A_1 x_n - i\beta_n x_n \to A_1 y - 0$ as $n \to \infty$, and so $A_1 y = 0$. Furthermore, since $x_n \in \mathcal{R}(A_1)$ it follows that $y \in \overline{\mathcal{R}(A_1)}$. Applying Lemma 1.1 with the a.p. semigroup T_1 we conclude that $y = 0$ which is a contradiction and proves our statement.

Using now the characterization (Proposition 1.1) of Bart and Goldberg [1] we obtain that $0 \in \sigma_p(A_1)$ and that the set of eigenvectors of A_1 is total in X_1. Next we decompose the spectrum $\sigma(A)$ into the spectral sets $\sigma_1 := \sigma(A) \setminus \{0\}$ and $\sigma_2 := \{0\}$. If $X = Y_1 \oplus Y_2$ is the decomposition of X induced by the decomposition of $\sigma(A)$ (see Taylor [17], Theorem 5.7-A), then Y_1 is a finite-dimensional space and the ascent of the eigenvalue 0 is 1. In fact, if $x \in Y_2$ and $A^k x = 0$ for some $k \in \mathbb{N}$, then

$$T(t)x = x + tAx + \cdots + \frac{t^{k-1}}{(k-1)!} A^{k-1} x.$$

Since $T(t)x$ is a bounded function on $\mathbb{R}_0^+$, it follows from the above expression that $Ax = 0$. Thus, the generalized eigenvectors of $A|_{Y_2}$ coincide with the eigenvectors and this implies that $\mathrm{Ker}(A|_{Y_2}) = Y_2$. Since the generalized eigenvectors of $A|_{Y_1}$ span Y_1, this shows (b) and (c).

We assume now conditions (a), (b) and (c) are verified. If X is a finite-dimensional space, then condition (a) implies that $T(t) - I$ is collectively compact. In the case X is an infinite-dimensional space the earlier argument shows that we can decompose X as $X = Y_1 \oplus Y_2$, where Y_1 is a finite-dimensional space and $Y_2 = \mathrm{Ker}(A)$. If we use subindices to indicate restriction to the corresponding subspace, we deduce that $T_1(t) - I$ is collectively compact and $A_2 = 0$. Therefore $T_2(t) = I$ and $T(t) = T_1(t) \oplus I$ so that $T(t) - I = (T_1(t) - I) \oplus 0$ is collectively compact. ■

Remarks.

(i) If $T(t) - I$ is collectively compact, the preceding argument shows that the ascent of eigenvalues of A located in $i\mathbb{R} \setminus \{0\}$ is equal to 1.

(ii) By Lemma 2.1, [5], and the proof of Theorem 2.1, one can see that $T(t) - I$ is collectively compact if and only if T is a.a.p. and A is compact.

3. Results for Cosine Operator Functions

Throughout this section $C(t)$ is a strongly continuous cosine operator function with infinitesimal generator A. We denote by $S(t)$ the sine function associated with C which is defined by

$$S(t)x := \int_0^t C(s)x\, ds, \quad x \in X, \quad t \in \mathbb{R}. \tag{3.1}$$

Theorem 3.1. *The operator-valued function $C(t) - I$ is collectively compact if and only if the following conditions hold*:

(a) *The cosine function C is uniformly bounded and its infinitesimal generator A is compact;*

(b) *The spectrum of A is a finite set of eigenvalues included in $\mathbb{R}_0^-$;*

(c) *The eigenvectors of A span the space X.*

Proof. Suppose that $C(t) - I$ is a collectively compact function. Then condition (a) follows from [7]. Moreover, it is well-known ([6]) that A is the infinitesimal generator of a holomorphic semigroup $T(t)$ defined by

$$T(t)x = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/4t} C(s)x\, ds, \quad x \in X, \quad t > 0. \tag{3.2}$$

Since $\{C(t)x - x : t \geq 0, \quad \|x\| \leq 1\}$ is totally bounded, from the above expression it is easy to see that $\{T(t)x - x : t \geq 0, \ \|x\| \leq 1\}$ is also a totally bounded set in X. Therefore $T(t) - I$ is a collectively compact function. On the other hand, if $C(t)$ is uniformly bounded then ([13]) $\sigma(A) \subseteq \mathbb{R}_0^-$ so that, collecting this property with Theorem 2.1, we obtain that $\sigma(A) = \sigma_p(A) \subseteq \mathbb{R}_0^-$ is a finite set. This shows (b). Moreover, the ascent of eigenvalues of A is equal to 1. In fact, if $\lambda = 0$ is an eigenvalue and $A^k x_0 = 0$ for some vector $x_0 \neq 0$, then using the series expansion of $C(t)x_0$ we obtain the vector polynomial

$$C(t)x_0 = x_0 + \frac{t^2}{2!} A x_0 + \frac{t^4}{4!} A^2 x_0 + \cdots + \frac{t^{2k-2}}{(2k-2)!} A^{k-1} x_0,$$

which is bounded only in the case $k = 1$. This shows the assertion for the eigenvalue 0. Similarly, if $\mu = -\lambda^2, \ \lambda \neq 0$, is an eigenvalue of A and $(A-\mu)^k x_0 = 0$ for some vector $x_0 \neq 0$, then $C(t)x_0$ is a linear combination of functions $t^i(\sin \lambda t)\, y_i$ and $t^i(\cos \lambda t)\, z_i$, with $y_i, \ z_i \in X$ for $i = 0, 1, \ldots, k-1$, which is bounded only in the case $k = 1$. Therefore, the generalized eigenvectors of A coincide with the eigenvectors and Theorem 2.1 proves condition (c).

Conversely, if we assume conditions (a), (b) and (c), then $C(t) - I$ is a compact operator for each $t \in \mathbb{R}$. Moreover, proceeding as above we can assert that the ascent of the eigenvalues of A is equal to 1. If $0 \in \sigma(A)$ and $X = X_1 \oplus X_2$ is the decomposition of X associated with the spectral sets $\sigma_1 := \sigma(A) \setminus \{0\}$ and $\sigma_2 = \{0\}$, then the compactness of A implies that X_1 is a finite-dimensional space and using (c) we easily find that $X_1 = \oplus_{i=1}^n \mathrm{Ker}(A - \mu_i)$, where $\mu_i = -\lambda_i^2, \ \lambda \neq 0, \ i = 1, 2 \ldots, n$, denote the non-zero eigenvalues of A, and $X_2 = \mathrm{Ker}(A)$. It follows that $A|_{X_2} = 0$ and $C_2(t) := C(t)|_{X_2} = I$. Consequently $C(t) = C_1(t) \oplus I$, where $C_1(t) = C(t)|_{X_1}$ is a uniformly bounded cosine operator function defined on a finite-dimensional space. This completes the proof. ∎

We will consider now the collective compactness of the function $S(t)$. We begin by establishing an elementary property of cosine operator functions that we will need in the proof of our main result.

Lemma 3.1. *If the sine function $S(\cdot)$ is collectively compact, then for each $a > 0$ the operator function $C(\cdot)S(a)$ is also collectively compact and a.p.*

Proof. The first assertion follows from the identity

$$S(t+a) - S(t-a) = 2C(t)S(a).$$

To prove the second one we begin by showing that the operator-valued function $S(\cdot)S(a)$ is a.p. To this end, we fix $x \in X$ and prove that the set of translations $\{H_t S(\cdot)S(a)x : t \in \mathbb{R}\}$ is relatively compact in the space $C_b(\mathbb{R}; X)$. Let $(t_n)_n$ be a real sequence. From the collective compactness of $S(\cdot)$ and $C(\cdot)S(a)$ we infer the existence of a subsequence $(t'_n)_n$ and elements $y,\ z \in X$ such that $S(t'_n)x \to y$ and $C(t'_n)S(a)x \to z$ as $n \to \infty$. Since $C(\cdot)S(a)$ and $S(\cdot)$ are uniformly bounded on $\mathbb{R}$, from the expression

$$S(t'_n + u)S(a)x = C(u)S(a)S(t'_n)x \; + \; S(u)C(t'_n)S(a)x$$

we conclude that the sequence of functions $S(t'_n + u)S(a)x$ converges uniformly to $C(u)S(a)y \; + \; S(u)z$. Bochner's characterization of almost periodicity implies that $S(\cdot)S(a)$ is a.p. In addition, from the identity

$$C(t)y - y = \int_0^t S(u)Ay\, du, \quad y \in D(A),$$

and the fact that the operator-valued function $C(\cdot)S(a)$ is uniformly bounded on $\mathbb{R}$, we infer that $C(\cdot)S(a)x$ is uniformly continuous on $\mathbb{R}$. The assertion is now a consequence of the fact that $C(t)S(a)x$ is the derivative of $S(t)S(a)x$ and the properties of almost periodic functions (see [19]). ∎

Our main result is:

Theorem 3.2. *The operator-valued function $S(t)$ is collectively compact if and only if the following conditions hold :*

(a) *The operator $R(\mu, A)$ is compact for every μ in the resolvent set of A;*

(b) *The spectrum of A is a discrete set of eigenvalues included in $\mathbb{R}^-$;*

(c) *The set of eigenvectors of A is total in X.*

Proof. Suppose that the operator function $S(t)$ is collectively compact. It is clear that $S(t)$ is uniformly bounded on $\mathbb{R}$, and from Proposition 2.3 in [18] it follows that $R(\mu, A)$ is compact. Moreover, it is well-known that in this case the spectrum of A is a discrete set of isolated eigenvalues with finite algebraic multiplicity. If $\mu \in \sigma(A)$, then $\mu \in \mathbb{R}$ and $\mu < 0$. To prove this assertion we assume the contrary. We select $x \in D(A)$, $x \neq 0$, such that $Ax = \mu x$. We must consider two cases. If $\mu = 0$, then $S(t)x = tx$; if $\mu \notin \mathbb{R}^-$, then we can write $\mu = \lambda^2$ with $\mathrm{Re}(\lambda) \neq 0$ and it is easy to see that $S(t)x = \frac{\sinh \lambda t}{\lambda} x$. Hence, in both cases the function $S(t)x$ is not bounded on $\mathbb{R}$ which is a contradiction.

To prove condition (c) we modify slightly the construction carried out in the proof of Theorem 1 in [2]. Initially we fix $a > 0$. Since by the previous lemma for every $x \in X$ the function $C(\cdot)S(a)x$ is a.p., we can define the Fourier coefficient

$$P_\lambda(a)x := \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} e^{-i\lambda s} C(s)S(a)x\, ds, \quad \lambda \in \mathbb{R}.$$

Proceeding as in [12] we can show that $P_\lambda(a)x \in D(A)$ and $AP_\lambda(a)x = -\lambda^2 P_\lambda(a)x$ for every $x \in X$. Thus if E denotes the set of eigenvectors of A, including the vector 0, then $P_\lambda(a)x \in E$. If we assume that E is not total in X, then there exists $x' \in X'$, $x' \neq 0$, such that $\langle x', y' \rangle = 0$ for all $y \in E$. In particular, this implies that $\langle x', P_\lambda(a)x \rangle = 0$ for every $\lambda \in \mathbb{R}$ and all $x \in X$. From the properties of scalar a.p. functions we conclude that $\langle x', C(t)S(a)x \rangle \equiv 0$. Taking $t = 0$ we obtain that $\langle x', S(a)x \rangle = 0$ for all $x \in X$. Since $\frac{1}{a} S(a)x \to x$ as $a \to 0^+$, it follows that $x' = 0$, which is contrary to our assumption.

Assuming now that conditions (a), (b) and (c) are fulfilled, we will prove that $S(t)$ is collectively compact. Initially we establish that $C(t)$ is uniformly bounded on $\mathbb{R}$. From Cioranescu and Ubilla [3] it follows that it is sufficient to show that the semigroup T defined by the expression (3.2) is uniformly bounded and that the space of exponential vectors of A is dense in X. Since T is a holomorphic semigroup, its growth bound $\omega_0(T) = \sup \mathrm{Re}\, \sigma(A) < 0$ (see Nagel [14]) which implies that $T(t)$ is uniformly bounded on $\mathbb{R}_0^+$. The second assertion is a direct consequence of (c) since every eigenvector is an exponential vector. As an immediate consequence of this property and (3.1) we deduce the existence of a positive constant M such that $\|S(t_2) - S(t_1)\| \leq M|t_2 - t_1|$ for every $t_1,\ t_2 \in \mathbb{R}$.

Next we observe that A^{-1} is a compact operator. From the identity

$$A^{-1}(C(b) - C(a))x = \int_a^b S(u)x\, du$$

we conclude that the two-parameter operator-valued function $\int_a^b S(u)\, du$, $0 \leq a \leq b$, is collectively compact. In order to prove that $S(t)$ is collectively compact we assume the contrary. Thus, there exist $\varepsilon > 0$, a real sequence $(t_n)_n$ and a sequence $(x_n)_n$, $x_n \in X$, $\|x_n\| = 1$, such that the sequence $y_n := S(t_n)(x_n)$ satisfies $\|y_n - y_m\| \geq \varepsilon$ for all $m \neq n$. Let us take $a := \varepsilon/2M$ and define $z_n := \frac{1}{a}\int_{t_n}^{t_n+a} S(u)x_n\,du$. Clearly $\{z_n : n \in \mathbb{N}\}$ is a relatively compact set. Moreover

$$\begin{aligned} \|z_n - y_n\| &\leq \frac{1}{a}\int_{t_n}^{t_n+a} \|S(u)x_n - S(t_n)x_n\|\,du \\ &\leq \frac{M}{a}\int_{t_n}^{t_n+a} (u - t_n)\,du = \frac{\varepsilon}{4}, \end{aligned}$$

which allows us to conclude that $\|z_n - z_m\| \geq \varepsilon/2$ for every $m \neq n$. Since this is a contradiction, we have completed the proof. ■

Remark. The last part of the above demonstration also serves to show that conditions (a) and (b) in the statement of the previous theorem can be substituted by

(a') For each $t > 0$, the operator $S(t)$ is compact;

(b') For each $x \in X$, the function $S(\cdot)x$ has relatively compact range.

Corollary 3.1. *If $S(t)$ is collectively compact, then $C(\cdot)$ and $S(\cdot)$ are a.p. operator-valued functions.*

Proof. From the demonstration of Theorem 3.2 we know that $C(t)$ is uniformly bounded on $\mathbb{R}$. Collecting this property with condition (c) in the statement of Theorem 3.2 we conclude that the hypotheses of Proposition 1.3 ([2], Theorem 1) are verified. Hence C is a.p. Furthermore, since for each $x \in X$ the function $S(t)x$ is a primitive of $C(t)x$ and the range of $S(t)x$ is relatively compact, the function $S(t)x$ is a.p. ([19], Theorem 7.1). ■

We conclude this section with an application to the second-order abstract Cauchy problem. We consider the inhomogeneous Cauchy problem

$$x''(t) = Ax(t) + f(t), \quad t \in \mathbb{R}, \quad x(0) = x_0,\ x'(0) = x_1, \tag{3.3}$$

where A is the infinitesimal generator of a cosine function $C(t)$ and f is a locally integrable function. It is well-known that the mild solution of (3.3) is given by

$$x(t; x_0, x_1, f) := C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)f(s)\,ds. \tag{3.4}$$

Next we will establish conditions to guarantee the compactness of the trajectory $\{x(t; x_0, x_1, f) : t \geq 0\}$, where $f \in \mathcal{L}^1([0,\infty); X)$. In order to state our result we introduce the following terminology : a subset E of $\mathcal{L}^1([0,\infty); X)$ is said equi-integrable at infinity if for every $\varepsilon > 0$ there exists $a > 0$ such that

$$\int_a^\infty \|f(s)\|\, ds \leq \varepsilon$$

for every $f \in E$. It is clear from the dominated convergence theorem that the singleton $E = \{f\}$ is equi-integrable at infinity for each function $f \in \mathcal{L}^1([0,\infty); X)$.

Proposition 3.1. *Let K be a compact subset of X, let E be a bounded and equi-integrable-at-infinity subset of $\mathcal{L}^1([0,\infty); X)$ and let M be a positive real number. If $S(t)$ is collectively compact, then the set $\{x(t; x_0, x_1, f) : t \geq 0,\ x_0 \in K, \|x_1\| \leq M,\ f \in E\}$ is relatively compact.*

Proof. It follows from Corollary 3.1 that $C(t)$ is a.p., which in turn implies that $\{C(t)x_0 : t \geq 0,\ x_0 \in K\}$ is relatively compact. Similarly, by the hypothesis, the set $\{S(t)x_1 : t \geq 0,\ \|x_1\| \leq M\}$ is relatively compact. Consequently, using (3.4), it remains to prove that the set $\{u(t,f) : t \geq 0,\ f \in E\}$ is relatively compact, where we have defined

$$u(t,f) := \int_0^t S(t-s)f(s)\, ds.$$

It is clear from the hypothesis that $S(t)$ is uniformly bounded. Hence for every $\varepsilon > 0$ there exists $a > 0$ such that

$$\| \int_a^t S(t-s)f(s)\, ds\| \leq \varepsilon, \quad t \geq a,\ f \in E.$$

On the other hand, denoting $v(t,f) := u(t,f) - \int_a^t S(t-s)f(s)\, ds$ for every $t \geq a$ we can write

$$\begin{aligned} v(t,f) &= \int_0^a S(t-s)f(s)\, ds \\ &= \int_0^a \left[S(t-a)C(a-s) + C(t-a)S(a-s)\right] f(s)\, ds \\ &= S(t-a)\int_0^a C(a-s)f(s)\, ds + C(t-a)u(a,f). \end{aligned} \tag{3.5}$$

The first term of the right-hand side of (3.5) is included in a compact set since $S(\cdot)$ is collectively compact and the set formed by $\int_0^a C(a-s)f(s)\, ds,\ f \in E$, is bounded. To study the second term of (3.5) let us define the operator $\Lambda : \mathcal{L}^1([0,a];X) \to X$ by the expression

$$\Lambda(f) := \int_0^a S(a-s)f(s)\,ds.$$

It has been proved in [11], Theorem 5, that Λ is a compact operator. Therefore $\{u(a,f) : f \in E\} = \{\Lambda(f) : f \in E\}$ is relatively compact. Turning to use the almost periodicity of $C(\cdot)$ we infer that the second term of the right-hand side of (3.5) is also included in a compact set, which completes the proof. ∎

The previous result remains valid under some weaker form of integrability for functions f. In fact, the identity

$$C(t)x - x = A\int_0^t S(t-s)x\,ds, \quad x \in X,\ t \in \mathbb{R},$$

shows that the argument used in the proof of Proposition 3.1 also serves for functions f of type $f = g + Az$ where $g \in E$, $z \in W$ and W is a compact subset of $D(A)$.

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Departamento de Matemática, Universidad de Santiago de Chile
Casilla 307, Correo 2, Santiago, Chile.