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ON A SUFFICIENT CONDITION AND AN ANGULAR ESTIMATION FOR Φ-LIKE FUNCTIONS

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Abstract. The object of the present paper is to investigate a sufficient condition and an angular estimation for Φ-like functions. Our result contains the previous results as special cases.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Then a function $f \in \mathcal{A}$ is called Φ-like in U if

$$\operatorname{Re} \frac{zf'(z)}{\Phi(f(z))} > 0 \quad (z \in U),$$

where $\Phi(w)$ is analytic in $f(U)$, $\Phi(0) = \Phi'(0) - 1 = 0$ and $\Phi(w) \neq 0$ in $f(U) - \{0\}$.

The definition of Φ-like functions was introduced by Brickman [1], and he proved that every Φ-like function is univalent in U. In particular, a classical starlike function is the special case of a Φ-like function with $\Phi(w) = w$.

In this paper, we give a sufficient condition and an angular estimation for Φ-like functions. Also we generalize the results given by Mocanu [2] and Nunokawa [4, 5].

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2. Main results

In proving our theorems, we need the following lemma due to Nunokawa [3, 4].

Lemma 1. *Let p be analytic in U, $p(0) = 1$ and $p(z) \neq 0$ in U. If there exists a point $z_0 \in U$ such that*

$$|\arg p(z)| < \frac{\pi\delta}{2} \quad \text{for} \quad |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\delta}{2},$$

where $\delta > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\delta k,$$

where

$$k \geq \frac{1}{2}\left(l + \frac{1}{l}\right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi\delta}{2}$$

and

$$k \leq -\frac{1}{2}\left(l + \frac{1}{l}\right) \quad \text{when} \quad \arg p(z_0) = \frac{\pi\delta}{2},$$

and where

$$p(z_0)^{\frac{1}{\delta}} = \pm il \ (l > 0).$$

With the help of Lemma 1, we derive

Theorem 1. *Let $a \geq 0$ or $a \leq -2b\,(b > 0)$. If $f \in \mathcal{A}$ satisfies the condition*

$$a\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\Phi(f(z)))'}{\Phi(f(z))}\right) + b\left(\frac{zf'(z)}{\Phi(f(z))}\right) \neq it \quad (z \in U), \tag{1}$$

where t is a real number with $|t| \geq \sqrt{a(a+2b)}$ and $\Phi(w)$ is analytic in $f(U)$, $\Phi(0) = \Phi'(0) - 1 = 0$ and $\Phi(w) \neq 0$ in $f(U) - \{0\}$, then $f(z)$ is Φ-like in U.

Proof. For the case $a = 0$ it is obvious and so we suppose $a \neq 0$. Let

$$p(z) = \frac{zf'(z)}{\Phi(f(z))},$$

where $p(0) = 1$. From the assumption (1), we easily have

$$p(z) \neq 0 \quad (z \in U).$$

In fact, if p has a zero of order m at $z = z_1 \in U$, then p can be written as

$$p(z) = (z - z_1)^m q(z) \quad (m \in N = \{1, 2, \cdots\}),$$

where q is analytic in U and $q(z_1) \neq 0$. Hence we have

$$\begin{aligned} &a\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\Phi(f(z)))'}{\Phi(f(z))}\right) + b\left(\frac{zf'(z)}{\Phi(f(z))}\right) = a\frac{zp'(z)}{p(z)} + bp(z) \\ &= a\frac{mz}{z - z_1} + a\frac{zq'(z)}{q(z)} + b(z - z_1)^m q(z). \end{aligned} \tag{2}$$

But the imaginary part of (2) can take any infinite values when z approaches z_1 in a suitable direction. This contradicts (1). Therefore we have $p(z) \neq 0$ $(z \in U)$. Therefore, if there exists a point $z_0 \in U$ such that

$$\mathrm{Re}p(z) > 0 \quad \text{for} \quad |z| < |z_0|,$$

$$\mathrm{Re}p(z_0) = 0 \quad \text{and} \quad p(z_0) = il \ (l \neq 0),$$

then we have $p(z_0) \neq 0$ $(l \neq 0)$. For the case $a \geq 0$, from Lemma 1 and (2), we have

$$a\frac{z_0 p'(z_0)}{p(z_0)} + bp(z_0) = i(ak + bl),$$

and

$$ak + bl \geq \frac{1}{2}\left((a + 2b)l + \frac{a}{l}\right) \geq \sqrt{(a + 2b)a} \text{ when } l > 0,$$

and

$$ak + bl \geq -\frac{1}{2}\left((a + 2b)|l| + \frac{a}{|l|}\right) \leq -\sqrt{(a + 2b)a} \text{ when } l < 0,$$

which contradicts (1). Therefore we have $\mathrm{Re}p(z) > 0$ in U. For the case $a \leq -2b$, applying the same method as the above, we easily have the same conclusion. This completes the proof of our theorem. ■

Taking $a = \alpha$, $b = 1$ and $\Phi(w) = w$ in Theorem 1, we have the following result obtained by Nunokawa [6].

Corollary 1. *Let $\alpha \geq 0$ or $\alpha \leq -2$. If $f \in \mathcal{A}$ satisfies the condition*

$$(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \neq it \quad (z \in U),$$

where t is a real number and $|t| \geq \sqrt{(\alpha + 2)\alpha}$, then f is starlike in U.

Letting $a = b = 1$ and $\Phi(w) = w$ in Theorem 1, we get the following

Corollary 2. *Let $f \in \mathcal{A}$ and suppose that there exists a real number R for which*

$$\left| \frac{zf''(z)}{f'(z)} - R \right| < \sqrt{(R+1)^2 + 3} \quad (z \in U).$$

Then f is starlike in U.

Remark. Corollary 2 is the corresponding result of Nunokawa [5] and an extension of a result by Mocanu [2].

Taking $\Phi(f(z)) = z$ in Theorem 1, we have

Corollary 3. *Let $a \geq 0$ or $a \leq -2b$ $(b > 0)$. If $f \in \mathcal{A}$ and satisfies the condition*

$$a\frac{zf''(z)}{f'(z)} + bf'(z) \neq it \quad (z \in U),$$

where t is a real number given by Theorem 1, then $\mathrm{Re} f'(z) > 0$ *(or f is close-to-convex) in U.*

Next, we prove

Theorem 2. *Let $a > 0$ and $b > 0$. If $f \in \mathcal{A}$ satisfies the condition*

$$\left| \arg \left\{ a\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\Phi(f(z)))'}{\Phi(f(z))}\right) + b\left(\frac{zf'(z)}{\Phi(f(z))}\right) \right\} \right| < \frac{\pi\delta}{2} \quad (0 < \delta \leq 1),$$

where $\Phi(w)$ is analytic in $f(U)$, $\Phi(0) = \Phi'(0) - 1 = 0$ and $\Phi(w) \neq 0$ in $f(U) - \{0\}$, then

$$\left| \arg \left(\frac{zf'(z)}{\Phi(f(z))} \right) \right| < \frac{\pi\eta}{2},$$

where η $(0 < \eta < 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{a(\frac{\eta}{1-\eta})(\frac{1-\eta}{1+\eta})^{\frac{1+\eta}{2}} \sin\frac{\pi(1-\eta)}{2}}{b + a(\frac{\eta}{1-\eta})(\frac{1-\eta}{1+\eta})^{\frac{1+\eta}{2}} \cos\frac{\pi(1-\eta)}{2}} \right). \tag{3}$$

Proof. Let

$$p(z) = \frac{zf'(z)}{\Phi(f(z))}.$$

By the similar method of the proof in Theorem 1, we can see that $p(z) \neq 0$ in U. If there exists a point $z_0 \in U$ such that

$$|\arg\, p(z)| < \frac{\pi\eta}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg\, p(z_0)| = \frac{\pi\eta}{2},$$

then, from Lemma 1, we have

$$\frac{z_0 p^{'}(z_0)}{p(z_0)} = i\eta k,$$

where

$$k \geq \frac{1}{2}\left(l + \frac{1}{l}\right) \text{ when } \arg\, p(z_0) = \frac{\pi\eta}{2}$$

and

$$k \leq -\frac{1}{2}\left(l + \frac{1}{l}\right) \text{ when } \arg\, p(z_0) = -\frac{\pi\eta}{2},$$

and where

$$p(z_0)^{\frac{1}{\eta}} = \pm il \quad (l > 0).$$

At first, suppose that $p(z_0)^{\frac{1}{\eta}} = il\,(l > 0)$. Then we have

$$\begin{aligned} &a\left(1 + \frac{z_0 f^{''}(z_0)}{f^{'}(z_0)} - \frac{z_0(\Phi(f(z_0)))^{'}}{\Phi(f(z_0))}\right) + b\left(\frac{z_0 f^{'}(z_0)}{\Phi(f(z_0))}\right) \\ &= bp(z_0) + a\frac{z_0 p^{'}(z_0)}{p(z_0)} = p(z_0)\left(b + a\frac{z_0 p^{'}(z_0)}{(p(z_0))^2}\right) \\ &= (il)^\eta\left(b + i\eta k\frac{a}{(il)^\eta}\right) = l^\eta e^{i\frac{\pi\eta}{2}}\left(b + e^{i\frac{\pi(1-\eta)}{2}}\eta k\frac{a}{l^\eta}\right), \end{aligned}$$

where

$$k \geq \frac{1}{2}\left(l + \frac{1}{l}\right).$$

Then we have

$$\eta k\frac{a}{l^\eta} \geq \frac{\eta a}{2}(l^{1-\eta} + l^{-1-\eta}).$$

Putting

$$g(l) = \frac{1}{2}(l^{1-\eta} + l^{-1-\eta}) \quad (l > 0),$$

we can show easily that $g(l)$ takes the minimum value at $l = \sqrt{(1+\eta)/(1-\eta)}$. Therefore we have

$$\begin{aligned}
&\arg\left\{a\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}-\frac{z_0(\Phi(f(z_0)))'}{\Phi(f(z_0))}\right)+b\left(\frac{z_0 f'(z_0)}{\Phi(f(z_0))}\right)\right\}\\
&=\arg\ p(z_0)+\arg\left(b+a\frac{z_0 p'(z_0)}{(p(z_0))^2}\right)\\
&=\frac{\pi\eta}{2}+\arg\left(b+e^{i\frac{\pi(1-\eta)}{2}}\eta k\frac{a}{l^\eta}\right)\\
&\geq \frac{\pi\eta}{2}+\tan^{-1}\left(\frac{a(\frac{\eta}{1-\eta})(\frac{1-\eta}{1+\eta})^{\frac{1+\eta}{2}}\sin\frac{\pi(1-\eta)}{2}}{b+a(\frac{\eta}{1-\eta})(\frac{1-\eta}{1+\eta})^{\frac{1+\eta}{2}}\cos\frac{\pi(1-\eta)}{2}}\right).
\end{aligned}$$

This contradicts the assumption of the theorem.

For the case $p(z_0)^{\frac{1}{\eta}} = -il\ (l > 0)$, applying the same method as the above, we have

$$\begin{aligned}
&\arg\left\{a\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}-\frac{z_0(\Phi(f(z_0)))'}{\Phi(f(z_0))}\right)+b\left(\frac{z_0 f'(z_0)}{\Phi(f(z_0))}\right)\right\}\\
&\leq -\frac{\pi\eta}{2}-\tan^{-1}\left(\frac{a(\frac{\eta}{1-\eta})(\frac{1-\eta}{1+\eta})^{\frac{1+\eta}{2}}\sin\frac{\pi(1-\eta)}{2}}{b+a(\frac{\eta}{1-\eta})(\frac{1-\eta}{1+\eta})^{\frac{1+\eta}{2}}\cos\frac{\pi(1-\eta)}{2}}\right),
\end{aligned}$$

which contradicts the assumption. Therefore we complete the proof of our theorem. ■

Putting $a = b = 1$ and $\Phi(f) = f$ in Theorem 2, we have the following result given by Nunokawa [4].

Corollary 4. *If $f \in \mathcal{A}$ satisfies the condition*

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi\delta}{2} \quad (0 < \delta \leq 1),$$

then

$$\left|\arg\left(\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi\eta}{2},$$

where η $(0 < \eta < 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{(\frac{\eta}{1-\eta})(\frac{1-\eta}{1+\eta})^{\frac{1+\eta}{2}}\sin\frac{\pi(1-\eta)}{2}}{1+(\frac{\eta}{1-\eta})(\frac{1-\eta}{1+\eta})^{\frac{1+\eta}{2}}\cos\frac{\pi(1-\eta)}{2}}\right).$$

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