

A WAVELET METHOD FOR THE FIRST KIND INTEGRAL EQUATIONS WITH KERNEL $k(x - y)$

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Abstract. We study the first kind integral equation

$$\int_0^{+\infty} k(x - y)\sigma(y)dy = g(x)$$

by the wavelet method. The integral equation is discretized with respect to two different wavelet bases. We then have two different linear systems. One of them is a Toeplitz system and the other one is a system with condition number $\kappa = O(1)$ after a diagonal scaling. By using the preconditioned conjugate gradient (PCG) method with the fast wavelet transform (FWT) and the fast Fourier transform (FFT), we can solve the systems in $O(n \log n)$ operations where n is the size of the systems.

1. INTRODUCTION

In this paper, we study the first kind integral equation

$$\int_0^{+\infty} k(x - y)\sigma(y)dy = g(x)$$

by the wavelet method. The integral equation is discretized with respect to two different orthonormal wavelet bases \mathcal{B}_1 and \mathcal{B}_2 . The \mathcal{B}_1 comes from the father wavelet $\varphi(x)$ and the \mathcal{B}_2 comes from the mother wavelet $\psi(x)$. After discretizing of the integral equation with respect to \mathcal{B}_1 and \mathcal{B}_2 on a finite

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interval, we then have two different n -by- n linear systems. One of them is a Toeplitz system

$$T_n x = b$$

(corresponding to \mathcal{B}_1), and the other one is a system $A_n y = d$ (corresponding to \mathcal{B}_2) with condition number

$$(1) \quad \kappa(D_n^{-1/2} A_n D_n^{-1/2}) = O(1)$$

after a diagonal scaling D_n . The relation between T_n and A_n is

$$A_n = W_n T_n W_n^{-1},$$

where W_n is the wavelet transform matrix between \mathcal{B}_1 and \mathcal{B}_2 .

We then solve the Toeplitz system $T_n x = b$ by solving its equivalent form

$$A_n \tilde{x} = \tilde{b}$$

with $\tilde{x} = W_n x$ and $\tilde{b} = W_n b$. For solving the system $A_n \tilde{x} = \tilde{b}$, we use the PCG method with the diagonal preconditioner D_n . More precisely, instead of solving the system $A_n \tilde{x} = \tilde{b}$, we solve the preconditioned system

$$D_n^{-1} A_n \tilde{x} = D_n^{-1} \tilde{b}.$$

The condition number of the preconditioned system is, by (1),

$$\kappa(D_n^{-1} A_n) = \kappa(D_n^{-1/2} A_n D_n^{-1/2}) = O(1).$$

When the PCG method is applied to solve the preconditioned system, the convergence rate will be linear, see [5]. By using the FWT (see [1] and [3]) and Strang's algorithm (see [2] and [7]), we can solve the system $A_n \tilde{x} = \tilde{b}$ and also $T_n x = b$ in $O(n \log n)$ operations.

The outline of the paper is as follows. In Section 2, we introduce the father wavelet basis \mathcal{B}_1 and the mother wavelet basis \mathcal{B}_2 . The integral equation is discretized with respect to the bases \mathcal{B}_1 and \mathcal{B}_2 . Two different linear systems are obtained. In Section 3, we use the PCG method to solve the linear systems obtained in Section 2. We then discuss the condition number and the operation cost of the PCG method.

2. DISCRETIZATION OF INTEGRAL EQUATION

Let $H^s(\mathbb{R})$ with $s \geq 0$ and $H^t(\mathbb{R})$ be two Sobolev spaces. The integral operator A from $H^s(\mathbb{R})$ to $H^t(\mathbb{R})$ is defined as

$$(2) \quad (A\sigma)(x) = \int_0^{+\infty} k(x-y)\sigma(y)dy,$$

where $k(x-y) \in L^2(\mathbb{R})$ is symmetric, i.e., $k(x-y) = k(y-x)$. For a given function $g \in H^t(\mathbb{R})$, we try to find $\sigma \in H^s(\mathbb{R})$ such that

$$A\sigma = g,$$

i.e.,

$$\int_0^{+\infty} k(x-y)\sigma(y)dy = g(x).$$

The equivalent variational form is: find $\sigma \in H^s(\mathbb{R})$ such that

$$(3) \quad B(\sigma, \mu) = F(\mu)$$

for all $\mu \in H^s(\mathbb{R})$, where

$$B(\sigma, \mu) = \langle A\sigma, \mu \rangle = \int_0^{+\infty} \int_0^{+\infty} k(x-y)\sigma(y)\mu(x)dydx$$

and

$$F(\mu) = \int_0^{+\infty} g(x)\mu(x)dx.$$

We assume that $B(\sigma, \mu)$ is a continuous elliptic bilinear form on $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, i.e., there exist two constants $\beta \geq \alpha > 0$, such that

$$\alpha \|\sigma\|_{H^s} \leq B(\sigma, \sigma) \quad \text{and} \quad B(\sigma, \mu) \leq \beta \|\sigma\|_{H^s} \|\mu\|_{H^s}.$$

For instance, when $s = 0$ and $k(x-y)$ is bounded with $k(x-y) \geq c > 0$, then obviously, $B(\sigma, \mu)$ is a continuous elliptic bilinear form on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

2.1 Wavelet Bases

Now we are going to discretize the integral equation with respect to two different orthonormal wavelet bases \mathcal{B}_1 and \mathcal{B}_2 . First of all, we introduce a function $\varphi(x) \in L^2(\mathbb{R})$ called the father wavelet (or scaling function), with a compact support $[0, a]$, $a > 0$; see [4]. The $\varphi(x)$ has the property that

$$(4) \quad \varphi(x-k), \quad k \in Z,$$

form an orthonormal sequence in $L^2(\mathbb{R})$. Let V_0 be the closed linear subspace of $L^2(\mathbb{R})$ generated by (4). The multiresolution analysis (MRA), depending on this $\varphi(x)$, is given as follows:

- (i) $f(x) \in V_0$ if and only if $f(2^j x) \in V_j$;
- (ii) $\cdots \subset V_{j-1} \subset V_j \subset V_{j+1} \subset \cdots$;
- (iii) $\overline{\cup_{-\infty}^{\infty} V_j} = L^2(\mathbb{R})$ and $\cap_{-\infty}^{\infty} V_j = 0$;
- (iv) The sequence (4) forms a Riesz basis of V_0 .

Let W_j denote the orthogonal complement of V_j in V_{j+1} , i.e., $V_{j+1} = V_j \oplus W_j$. From MRA (iii), we also have $\oplus_{-\infty}^{\infty} W_j = L^2(\mathbb{R})$. There exists at least one function $\psi(x) \in W_0$ such that

$$\psi(x - k), \quad k \in Z,$$

is an orthonormal basis of W_0 (see [3] and [6]). The $\psi(x)$ is called the mother wavelet. We then construct the following two wavelet sequences:

$$\varphi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k), \quad j, k \in Z,$$

and

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad j, k \in Z.$$

The wavelet sequence $\{\psi_{j,k}(x)\}$ forms a Riesz basis of $H^s(\mathbb{R})$ for $s \geq 0$ (see [6]).

The operator A defined by (2) can be projected on the subspace V_J (J is fixed) with respect to the following two bases in V_J :

$$\mathcal{B}_1 = (\varphi_{J,k}(x))_k, \quad k \in Z,$$

and

$$\mathcal{B}_2 = \bigcup_{-\infty < j \leq J-1} (\psi_{j,k}(x))_k, \quad k \in Z.$$

Here \mathcal{B}_1 comes from the father wavelet $\varphi(x)$ and \mathcal{B}_2 comes from the mother wavelet $\psi(x)$.

2.2 Projection of A with respect to \mathcal{B}_1 and \mathcal{B}_2

Let A_J denote the projection of A on V_J . The matrix representation of A_J corresponding to the basis \mathcal{B}_1 has the elements given by

$$(5) \quad t_{p,q} = \langle A(\varphi_{J,p}), \varphi_{J,q} \rangle = \int_0^{+\infty} \int_0^{+\infty} k(x-y) \varphi_{J,p}(x) \varphi_{J,q}(y) dy dx, \quad \forall p, q.$$

For all $\sigma, \mu \in H^s(\mathbb{R})$, let σ_J, μ_J denote the projections of σ, μ on V_J respectively. Then equation (3) becomes

$$(6) \quad \int_0^{+\infty} \int_0^{+\infty} k(x-y)\sigma_J(y)\mu_J(x)dydx = \int_0^{+\infty} g(x)\mu_J(x)dx.$$

Let

$$(7) \quad \sigma_J = \sum_p x_p \varphi_{J,p} \quad \text{and} \quad \mu_J = \varphi_{J,q}, \quad \forall q.$$

Substituting (7) into (6), we have the following linear system

$$(8) \quad T_\infty x = b,$$

where $(T_\infty)_{p,q} = t_{p,q}$ is given by (5), and

$$(x)_p = x_p, \quad (b)_q = \int_0^{+\infty} g(x)\varphi_{J,q}(x)dx.$$

Since $k(x-y)$ is symmetric and $\varphi(x)$ has the compact support $[0, a]$, we have

$$\begin{aligned} t_{p,q} &= \int_0^{+\infty} \int_0^{+\infty} k(x-y)\varphi_{J,p}(x)\varphi_{J,q}(y)dydx \\ &= 2^J \int_0^{+\infty} \int_0^{+\infty} k(x-y)\varphi(2^J x - p)\varphi(2^J y - q)dydx \\ &= 2^J \int_{2^{-J}p}^{2^{-J}(a+p)} \int_{2^{-J}q}^{2^{-J}(a+q)} k(x-y)\varphi(2^J x - p)\varphi(2^J y - q)dydx \\ &= 2^{-J} \int_0^a \int_0^a k[2^{-J}(x-y+p-q)]\varphi(x)\varphi(y)dydx \\ &= t_{p-q} = t_{q,p}. \end{aligned}$$

Hence T_∞ is a symmetric Toeplitz matrix.

The matrix representation of A_J corresponding to the basis \mathcal{B}_2 has the elements given by

$$(9) \quad a_{p,q;s,t} = \langle A(\psi_{p,q}), \psi_{s,t} \rangle = \int_0^{+\infty} \int_0^{+\infty} k(x-y)\psi_{p,q}(x)\psi_{s,t}(y)dydx$$

for $-\infty < p, s < J$ and $-\infty < q, t < \infty$. Let

$$(10) \quad \sigma_J = \sum_{p,q} y_{p,q} \psi_{p,q} \quad \text{and} \quad \mu = \psi_{p,q}, \quad -\infty < p < J, \quad \forall q.$$

Substituting (10) into (6), we have the following linear system

$$(11) \quad A_\infty y = d,$$

where $(A_\infty)_{p,q;s,t} = a_{p,q;s,t}$ is given by (9), $y = (y_{p,q})^T$ and $d = (d_{p,q})^T$ are vectors with $d_{p,q} = \int_0^{+\infty} g(x)\psi_{p,q}(x)dx$.

3. CONDITION NUMBER AND OPERATION COST

We have two different linear systems obtained in the last section. One of them is the Toeplitz system (8) (corresponding to \mathcal{B}_1) and the other one is the system (11) (corresponding to \mathcal{B}_2). Now we consider the condition number of the system (11). Following the idea in [8], we first introduce the following Lemma (see [6] and [8]).

Lemma 1. *Let $f = \sum_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k}$. Then $f \in H^s(\mathbb{R})$ if and only if*

$$\sum_{j,k} |\langle f, \psi_{j,k} \rangle|^2 (1 + 4^{js}) < +\infty, \quad -r < s < r$$

where r is the regularity of the MRA. Moreover, since $\{\psi_{j,k}\}$ is a Riesz basis of $H^s(\mathbb{R})$, we also have

$$(12) \quad C_1 \sum_{j,k} |\langle f, \psi_{j,k} \rangle|^2 (1 + 4^{js}) \leq \|f\|_{H^s}^2 \leq C_2 \sum_{j,k} |\langle f, \psi_{j,k} \rangle|^2 (1 + 4^{js}),$$

where $C_2 \geq C_1 > 0$ are constants.

Let $\phi \in V_J$ with $\phi = \sum_{j,k} w_{j,k} \psi_{j,k}$. We have

$$(13) \quad \langle A(\phi), \phi \rangle = \sum_{j,k} \sum_{s,t} w_{j,k} w_{s,t} \langle A(\psi_{j,k}), \psi_{s,t} \rangle = w^T A_\infty w,$$

where $w = (w_{j,k})^T$ is a vector. By the assumption that $B(\sigma, \mu) = \langle A\sigma, \mu \rangle$ is a continuous elliptic bilinear form on the space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$, we have

$$(14) \quad C_3 \|\phi\|_{H^s}^2 \leq \langle A(\phi), \phi \rangle \leq C_4 \|\phi\|_{H^s}^2,$$

where $C_4 \geq C_3 > 0$ are constants. Combining (13) and (14), we have

$$C_3 \|\phi\|_{H^s}^2 \leq w^T A_\infty w \leq C_4 \|\phi\|_{H^s}^2.$$

By using (12), one can easily obtain

$$C_5 \sum_{j,k} |2^{js} w_{j,k}|^2 \leq w^T A_\infty w \leq C_6 \sum_{j,k} |2^{js} w_{j,k}|^2,$$

where $C_6 \geq C_5 > 0$ are constants. After a diagonal scaling D , we have

$$C_5 \|w\|^2 \leq w^T D^{-1/2} A_\infty D^{-1/2} w \leq C_6 \|w\|^2,$$

where $\|\cdot\|$ is the l_2 -norm. Thus, the condition number of A_∞ after a diagonal scaling is

$$\kappa(D^{-1/2}A_\infty D^{-1/2}) = O(1).$$

The relation between T_∞ given by (8) and A_∞ given by (11) is

$$A_\infty = WT_\infty W^{-1},$$

where W is the wavelet transform matrix between two orthonormal wavelet bases \mathcal{B}_1 and \mathcal{B}_2 . We then solve the Toeplitz system (8) by solving its equivalent form

$$(WT_\infty W^{-1})Wx = Wb,$$

i.e.,

$$A_\infty \tilde{x} = \tilde{b},$$

where $\tilde{x} = Wx$ and $\tilde{b} = Wb$. For solving the system $A_\infty \tilde{x} = \tilde{b}$, we use the PCG method with the diagonal preconditioner D . Instead of solving the system $A_\infty \tilde{x} = \tilde{b}$, we solve the preconditioned system

$$D^{-1}A_\infty \tilde{x} = D^{-1}\tilde{b}.$$

Since the condition number of the preconditioned system is

$$\kappa(D^{-1}A_\infty) = \kappa(D^{-1/2}A_\infty D^{-1/2}) = O(1),$$

when the PCG method is applied to solve the preconditioned system, the convergence rate will be linear (see [5]).

In practice, we usually use a finite interval instead of $[0, +\infty)$. We then have an n -by- n system

$$(15) \quad T_n x = b,$$

where T_n is the finite section of T_∞ . Let W_n be the finite section of the wavelet transform matrix W . The system (15) can be solved by solving its equivalent form

$$(W_n T_n W_n^{-1})W_n x = W_n b,$$

i.e.,

$$(16) \quad A_n \tilde{x} = \tilde{b},$$

where A_n is the finite section of A_∞ , $\tilde{x} = W_n x$ and $\tilde{b} = W_n b$. The PCG method is applied to solve the system (16) with the diagonal preconditioner D_n which is the finite section of D . In each iteration of the PCG method, we have to compute the matrix-vector multiplication $A_n v$ for some vector v and solve the

system $D_n y = z$ (see [5]). For $A_n v$, since $A_n v = W_n T_n W_n^{-1} v$, by using the FWT, $u = W_n^{-1} v$ could be computed in $O(n)$ operations. The $T_n u$ could be computed using the FFT in $O(n \log n)$ operations by first embedding T_n into a $2n$ -by- $2n$ circulant matrix (see [2] and [7]). By using FWT again, we note that the operation cost for $A_n v$ will remain $O(n \log n)$. It requires only $O(n)$ operations to solve the system $D_n y = z$. Thus, the total operation cost per iteration is $O(n \log n)$. Since the number of iterations is independent of n , we therefore can solve the system (16) and also the system (15) in $O(n \log n)$ operations. For readers interested in the actual numerical implementation of our algorithm, we refer to [1].

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