TAIWANESE JOURNAL OF MATHEMATICS
Vol. 2, No. 3, pp. 297-302, September 1998

ON CERTAIN CLASSES OF STRONGLY STARLIKE FUNCTIONS

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Abstract. By using the method of differential inequalities we obtained some new and better results for certain classes of strongly starlike functions introduced recently in [3].

1. Introduction

Let A denote the class of functions f which are analytic in the unit disc $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Also, as usual, let

$$S^* = \left\{ f \in A : \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > 0,\ z \in U \right\}$$

and

$$\tilde{S}^*(\alpha) = \left\{ f \in A : \left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha\pi}{2},\ z \in U \right\}$$

be the classes of starlike and strongly starlike functions of order α $(0 < \alpha \leq 1)$, respectively. We note that $\tilde{S}^*(\alpha) \subset S^*$ for $0 < \alpha < 1$ and $\tilde{S}^*(1) \equiv S^*$.

Let $H(\alpha)$ denote a class of functions $f \in A$ for which

$$\mathrm{Re}\left\{ \frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \right\} > 0$$

for $a \geq 0$ and $\frac{f(z)}{z} \neq 0$, $z \in U$ ([3]). In [3] it was shown that $H(\alpha) \subset S^*$ and $H(1) \subset \tilde{S}^*(\frac{1}{2})$. In the paper [2] the authors gave a better result of the type $H(1) \subset \tilde{S}^*(\beta)$, $\beta < \frac{1}{2}$.

Received October 30, 1996.
Communicated by S.-Y. Shaw.
1991 *Mathematics Subject Classification*: 30C45.
Key words and phrases: Analytic function, starlike function, strongly starlike function, differential inequalities.

In this paper, by using the method of differential inequalities, we also give a better result but in other direction and some new results. First, we cite the following result on differential inequalities.

Lemma 1([1]). *Let $\Phi(u,v)$ be a complex function, $\Phi : D \to \mathbb{C}$, $D \subset \mathbb{C}\times\mathbb{C}$ ($\mathbb{C}$ is the complex plane), and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\Phi(u,v)$ satisfies the following conditions:*
(a) $\Phi(u,v)$ *is continuous in* D;
(b) $(1,0) \in D$ *and* $\mathrm{Re}\{\Phi(1,0)\} > 0$;
(c) $\mathrm{Re}\{\Phi(iu_2, v_1)\} \le 0$ *for all* $(iu_2, v_1) \in D$ *and such that* $v_1 \le -(1+u_2^2)/2$.
Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in U such that $(p(z), zp'(z)) \in D$ for all $z \in U$. If $\mathrm{Re}\ \{\Phi(p(z), zp'(z))\} > 0$, $z \in U$, *then* $\mathrm{Re}\ \{p(z)\} > 0$, $z \in U$.

2. Applications of Differential Inequalities

Let us consider the following implication

$$(1)\quad \mathrm{Re}\left\{\alpha\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > \beta \quad \Rightarrow \quad \mathrm{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)^{\gamma}\right\} > 0,\ z \in U,$$

where $\alpha \ge 0$, $\beta < 1$, $\gamma \ge 1$.

If we put $p(z) = \left(\frac{zf'(z)}{f(z)}\right)^{\gamma}$, then (1) is equivalent to

$$(2)\quad \begin{aligned} &\mathrm{Re}\left\{\frac{\alpha}{\gamma}p(z)^{\frac{1}{\gamma}-1}zp'(z) + \alpha p(z)^{\frac{2}{\gamma}} + (1-\alpha)p(z)^{\frac{1}{\gamma}} - \beta\right\} > 0 \\ &\Rightarrow\ \mathrm{Re}\{p(z)\} > 0,\ z \in U. \end{aligned}$$

Set

$$\Phi(u,v) = \frac{\alpha}{\gamma}u^{\frac{1}{\gamma}-1}v + \alpha u^{\frac{2}{\gamma}} + (1-\alpha)u^{\frac{1}{\gamma}} - \beta$$

(we note that we put $p(z) = u$, $zp'(z) = v$). It is easy to show that for $\alpha \ge 0$, $\beta < 1$, $\gamma \ge 1$ we have
(a') $\Phi(u,v)$ is continuous in $D = (\mathbb{C}\setminus\{0\})\times\mathbb{C}$;
(b') $(1,0) \in D$ and $\mathrm{Re}\{\Phi(1,0)\} = 1 - \beta > 0$;
i.e. the conditions (a) and (b) of Lemma 1 are satisfied, while for $(iu_2, v_1) \in D$ such that $v_1 \le -(1+u_2^2)/2$ we obtain

$$(3)\quad \begin{aligned} \mathrm{Re}\{\Phi(iu_2,v_1)\} = &\frac{\alpha}{\gamma}|u_2|^{\frac{1}{\gamma}-1}\cos\left(\left(\frac{1}{\gamma}-1\right)\frac{\pi}{2}\right)v_1 + \alpha|u_2|^{\frac{2}{\gamma}}\cos\frac{\pi}{\gamma} \\ &+(1-\alpha)|u_2|^{\frac{1}{\gamma}}\cos\tfrac{\pi}{2\gamma} - \beta \le -\frac{\alpha}{2\gamma}(1+u_2^2)|u_2|^{\frac{1}{\gamma}-1}\sin\frac{\pi}{2\gamma} \\ &+\alpha|u_2|^{\frac{2}{\gamma}}\cos\frac{\pi}{\gamma} + (1-\alpha)|u_2|^{\frac{1}{\gamma}}\cos\frac{\pi}{2\gamma} - \beta \end{aligned}$$

or if we put $|u_2| = t$, $t > 0$:

$$\mathrm{Re}\,\{\Phi(iu_2, v_1)\} \le \Phi_1(t), \tag{4}$$

where

$$\Phi_1(t) = -\frac{\alpha}{2\gamma}(1+t^2)t^{\frac{1}{\gamma}-1}\sin\frac{\pi}{2\gamma} + \alpha t^{\frac{2}{\gamma}}\cos\frac{\pi}{\gamma} + (1-\alpha)t^{\frac{1}{\gamma}}\cos\frac{\pi}{2\gamma} - \beta. \tag{5}$$

If for some choice of constants α, β, γ we obtain that for every $t > 0$ we have $\Phi_1(t) \le 0$, then from (4) and Lemma 1 we can conclude that appropriate implication (1) is true.

Further we will consider some of such cases.

For $\beta = \frac{1}{2}$ and $\gamma = 1$ we get

Theorem 1. *If $f \in A$, $f(z)/z \ne 0$ for $z \in U$, and satisfies the condition*

$$\mathrm{Re}\left\{\alpha\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > -\frac{\alpha}{2},\ z \in U,$$

then $f \in S^*$ for any real $\alpha \ge 0$.

Proof. For $\beta = -\frac{\alpha}{2}$ and $\gamma = 1$ from (5) we obtain $\Phi_1(t) = -\frac{3}{2}\alpha t^2 \le 0$ for real t, and from the previous remarks and (1) we have the statement of the theorem. ■

This is the earlier result given in [3].

For $\alpha = \frac{2}{3}$, $\beta = 0$ and $\gamma = 2$ we get

Theorem 2. *If $f \in A$, $f(z)/z \ne 0$ for $z \in U$, and satisfies the condition*

$$\mathrm{Re}\left\{\frac{2}{3}\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > 0,\ z \in U,$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4},\ z \in U,$$

i.e., $H\left(\frac{2}{3}\right) \subset \tilde{S}^*\left(\frac{1}{2}\right)$.

Proof. For $\alpha = \frac{2}{3}$, $\beta = 0$ and $\gamma = 2$, from (5) we have

$$\Phi_1(t) = -\frac{\sqrt{2}}{12\sqrt{t}}(1-t)^2 \le 0$$

for all $t > 0$. Now, the result of this theorem is the consequence of the implication (1). ■

Remark 1. In [3] it has been shown that $H(1) \subset \tilde{S}^*\left(\frac{1}{2}\right)$. Since $H(1) \subset H\left(\frac{2}{3}\right)$ (see Theorem 3 in [3]), it means that our previous result is better. In [2] it has been given the result $H(1) \subset \tilde{S}^*(\beta) \subset \tilde{S}^*\left(\frac{1}{2}\right)$ for certain $\beta < \frac{1}{2}$.

If we put $\beta = 0$ and $\gamma = 2$, then we have

Theorem 3. *Let $f \in A$, $f(z)/z \neq 0$ for $z \in U$ and satisfies the condition*

$$\operatorname{Re}\left\{\alpha_1 \frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)}\right\} > 0, \ z \in U,$$

where

$$\alpha_1 = \frac{3}{3 + 2\sqrt{3+\sqrt{3}}} = 0,34380\cdots. \tag{6}$$

Then

$$\left|\arg \frac{z f'(z)}{f(z)}\right| < \frac{\pi}{3}, \ z \in U,$$

i.e., $H(\alpha_1) \subset \tilde{S}^*\left(\frac{2}{3}\right)$.

Proof. For $\beta = 0$ and $\gamma = \frac{3}{2}$, from (5) we have

$$\Phi_1(t) = -\frac{t^{-\frac{1}{3}}}{6}\Phi_2(t), \tag{7}$$

where

$$\Phi_2(t) = \alpha\sqrt{3}t^2 + 3\alpha t^{\frac{5}{3}} - 3(1-\alpha)t + \alpha\sqrt{3}. \tag{8}$$

If $0 < t \leq 1$, then $t^{\frac{5}{3}} \geq t^2$, and from (8) we get

$$\Phi_2(t) \geq \alpha(3+\sqrt{3})t^2 - 3(1-\alpha)t + \alpha\sqrt{3} \geq 0$$

for $\alpha \geq \alpha_1$, where α_1 is given by (6). For $t > 1$ we obtain $t^{\frac{5}{3}} > t$, and from (8) we have

$$\Phi_2(t) \geq \alpha\sqrt{3}t^2 - 3(1-2\alpha)t + \alpha\sqrt{3}. \tag{9}$$

If $\alpha > \frac{1}{2}$, then all members on the right side in (9) are positive and we have $\Phi_2(t) > 0$ for all $t > 0$. If $0 < \alpha \leq \frac{1}{2}$, then the trinom on the right side in (9) is non-negative for all $t > 1$ if $\alpha \geq \frac{3-\sqrt{3}}{4} = 0.31698\cdots$. In any case, if $\alpha \geq \alpha_1$, then $\Phi_2(t) \geq 0$ for all $t > 0$, and from (7) we conclude that $\Phi_1(t) \leq 0$ for all $t > 0$, which from (1) give the statement of this theorem. ∎

For $\beta = 0$, $\gamma = 3$ we have the following

Theorem 4. *Let $f \in A$, $f(z)/z \neq 0$ for $z \in U$, and let*

$$\operatorname{Re}\left\{\alpha_2 \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right\} > 0,\ z \in U,$$

where

$$\alpha_2 = \frac{9}{2}(\sqrt{3}-1) = 3.29422\cdots. \tag{10}$$

Then

$$\left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\pi}{6},\ z \in U,$$

i.e., $H(\alpha_2) \subset \tilde{S}^*\left(\frac{1}{3}\right)$.

Proof. For $\beta = 0$ and $\gamma = 3$, from (5) we have

$$\Phi_1(t) = -\frac{t^{-\frac{2}{3}}}{12}\Phi_3(t),$$

where

$$\Phi_3(t) = \alpha t^2 - 6\alpha t^{\frac{4}{3}} - 6\sqrt{3}(1-\alpha)t + \alpha. \tag{11}$$

First, let us suppose that $t \geq 1$. Since

$$\Phi'_3(t) = 2\alpha t - 8\alpha t^{\frac{1}{3}} - 6\sqrt{3}(1-\alpha) \quad \text{and} \quad \Phi''_3(t) = 2\alpha - \frac{8}{3}\alpha t^{-\frac{2}{3}},$$

we easily conclude that $\Phi''_3(t) \geq 0$ for $t \geq \frac{8\sqrt{3}}{9} = 1.53960\cdots$, and from there that $\Phi'_3(t) \geq \Phi'_3\left(\frac{8\sqrt{3}}{9}\right) \geq 0$ for $\alpha \geq \frac{27}{11} = 2.45454\cdots$. Therefore, for such α and t we have that $\Phi_3(t) \geq \Phi_3\left(\frac{8\sqrt{3}}{9}\right) > 0$. If $1 \leq t \leq \frac{8\sqrt{3}}{9}$, then

$$\Phi'_3(t) \geq 2\alpha - 8\alpha\left(\frac{8\sqrt{3}}{9}\right)^{\frac{1}{3}} - 6\sqrt{3}(1-\alpha) \geq 0$$

for $\alpha > \alpha_2 = \frac{9}{2}(\sqrt{3}-1)$, and for such t and α we have $\Phi_3(t) \geq \Phi_3(1) > 0$.

In the case $0 < t < 1$ we have that $t^{\frac{4}{3}} < t$ and so

$$\Phi_3(t) \geq \alpha t^2 - 6\left[\sqrt{3} - \alpha(\sqrt{3}-1)\right]t + \alpha. \tag{12}$$

It is easy to check that for $\alpha \geq \alpha_2$, where α_2 is given by (10) and $0 < t < 1$ the trinom on the right side in (12) is positive. It follows that $\Phi_3(t) > 0$. Therefore, the conclusion is similar as in the previous theorem. ∎

Acknowledgements

1. This research was completed at the Department of Mathematics, Faculty of Technology and Metallurgy, Belgrade while the second author was visiting from the Department of Mathematics, Walchand College of Engineering, Sangli, India, and he would like to thank DST, New Delhi, India for granting BOYSCAST Fellowship.

2. The work of the first author was supported by Grant No.04M03 of MNTRS through Math. Institute SANU.

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