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SUPERCYCLIC AND CESÀRO HYPERCYCLIC WEIGHTED TRANSLATIONS ON GROUPS

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Abstract. Let G be a locally compact group and let $1 \leq p < \infty$. We characterize supercyclic weighted translation operators on the Lebesgue space $L^p(G)$ in terms of the weight. Using this result, the characterization for Cesàro hypercyclic weighted translation operators is given. We also determine when scalar multiples of weighted translation operators are hypercyclic and topologically mixing, and show, for any weighted translation operator T, βT is mixing for all $\beta \in (1, 4)$ if T and $4T$ are mixing.

1. Introduction

Let T be a continuous linear self-map on a Banach space X, and denote by T^n the n-th iterate of T. If there exists a vector $x \in X$ such that the orbit $\{x, Tx, \ldots, T^n x, \ldots\}$ is dense in X, then T resp. x is called *hypercyclic* resp. a *hypercyclic vector* of T. Accordingly, T is *supercyclic* if there exists a vector $x \in X$ such that the orbit

$$\{\alpha T^n x : \alpha \in \mathbb{C}, n \in \mathbb{N} \cup \{0\}\} = \bigcup_{\mathbb{N}\cup\{0\}} \mathbb{C}T^n x$$

is dense in X in which x is called a *supercyclic vector* and T has a dense set of supercyclic vectors [14]. Hypercyclicity and supercyclicity have been studied by many authors; we refer to [5, 6, 12, 14] for surveys.

Recently, we characterized in [2, 3, 4] hypercyclic and chaotic weighted translations on locally compact groups and their homogeneous spaces. In [14], Salas gave a Supercyclic Criterion and characterized the supercyclic bilateral weighted shifts in terms of weights. In this paper, we adapt an approach different from those in [3, 4, 13, 14] where Salas' idea was used, and give sufficient and necessary conditions for weighted translations on groups to be supercyclic. This result entails two further consequences. One is

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the characterization of Cesàro hypercyclic weighted translations, the other is the weight conditions for scalar multiples of weighted translation operators to be hypercyclic and topologically mixing.

In what follows, let G be a locally compact group with identity e and a right-invariant Haar measure λ which is the counting measure if G is discrete. It is known that a complex Banach space admits a supercyclic operator if it is one dimensional, or infinite-dimensional and separable. Hence we assume G is second countable and denote by $L^p(G)$ $(1 \leq p < \infty)$ the complex Lebesgue space, with respect to λ.

A continuous function $w : G \to (0, \infty)$ is called a *weight* on G. Let $a \in G$ and let δ_a be the unit point mass at a. A *weighted translation* on G is a weighted convolution operator $T_{a,w} : L^p(G) \longrightarrow L^p(G)$ defined by

$$T_{a,w}(f) := wT_a(f) \qquad (f \in L^p(G))$$

where w is a weight on G and $T_a(f) = f * \delta_a \in L^p(G)$ is the convolution:

$$(f * \delta_a)(x) := \int_G f(xy^{-1})d\delta_a(y) = f(xa^{-1}) \qquad (x \in G)$$

which is just the right translation of f by a^{-1}.

It has been shown in [4] that $T_{a,w}$ is not hypercyclic if a is a torsion element. The following lemma reveals that if a is a torsion element with the simple weight $w = 1$, then T_a is not supercyclic.

An element a in a group G is called a *torsion element* if it is of finite order. In a locally compact group G, an element $a \in G$ is called *periodic* [7] (or *compact* [8, 9.9]) if the closed subgroup $G(a)$ generated by a is compact. We call an element in G *aperiodic* if it is not periodic. For discrete groups, periodic and torsion elements are identical, in other words, aperiodic elements are the non-torsion elements.

Lemma 1.1. *Let G be a locally compact group and let a be a torsion element in G. Let $1 \leq p < \infty$ and T_a be a translation on $L^p(G)$. Then T_a is not supercyclic.*

Proof. Let f be any nonzero vector in $L^p(G)$ and let a have order d say. Then $T_a^d f = f * \delta_a^d = f * \delta_{a^d} = f$. Now assume $T_a^n f \neq 0$ for all $1 \leq n \leq d-1$. Then the orbit

$$\left\{ \frac{T_a^n f}{\|T_a^n f\|} : n \geq 0 \right\} = \left\{ \frac{f}{\|f\|}, \frac{f * \delta_a}{\|f * \delta_a\|}, \frac{f * \delta_{a^2}}{\|f * \delta_{a^2}\|}, \cdots, \frac{f * \delta_{a^{d-1}}}{\|f * \delta_{a^{d-1}}\|} \right\}$$

can not be dense in the unit sphere. Hence T_a is not supercyclic. ■

2. Supercyclic Weighted Translations

It has been shown in [4, Lemma 2.1] that an element a in a locally compact group G is aperiodic if, and only if, for any compact subset $K \subset G$, there exists $N \in \mathbb{N}$

such that $K \cap Ka^n = \emptyset$ (equivalently, $K \cap Ka^{-n} = \emptyset$) for $n > N$. We note that [4, Remark 2.2] in many familiar non-discrete groups, including the additive group $\mathbb{R}^n$, the Heisenberg group and the affine group, all elements except the identity are aperiodic.

We are now ready to give the sufficient and necessary conditions for supercyclic weighed translations. We recall that a sequence of operators (T_n) is *topologically transitive* if for any nonempty open sets U and V, we have $T_n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$. If (T_n) satisfies the stronger condition that $T_n(U) \cap V \neq \emptyset$ for some n onwards, then (T_n) is said to be *topologically mixing*. We will make use of the equivalence of dense hypercyclicity and topological transitivity [5]. A sequence of operators (T_n) on a Banach space X is *hypercyclic* if there exists a vector $x \in X$ such that the orbit $\{T_n x\}_{n \geq 0}$ is dense in X. If (T_n) has a dense set of hypercyclic vectors, then (T_n) is called *densely hypercyclic*. In the result below, we will show that $(T_n = \alpha_n T^n_{a,w})$ is topologically transitive, which implies that $(\alpha_n T^n_{a,w})$ is hypercyclic, and therefore $T_{a,w}$ is supercyclic.

Theorem 2.1. *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \leq p < \infty$ and $T_{a,w}$ be a weighted translation on $L^p(G)$. The following conditions are equivalent.*

(i) *$T_{a,w}$ is supercyclic.*

(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K and there exists a sequence $(\alpha_n) \subset \mathbb{C} \setminus \{0\}$ such that $\lambda(K) = \lim\limits_{k\to\infty} \lambda(E_k)$ and both sequences*

$$\varphi_n := |\alpha_n| \prod_{s=1}^{n} w * \delta^s_{a^{-1}} \quad \text{and} \quad \widetilde{\varphi}_n := \left(|\alpha_n| \prod_{s=0}^{n-1} w * \delta^s_a \right)^{-1}$$

admit respectively subsequences (φ_{n_k}) and $(\widetilde{\varphi}_{n_k})$ satisfying

$$\lim_{k\to\infty} \|\varphi_{n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{n_k}|_{E_k}\|_\infty = 0.$$

Proof. (ii) $\Rightarrow$ (i). We show that $(\alpha_n T^n_{a,w})_{n\in\mathbb{N}}$ is topologically transitive. Let U and V be non-empty open subsets of $L^p(G)$. Since the space $C_c(G)$ of continuous functions on G with compact support is dense in $L^p(G)$, we can pick $f, g \in C_c(G)$ with $f \in U$ and $g \in V$. Let K be the union of the compact supports of f and g. Let $E_k \subset K$ and the subsequences (φ_{n_k}) and $(\widetilde{\varphi}_{n_k})$ satisfy condition (ii).

By aperiodicity of a, there exists $M \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \emptyset$ for all $n > M$.

First, we show that $\|\alpha_{n_k} T^{n_k}_{a,w}(f\chi_{E_k})\|_p \to 0$ as $k \to \infty$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $n_k > M$ and $\varphi^p_{n_k} < \frac{\varepsilon}{\|f\|^p_p}$ on E_k for $k > N$. Hence

$$\begin{aligned} &\|\alpha_{n_k} T^{n_k}_{a,w}(f\chi_{E_k})\|^p_p \\ &= \int_{E_k a^{n_k}} |\alpha_{n_k} w(x) w(xa^{-1}) \cdots w(xa^{-(n_k-1)})|^p |f(xa^{-n_k})|^p d\lambda(x) \end{aligned}$$

$$= \int_{E_k} |\alpha_{n_k} w(xa^{n_k}) w(xa^{n_k-1}) \cdots w(xa)|^p |f(x)|^p d\lambda(x)$$
$$= \int_{E_k} \varphi_{n_k}^p(x) |f(x)|^p d\lambda(x) < \varepsilon$$

for $k > N$.

We define a self-map $S_{a,w}$ on the subspace $L_c^p(G)$ of functions in $L^p(G)$ with compact support by

$$S_{a,w}(h) = \frac{h}{w} * \delta_{a^{-1}} \qquad (h \in L_c^p(G))$$

so that

$$T_{a,w} S_{a,w}(h) = h \qquad (h \in L_c^p(G)).$$

Applying similar arguments to the iterates $S_{a,w}^{n_k}$, using the sequence $(\widetilde{\varphi}_{n_k})$, yields

$$\lim_{k\to\infty} \|\frac{1}{\alpha_{n_k}} S_{a,w}^{n_k}(g\chi_{E_k})\|_p^p$$
$$= \lim_{k\to\infty} \int_{E_k a^{-n_k}} \frac{1}{|\alpha_{n_k} w(xa) w(xa^2) \cdots w(xa^{n_k})|^p} |g(xa^{n_k})|^p d\lambda(x) = 0.$$

For each $k \in \mathbb{N}$, we let

$$v_k = f\chi_{E_k} + \frac{1}{\alpha_{n_k}} S_{a,w}^{n_k}(g\chi_{E_k}) \in L^p(G).$$

Then

$$\|v_k - f\|_p^p \le \|f\|_\infty^p \lambda(K \setminus E_k) + \|\frac{1}{\alpha_{n_k}} S_{a,w}^{n_k}(g\chi_{E_k})\|_p^p$$

and

$$\|\alpha_{n_k} T_{a,w}^{n_k} v_k - g\|_p^p \le \|\alpha_{n_k} T_{a,w}^{n_k}(f\chi_{E_k})\|_p^p + \|g\|_\infty^p \lambda(K \setminus E_k).$$

Hence $\lim_{k\to\infty} v_k = f$ and $\lim_{k\to\infty} \alpha_{n_k} T_{a,w}^{n_k} v_k = g$ which imply $(\alpha_{n_k} T_{a,w}^{n_k})(U) \cap V \neq \emptyset$ for some k.

(i) $\Rightarrow$ (ii). We adapt an approach in [9] to our setting. Let $T_{a,w}$ be supercyclic. Let $K \subset G$ be a compact set with $\lambda(K) > 0$. By aperiodicity of a, there is some M such that $K \cap Ka^{\pm n} = \emptyset$ for $n > M$. Let $\chi_K \in L^p(G)$ be the characteristic function of K. Let $\varepsilon \in (0,1)$. By density of the supercyclic vectors for $T_{a,w}$, there exist a supercyclic vector $f \in L^p(G)$, some $m > M$ and $\alpha \in \mathbb{C} \setminus \{0\}$ such that

$$\|f - \chi_K\|_p < \varepsilon^2 \quad \text{and} \quad \|\alpha T_{a,w}^m f + \chi_K\|_p < \varepsilon^2.$$

By the continuity of the mapping $h \in L^p(G, \mathbb{C}) \mapsto Re\ h \in L^p(G, \mathbb{R})$ and the fact $T_{a,w}$ commutes with it, we can assume without loss of generality that f is real-valued and

α is real. For a Borel subsets F of G, we have $\|T_{a,w}^n h\chi_F\|_p \le \|T_{a,w}^n h\|_p$ for arbitrary n and $h \in L^p(G,\mathbb{R})$. Obviously the mapping $h \in L^p(G,\mathbb{R}) \mapsto h^+ \in L^p(G,\mathbb{R})$, where $h^+ = \max\{0,h\}$, satisfies $\|(h+g)^+\|_p \le \|h^+ + g^+\|_p$ and commutes with $T_{a,w}$ so that we have

$$\begin{aligned}\|\alpha T_{a,w}^m(f^+\chi_F)\|_p &\le \|(\alpha T_{a,w}^m f)^+\|_p = \|(\alpha T_{a,w}^m f - (-\chi_K) + (-\chi_K))^+\|_p \\ &\le \|(\alpha T_{a,w}^m f - (-\chi_K))^+\|_p + \|(-\chi_K)^+\|_p \\ &= \|(\alpha T_{a,w}^m f - (-\chi_K))^+\|_p \le \|\alpha T_{a,w}^m f + \chi_K\|_p < \varepsilon^2\end{aligned}$$

and

$$\begin{aligned}\|f^-\chi_F\|_p &\le \|f^-\|_p = \|(f-\chi_K+\chi_K)^-\|_p \\ &\le \|(f-\chi_K)^-\|_p + \|\chi_K^-\|_p \\ &= \|f-\chi_K\|_p < \varepsilon^2\end{aligned}$$

where $f^- = \max\{0,-f\}$. Let $A = \{x \in K : |f(x)-1| \ge \varepsilon\}$. Then

$$\varepsilon^{2p} > \|f-\chi_K\|_p^p \ge \int_A |f(x)-1|^p d\lambda(x) \ge \varepsilon^p \lambda(A),$$

giving $\lambda(A) < \varepsilon^p$. Similarly, let $B = \{x \in K : |\alpha T_{a,w}^m f(x) + 1| \ge \varepsilon\}$. Then

$$\varepsilon^{2p} > \|\alpha T_{a,w}^m f + \chi_K\|_p^p \ge \int_B |\alpha T_{a,w}^m f(x)+1|^p d\lambda(x) \ge \varepsilon^p \lambda(B),$$

which implies $\lambda(B) < \varepsilon^p$. Now setting $E := \{x \in K : |f(x)-1| < \varepsilon\} \cap \{x \in K : |\alpha T_{a,w}^m f(x)+1| < \varepsilon\}$, it follows that $\lambda(K \setminus E) < 2\varepsilon^p$,

$$f(x) > 1-\varepsilon > 0 \qquad \text{and} \qquad \alpha T_{a,w}^m f(x) < \varepsilon - 1 < 0 \qquad (x \in E).$$

Hence we have

$$\begin{aligned}\varepsilon^{2p} &> \|\alpha T_{a,w}^m f^+ \chi_{Ea^m}\|_p^p \\ &= \int_{Ea^m} |\alpha T_{a,w}^m f^+(x)|^p d\lambda(x) \\ &= \int_{Ea^m} |\alpha w(x)w(xa^{-1})\cdots w(xa^{-(m-1)})|^p |f^+(xa^{-m})|^p d\lambda(x) \\ &= \int_E |\alpha w(xa^m)w(xa^{m-1})\cdots w(xa)|^p |f^+(x)|^p d\lambda(x) \\ &> (1-\varepsilon)^p \int_E \varphi_m^p(x) d\lambda(x)\end{aligned}$$

which implies $\|\varphi_m|_E\|_\infty \to 0$ as $m \to \infty$. Similarly,

$$\begin{aligned}
\varepsilon^{2p} &> \|f^- \chi_{Ea^{-m}}\|_p^p = \left\| \left(\frac{1}{\alpha} S_{a,w}^m \alpha T_{a,w}^m \right) f^- \chi_{Ea^{-m}} \right\|_p^p \\
&= \int_{Ea^{-m}} \left| \frac{1}{\alpha} S_{a,w}^m (\alpha T_{a,w}^m f^-)(x) \right|^p d\lambda(x) \\
&= \int_{Ea^{-m}} \frac{1}{|\alpha w(xa) w(xa^2) \cdots w(xa^m)|^p} |\alpha T_{a,w}^m f^-(xa^m)|^p d\lambda(x) \\
&= \int_E \frac{1}{|\alpha w(x) w(xa^{-1}) \cdots w(xa^{-(m-1)})|^p} |(\alpha T_{a,w}^m f)^-(x)|^p d\lambda(x) \\
&> (1-\varepsilon)^p \int_E \widetilde{\varphi}_m^p(x) d\lambda(x)
\end{aligned}$$

which implies $\|\widetilde{\varphi}_m|_E\|_\infty \to 0$ as $m \to \infty$. ■

The two weight conditions below are equivalent, which is identical with [12, Proposition 2.8] if $G = \mathbb{Z}$.

Proposition 2.2. *Let G be a locally compact group and let a be an aperiodic element in G. Let K be a compact subset of G, and let $(E_k) \subset K$ be a sequence of Borel sets satisfying $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$. The following conditions are equivalent.*

(i) *There exists a sequence $(\alpha_n) \subset \mathbb{C} \setminus \{0\}$ such that both sequences*

$$\varphi_n := |\alpha_n| \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \quad \text{and} \quad \widetilde{\varphi}_n := \left(|\alpha_n| \prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit respectively subsequences (φ_{n_k}) and $(\widetilde{\varphi}_{n_k})$ satisfying

$$\lim_{k\to\infty} \|\varphi_{n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{n_k}|_{E_k}\|_\infty = 0.$$

(ii) *Both sequences*

$$w_n := \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \quad \text{and} \quad \widetilde{w}_n := \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit respectively subsequences (w_{n_k}) and $(\widetilde{w}_{n_k})$ satisfying

$$\lim_{k\to\infty} \|w_{n_k} \cdot \widetilde{w}_{n_k}|_{E_k}\|_\infty = 0.$$

Proof. It is clear that (i) implies (ii). Now assume (ii) holds. For $\varepsilon > 0$, there exists some $m \in \mathbb{N}$ such that $w_m(x)\widetilde{w}_m(y) < \varepsilon^2$ for all $x, y \in E$. Let $|\alpha_m| = \frac{1}{\varepsilon} \sup_{y\in E} \widetilde{w}_m$. Then $\frac{1}{|\alpha_m|}\widetilde{w}_m(y) \le \varepsilon$ for all $y \in E$. On the other hand, $|\alpha_m| w_m(x) < \varepsilon$ for all $x \in E$. ■

If G is discrete, then $E = K$ in the proof of Theorem 2.1. Hence we have the following characterization of supercyclic translation operators on discrete groups.

Corollary 2.3. *Let G be a discrete group and let a be a torsion free element in G. Let $1 \leq p < \infty$ and $T_{a,w}$ be a weighted translation on $\ell^p(G)$. Then the following conditions are equivalent.*

(i) *$T_{a,w}$ is supercyclic.*

(ii) *For each finite subset $K \subset G$, there exists a sequence $(\alpha_n) \subset \mathbb{C} \setminus \{0\}$ such that both sequences*

$$\varphi_n := |\alpha_n| \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \quad \textit{and} \quad \widetilde{\varphi}_n := \left(|\alpha_n| \prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit respectively subsequences (φ_{n_k}) and $(\widetilde{\varphi}_{n_k})$ satisfying

$$\lim_{k\to\infty} \|\varphi_{n_k}|_K\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{n_k}|_K\|_\infty = 0.$$

(iii) *For each finite subset $K \subset G$, both sequences*

$$w_n := \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \quad \textit{and} \quad \widetilde{w}_n := \left(\prod_{s=0}^{n-1} w * \delta_a^s \right)^{-1}$$

admit respectively subsequences (w_{n_k}) and $(\widetilde{w}_{n_k})$ satisfying

$$\lim_{k\to\infty} \|w_{n_k} \cdot \widetilde{w}_{n_k}|_K\|_\infty = 0.$$

Example 2.4. Let $(w_j)_{j\in\mathbb{Z}}$ be a bounded sequence of positive numbers. The operator $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is a bilateral weighted backward shift with weight sequence (w_j) if $Te_j = w_j e_{j-1}$ for every $j \in \mathbb{Z}$. Here $(e_j)_{j\in\mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. The bilateral weighted backward shift T with positive weight sequence (w_j) studied in [14] is, in our notation, the weight translation $T_{-1,w*\delta_{-1}}$ on $\ell^2(\mathbb{Z})$ with weight $w(j) = w_j$. Then by Corollary 2.3, the operator $T = T_{-1,w*\delta_{-1}}$ is supercyclic if, and only if, given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrarily large n such that for all $|h|, |j| \leq q$, we have

$$\frac{\prod_{s=0}^{n-1} w(j-s)}{\prod_{s=1}^{n} w(h+s)} = w_n(j) \cdot \widetilde{w}_n(h) < \varepsilon$$

which is the condition in [14, Theorem 3.1].

Given some $N \in \mathbb{N}$, let (T_{a_l,w_l}) be a sequence of weighted translation operators on $L^p(G)$, defined by sequences of aperiodic elements (a_l) in G and positive weights (w_l) for $1 \leq l \leq N$. For simplification, we write T_l for T_{a_l,w_l}. Then we have the following result.

Corollary 2.5. *Let T_l be a weighted translation defined above for $1 \leq l \leq N$. Then the following conditions are equivalent.*

(i) $T_1 \oplus T_2 \oplus \cdots \oplus T_N$ *is supercyclic.*

(ii) *For* $1 \leq l \leq N$ *and each compact subset* $K \subset G$ *with* $\lambda(K) > 0$*, there is a sequence of Borel sets* (E_k) *in* K *and there exist sequences* $(\alpha_{l,n}) \subset \mathbb{C} \setminus \{0\}$ *such that* $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ *and both sequences*

$$\varphi_{l,n} := \frac{1}{\alpha_{l,n}} \prod_{s=1}^{n} w_l * \delta^s_{a_l^{-1}} \quad \text{and} \quad \widetilde{\varphi}_{l,n} := \left(\frac{1}{\alpha_{l,n}} \prod_{s=0}^{n-1} w_l * \delta^s_{a_l} \right)^{-1}$$

admit respectively subsequences (φ_{l,n_k}) *and* $(\widetilde{\varphi}_{l,n_k})$ *satisfying*

$$\lim_{k\to\infty} \|\varphi_{l,n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{l,n_k}|_{E_k}\|_\infty = 0.$$

(iii) *For* $1 \leq l \leq N$ *and each compact subset* $K \subset G$ *with* $\lambda(K) > 0$*, there is a sequence of Borel sets* (E_k) *in* K *such that* $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ *and both sequences*

$$w_{l,n} := \prod_{s=1}^{n} w_l * \delta^s_{a_l^{-1}} \quad \text{and} \quad \widetilde{w}_{l,n} := \left(\prod_{s=0}^{n-1} w_l * \delta^s_{a_l} \right)^{-1}$$

admit respectively subsequences (w_{l,n_k}) *and* $(\widetilde{w}_{l,n_k})$ *satisfying*

$$\lim_{k\to\infty} \|w_{l,n_k} \cdot \widetilde{w}_{l,n_k}|_{E_k}\|_\infty = 0.$$

Proof. Since condition (ii) and (iii) are equivalent, we first show condition (ii) implies (i). We show that $(\frac{1}{\alpha_{l,n}} T_l^n)_{n\in\mathbb{N}}$ is topologically transitive for all l. Let U_l and V_l be non-empty open subsets of $L^p(G)$. Since the space $C_c(G)$ of continuous functions on G with compact support is dense in $L^p(G)$, we can pick $f_l, g_l \in C_c(G)$ with $f_l \in U_l$ and $g_l \in V_l$. Let K be the union of the compact supports of all f_l and g_l. Let $E_k \subset K$ and the subsequences (φ_{l,n_k}) and $(\widetilde{\varphi}_{l,n_k})$ satisfy condition (ii). Then, by the similar argument in the proof of Theorem 2.1, we have $(\frac{1}{\alpha_{l,n_k}} T_l^{n_k})(U_l) \cap V_l \neq \emptyset$ for each l.

(i) $\Rightarrow$ (ii). Let $T_1 \oplus T_2 \oplus \cdots \oplus T_N$ be supercyclic. Let $K \subset G$ be a compact set with $\lambda(K) > 0$. By aperiodicity of a, there is some M such that $K \cap Ka^{\pm n} = \emptyset$ for $n > M$. Let $\chi_K \in L^p(G)$ be the characteristic function of K. Let $\varepsilon \in (0,1)$. Then, for each l, there exist a vector $f_l \in L^p(G)$, some α_l and some $m > M$ such that

$$\|f_l - \chi_K\|_p < \varepsilon^2 \quad \text{and} \quad \|\alpha_l T_l^m f_l + \chi_K\|_p < \varepsilon^2.$$

Repeating the argument in the proof of Theorem 2.1, we can obtain the weight condition for each l. ∎

Example 2.6. Let $G = \mathbb{Z}$ and $a_l = -1$ for each l in Corollary 2.5. We consider a sequence of weighted translations (T_l) on $\ell^2(\mathbb{Z})$, defined by $T_l = T_{-1,w_l*\delta_{-1}}$ where

(w_l) is a sequence of positive weights. Then T_l is a bilateral weighted shift on $\ell^2(\mathbb{Z})$, that is, $T_l e_j = w_{l,j} e_{j-1}$ with $w_{l,j} = w_l(j)$ for each l. Given some $N \in \mathbb{N}$, by Corollary 2.5, $T_1 \oplus T_2 \oplus \cdots \oplus T_N$ is supercyclic if, and only if, given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrarily large n such that for all l and $|h|, |j| \le q$, we have

$$\frac{\prod_{s=0}^{n-1} w_l(j-s)}{\prod_{s=1}^{n} w_l(h+s)} = w_{l,n}(j) \cdot \widetilde{w}_{l,n}(h) < \varepsilon$$

which is the same with [14, Corollary 3.3].

Remark 2.7. A weighted translation operator $T_{a,w}$ is supercyclic if, and only if, $T_{a,w} \oplus T_{a,w}$ is supercyclic by Theorem 2.1 and Corollary 2.5.

León-Saavedra introduced in [10] the notion of Cesàro hypercyclicity. Let T be an operator on the Banach space X and define the operator $M_n(T) = \frac{1}{n}\sum_{s=0}^{n-1} T^s$ for $n \in \mathbb{N}$. Then T is called *Cesàro hypercyclic* if there is a vector $x \in X$ such that $\{M_n(T)x : n \ge 1\}$ is dense in X. León-Saavedra proved that T is Cesàro hypercyclic if, and only if, the family $\{\frac{1}{n}T^n : n \ge 1\}$ is hypercyclic. Therefore, by Theorem 2.1, we have the characterization for Cesàro hypercyclic weighted translations by letting $\alpha_n = \frac{1}{n}$ in the proof of Theorem 2.1.

Corollary 2.8. *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \le p < \infty$ and $T_{a,w}$ be a weighted translation on $L^p(G)$. The following conditions are equivalent.*

(i) *$T_{a,w}$ is Cesàro hypercyclic.*

(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K such that $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ and both sequences*

$$\varphi_n := \frac{1}{n}\prod_{s=1}^{n} w * \delta_{a^{-1}}^{s} \quad \text{and} \quad \widetilde{\varphi}_n := \left(\frac{1}{n}\prod_{s=0}^{n-1} w * \delta_a^s\right)^{-1}$$

admit respectively subsequences (φ_{n_k}) and $(\widetilde{\varphi}_{n_k})$ satisfying

$$\lim_{k\to\infty} \|\varphi_{n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{n_k}|_{E_k}\|_\infty = 0.$$

Since Cesàro hypercyclicity is a special case of supercyclicity, we have the following result by letting $\alpha_{l,n} = \frac{1}{n}$ for each l in Corollary 2.5.

Corollary 2.9. *Let T_l be a weighted translation defined as in Corollary 2.5 for $1 \le l \le N$. Then the following conditions are equivalent.*

(i) *$T_1 \oplus T_2 \oplus \cdots \oplus T_N$ is Cesàro hypercyclic.*

(ii) *For $1 \leq l \leq N$ and each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K such that $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ and both sequences*

$$\varphi_{l,n} := \frac{1}{n}\prod_{s=1}^{n} w_l * \delta^s_{a_l^{-1}} \quad \text{and} \quad \widetilde{\varphi}_{l,n} := \left(\frac{1}{n}\prod_{s=0}^{n-1} w_l * \delta^s_{a_l}\right)^{-1}$$

admit respectively subsequences (φ_{l,n_k}) and $(\widetilde{\varphi}_{l,n_k})$ satisfying

$$\lim_{k\to\infty} \|\varphi_{l,n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{l,n_k}|_{E_k}\|_\infty = 0.$$

Example 2.10. Let T be the bilateral weighted backward shift on $\ell^2(\mathbb{Z})$ as defined in Example 2.4. Then by Corollary 2.8, $T = T_{-1,w*\delta_{-1}}$ is Cesàro hypercyclic if, and only if, given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrarily large n such that for all $|j| \leq q$, we have

$$\frac{\prod_{s=0}^{n-1} w(j-s)}{n} < \varepsilon \quad \text{and} \quad \frac{\prod_{s=1}^{n} w(j+s)}{n} > \varepsilon.$$

This weight condition above is the same with the condition in [10, Proposition 3.4]. Moreover, if we consider the operator $T_l = T_{-1,w_l*\delta_{-1}}$ for each l as defined in Example 2.6, then by Corollary 2.9, $T_1 \oplus T_2 \oplus \cdots \oplus T_N$ is Cesàro hypercyclic if, and only if, given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrarily large n such that for all l and $|j| \leq q$, we have

$$\frac{\prod_{s=0}^{n-1} w_l(j-s)}{n} < \varepsilon \quad \text{and} \quad \frac{\prod_{s=1}^{n} w_l(j+s)}{n} > \varepsilon.$$

Remark 2.11. By Corollary 2.8 and 2.9, a weighted translation operator $T_{a,w}$ is Cesàro hypercyclic if, and only if, $T_{a,w} \oplus T_{a,w}$ is Cesàro hypercyclic.

In [11], León-Saavedra and Müller constructed a hypercyclic bilateral weighted shift T on $\ell^2(\mathbb{Z})$ such that βT is not hypercyclic for all $|\beta| \neq 1$. Recently, the following question was posed in [1] and a negative answer was given.

Question: *Let $T : X \to X$ be a bounded operator on a Banach space X. Suppose there are numbers $0 < t_1 < t_2$ such that $t_1 T$ and $t_2 T$ are hypercyclic. Is it true that tT is hypercyclic for every $t \in [t_1, t_2]$?*

In the following, we study complex multiples of weighted translations, and give an affirmative answer for the notion of topological mixing in the above **Question**. We note that, for $\beta T_{a,w} = T_{a,\beta w}$ to be hypercyclic, we must have $|\beta| > \frac{1}{\|w\|_\infty}$, for otherwise, we have $\|\beta T_{a,w}\| \leq |\beta|\|w\|_\infty \leq 1$. Letting $\alpha_n = \beta^n$ for each $n \in \mathbb{N}$ in the proof of Theorem 2.1, one has the following result immediately.

Corollary 2.12. *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \leq p < \infty$ and $T_{a,w}$ be a weighted translation on $L^p(G)$. For $\beta \in \mathbb{C} \setminus \{0\}$, the following conditions are equivalent.*

(i) *$\beta T_{a,w}$ is hypercyclic.*

(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K such that $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ and both sequences*

$$\varphi_{\beta,n} := |\beta^n| \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \quad \text{and} \quad \widetilde{\varphi}_{\beta,n} := \left(|\beta^n| \prod_{s=0}^{n-1} w * \delta_a^s\right)^{-1}$$

admit respectively subsequences (φ_{β,n_k}) and $(\widetilde{\varphi}_{\beta,n_k})$ satisfying

$$\lim_{k\to\infty} \|\varphi_{\beta,n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{\beta,n_k}|_{E_k}\|_\infty = 0.$$

Using similar arguments as in the proof of [4, Corollary 2.6], we can also characterize scalar multiples of weighted translations to be topologically mixing.

Corollary 2.13. *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \leq p < \infty$ and $T_{a,w}$ be a weighted translation on $L^p(G)$. For $\beta \in \mathbb{C} \setminus \{0\}$, the following conditions are equivalent.*

(i) *$\beta T_{a,w}$ is topologically mixing.*

(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_n) in K such that $\lambda(K) = \lim_{n\to\infty} \lambda(E_n)$ and both sequences*

$$\varphi_{\beta,n} := |\beta^n| \prod_{s=1}^{n} w * \delta_{a^{-1}}^s \quad \text{and} \quad \widetilde{\varphi}_{\beta,n} := \left(|\beta^n| \prod_{s=0}^{n-1} w * \delta_a^s\right)^{-1}$$

satisfy

$$\lim_{n\to\infty} \|\varphi_{\beta,n}|_{E_n}\|_\infty = \lim_{n\to\infty} \|\widetilde{\varphi}_{\beta,n}|_{E_n}\|_\infty = 0.$$

Remark 2.14. If $\beta = 1$ in Corollary 2.12 and 2.13, then these results are identical with [4, Theorem 2.3, Corollary 2.6]. Moreover, by Corollary 2.12 and 2.13, one can see that hypercyclicity and topological mixing are preserved by rotations for the weighted translations in this paper. We note that it is even true in [11] that the rotation of a hypercyclic operator is hypercyclic.

Proposition 2.15. *Let G be a locally compact group and let a be an aperiodic element in G. Let $1 \leq p < \infty$ and $T_{a,w}$ be a weighted translation on $L^p(G)$. If $\beta_1 T_{a,w}$ and $\beta_2 T_{a,w}$ are topologically mixing for some β_1 and $\beta_2 \in \mathbb{C} \setminus \{0\}$ with $|\beta_1| < |\beta_2|$, then $\beta T_{a,w}$ is topologically mixing for every β satisfying $|\beta_1| < |\beta| < |\beta_2|$.*

Proof. For a compact set $K \subset G$, by Corollary 2.13, there are sequences of Borel sets (E_n) and (F_n) in K such that $\lambda(K) = \lim_{n\to\infty} \lambda(E_n) = \lim_{n\to\infty} \lambda(F_n)$,

$$\lim_{n\to\infty} \|\varphi_{\beta_1,n}|_{E_n}\|_\infty = \lim_{n\to\infty} \|\widetilde{\varphi}_{\beta_1,n}|_{E_n}\|_\infty = 0$$

and

$$\lim_{n\to\infty} \|\varphi_{\beta_2,n}|_{F_n}\|_\infty = \lim_{n\to\infty} \|\widetilde{\varphi}_{\beta_2,n}|_{F_n}\|_\infty = 0.$$

Without loss of generality, we may assume $E_n \cap F_n \neq \emptyset$ for each $n \in \mathbb{N}$. We note that $\lambda(K) = \lim_{n\to\infty} \lambda(E_n \cap F_n)$ and

$$\varphi_{\beta_1,n}(x) < \varphi_{\beta,n}(x) < \varphi_{\beta_2,n}(x),$$

$$\widetilde{\varphi}_{\beta_1,n}(x) > \widetilde{\varphi}_{\beta,n}(x) > \widetilde{\varphi}_{\beta_2,n}(x)$$

for $x \in E_n \cap F_n$. Therefore we have

$$\lim_{n\to\infty} \|\varphi_{\beta,n}|_{E_n\cap F_n}\|_\infty = \lim_{n\to\infty} \|\widetilde{\varphi}_{\beta,n}|_{E_n\cap F_n}\|_\infty = 0$$

which implies $\beta T_{a,w}$ is topologically mixing. ■

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