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THE SUBGRADIENT METHOD FOR SOLVING VARIATIONAL INEQUALITIES WITH COMPUTATIONAL ERRORS IN A HILBERT SPACE

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Abstract. In a Hilbert space, we study the asymptotic behavior of the subgradient method for solving a variational inequality, under the presence of computational errors. Most results known in the literature establish convergence of optimization algorithms, when computational errors are summable. In the present paper, the convergence of the subgradient method to the solution of a variational inequalities is established for nonsummable computational errors. We show that the the subgradient method generates good approximate solutions, if the sequence of computational errors is bounded from above by a constant.

1. INTRODUCTION

The study of gradient-type methods and variational inequalities are important topics is optimization theory. See, for example, [1, 3-15, 17, 18] and the references mentioned therein.

In the present paper we study the asymptotic behavior of the gradient method for solving a variational inequality in a Hilbert space, under the presence of computational errors. Most results known in the literature establish convergence of optimization algorithms, when computational errors are summable. In the present paper, the convergence of the gradient method for solving variational inequalities is established for nonsummable computational errors. We show that the gradient method generates good approximate solution, if the sequence of computational errors is bounded from above by a constant. Note that results of this type were obtained in [17, 18] for convex constrained minimization problems.

Our goal is to obtain an ϵ-approximate solution of the problem in the presence of computational errors, where ϵ is a given positive number. Clearly, in practice it is sufficient to find an ϵ-approximate solution instead of constructing a minimizing sequence. On the other hand in practice computations introduce numerical errors and

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if one uses methods in order to solve minimization problems these methods usually provide only approximate solutions of the problems.

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ which induces a complete norm $\| \cdot \|$. For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \ \|x - y\| \leq r\}.$$

Let C be a nonempty closed (in the normed topology) convex subset of X.

It is well-known that the following proposition holds.

Proposition 1.1. *For each $x \in X$ there is a unique point $P_C(x) \in C$ satisfying*

$$\|x - P_C(x)\| = \inf\{\|x - y\| : \ y \in C\}.$$

Moreover,

$$\|P_C(x) - P_C(y)\| \leq \|x - y\| \text{ for all } x, y \in X.$$

Consider a single-valued mapping $f : X \to X$. We say that f is monotone on C if

$$\langle f(x) - f(y), x - y \rangle \geq 0 \text{ for all } x, y \in C.$$

We say that f is pseudo-monotone on C if for each $x, y \in C$ the inequality

$$\langle f(y), x - y \rangle \geq 0 \text{ implies that } \langle f(x), x - y \rangle \geq 0.$$

Clearly, if f is monotone on C, then f is pseudo-monotone on C.

For each $x \in X$ and each nonempty set $E \subset X$ set

$$\rho(x, E) = \inf\{\|x - y\| : \ y \in E\}.$$

For each nonempty set $D \subset X$ and each $r > 0$ set

$$D_r = \{x \in X : \ \rho(x, D) \leq r\}.$$

Fix $\bar{r} \in (0, 1)$. We suppose that

$$u_* \in C \text{ and } \langle f(u_*), u - u_* \rangle \geq 0 \text{ for all } u \in C. \tag{1.1}$$

Clearly, u_* is a solution of the corresponding variational inequality:

$$\text{To find } x \in C \text{ such that } \langle f(x), y - x \rangle \geq 0 \text{ for all } y \in C.$$

We say that f is strongly monotone with a constant $\alpha > 0$ [10] on $C_{\bar{r}}$ if

$$\langle f(x) - f(y), x - y \rangle \geq \alpha \|x - y\|^2 \text{ for all } x, y \in C_{\bar{r}}.$$

We suppose that there is $\alpha \in (0,1)$ such that

$$\langle f(x), x - u_* \rangle \geq \alpha \|x - u_*\|^2 \text{ for all } x \in C_{\bar{r}}. \tag{1.2}$$

Relation (1.2) implies that u_* is a unique solution of the corresponding variational inequality.

Remark 1. Note that by (1.1), equation (1.2) holds if f is strongly monotone with a constant α on $C_{\bar{r}}$.

Remark 2. The assumption that $\alpha \in (0,1)$ is not restrictive. If (1.2) holds with $\alpha = \alpha_0$, then (1.2) is true for any $\alpha \in (0, \alpha_0)$. More precisely, if (1.2) is true with $\alpha = \alpha_0$, where $\alpha_0 > 1$, then (1.2) and the assertion of Theorem 1.1 hold with $\alpha = 1/2$ (see (1.4), (1.6) and (1.7)).

In this paper we consider the single-valued mapping $f : X \to X$ which satisfy (1.1) and (1.2) and suppose that f is bounded on bounded subsets of $C_{\bar{r}}$.

In this paper, in order to solve the variational inequality (to find u_*), we use the algorithm known in the literature as the subgradient method. In each iteration of this algorithm, in order to get the next iterate x_{k+1}, an orthogonal projection onto C is calculated, according to the following iterative step. Given the current iterate x_k calculate $x_{k+1} = P_C(x_k - \tau_k f(x_k))$, where τ_k is some positive number. This algorithm generates sequences which converge to u_*. In this paper, we study the behavior of the sequences generated by the algorithm taking into account computational errors which are always present in practice. Namely, in practice the algorithm generates a sequence $\{x_k\}_{k=0}^{\infty}$ such that for each integer $k \geq 0$,

$$\|x_{k+1} - P_C(x_k - \tau_k f(x_k))\| \leq \delta,$$

with a constant $\delta > 0$ which depends only on our computer system. Surely, in this situation one cannot expect that the sequence $\{x_k\}_{k=0}^{\infty}$ converges to the point u_*. The goal of our paper is to understand what subset of C attracts all sequences $\{x_k\}_{k=0}^{\infty}$ generated by the algorithm. Our main result (Theorem 1.1) shows that this subset of C is an ϵ-neighborhood of u_* with some $\epsilon > 0$ depending on δ (see (1.7)).

As it was pointed out by the referee analogous asymptotic behavior of optimization methods was also considered in [16].

In this paper we prove the following result.

Theorem 1.1. *Let* $\epsilon \in (0,1)$, $M_0 > 1$, $M_1 > 1$,

$$f(B(u_*, M_0 + 1)) \subset B(0, M_1), \tag{1.3}$$

$$0 < \tau_1^* < \tau_0^* < 2^{-1}\alpha M_1^{-2}\epsilon^2/4, \tag{1.4}$$

$$\{\tau_k\}_{k=0}^{\infty} \subset [\tau_1^*, \tau_0^*], \tag{1.5}$$

a natural number k_0 satisfy

$$k_0 > (M_0+1)^2\alpha^{-1}(\tau_1^*)^{-1}16\epsilon^{-2} \tag{1.6}$$

and let a positive number δ satisfy

$$\delta < \bar{r}/2, \tag{1.7}$$
$$4\delta(M_0+2+M_1) < \tau_1^*\alpha(\epsilon/4)^2.$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset X$,

$$x_0 \in B(u_*, M_0) \tag{1.8}$$

and that for each integer $i \geq 0$,

$$\|x_{i+1} - P_C(x_i - \tau_i f(x_i))\| \leq \delta. \tag{1.9}$$

Then

$$\|x_i - u_*\| \leq \epsilon \text{ for all integers } i \geq k_0.$$

Note that under the assumptions of Theorem 1.1 its assertion also holds for exact iterates $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying (1.8) and such that for each each integer $i \geq 0$,

$$x_{i+1} = P_C(x_i - \tau_i f(x_i)).$$

The paper is organized as follows. Section 2 contains auxiliary results. Theorem 1.1 is proved in Section 3.

Since the mapping f is single-valued our result does not cover the case with subgradient operators for nonsmooth convex functionals but it does cover the case with gradient operators for smooth convex functionals. Convergence of the projected subgradient method for nonsmooth convex optimization in the presence of nonsummable computational errors was studied in [17].

2. Auxiliary Results

We use the assumptions, notation and the definitions introduced in Section 1.

In the sequel we use the following well-known result.

Proposition 2.1. *Assume that D is a nonempty convex closed subset of X and that $x \in X \setminus D$. Then for each $z \in D$,*

$$\langle z - P_D(x), x - P_D(x)\rangle \leq 0.$$

Lemma 2.1. *Let*

$$u \in C_{\bar{r}},\ \tau > 0,\ v = P_C(u - \tau f(u)). \tag{2.1}$$

Then

$$\|v - u_*\|^2 \le (1 - 2\alpha\tau)\|u - u_*\|^2 + \tau^2 \|f(u)\|^2.$$

Proof. By (1.2) and (2.1),

$$\langle f(u), u - u_* \rangle \ge \alpha \|u - u_*\|^2. \tag{2.2}$$

By (1.1), (2.1), (2.2) and Proposition 1.1,

$$\begin{aligned}\|v - u_*\|^2 &= \|P_C(u - \tau f(u)) - u_*\|^2 \le \|(u - u_*) - \tau f(u)\|^2 \\ &\le \|u - u_*\|^2 + \tau^2\|f(u)\|^2 - 2\tau\langle f(u), u - u_*\rangle \\ &\le \|u - u_*\|^2 + \tau^2\|f(u)\|^2 - 2\tau\alpha\|u - u_*\|^2 \\ &= (1 - 2\alpha\tau)\|u - u_*\|^2 + \tau^2\|f(u)\|^2.\end{aligned}$$

Lemma 2.1 is proved. ■

Lemma 2.2. *Let* $M_0 > 0,\ M_1 > 0,$

$$\tau \in (0, 1),\ \delta \in (0, \bar{r}), \tag{2.3}$$

$$f(B(u_*, M_0)) \subset B(0, M_1), \tag{2.4}$$

$$u \in C_{\bar{r}} \cap B(u_*, M_0) \tag{2.5}$$

and let

$$v \in X,\ \|v - P_C(u - \tau f(u))\| \le \delta. \tag{2.6}$$

Then

$$\|v - u_*\|^2 \le \delta^2 + 2\delta(M_0 + M_1) + (1 - \tau\alpha)\|u - u_*|^2 + \tau^2 M_1^2.$$

Proof. Put

$$y = P_C(u - \tau f(u)). \tag{2.7}$$

By (2.7), (2.5), (2.3), (2.4) and Lemma 2.1,

$$\|y - u_*\|^2 \le (1 - \alpha\tau)\|u - u_*\|^2 + \tau^2\|f(u)\|^2 \le (1 - \alpha\tau)\|u - u_*\|^2 + \tau^2 M_1^2. \tag{2.8}$$

By (2.7), (2.6), (2.8), (2.5) and (2.3),

$$\begin{aligned}\|v - u_*\|^2 &= \|v - y + y - u_*\|^2 \le \|v - y\|^2 + \|y - u_*\|^2 + 2\|v - y\|\|y - u_*\| \\ &\le \delta^2 + 2\delta(\|u - u_*\| + \tau M_1) \\ &\quad + (1 - \tau\alpha)\|u - u_*\|^2 + \tau^2 M_1^2 \\ &\le \delta^2 + 2\delta(M_0 + M_1) + (1 - \alpha\tau)\|u - u_*\|^2 + \tau^2 M_1^2.\end{aligned}$$

Lemma 2.2 is proved. ■

Lemma 2.3. *Let* $M_0 > 1$, $M_1 > 1$,

$$f(B(u_*, M_0 + 1)) \subset B(0, M_1),$$
$$0 < \tau_1^* < \tau_0^*, \ \tau_0^* \le 2^{-1}\alpha M_1^{-2}, \tag{2.9}$$

$$\{\tau_k\}_{k=0}^{\infty} \subset [\tau_1^*, \tau_0^*], \tag{2.10}$$

a positive number δ *satisfy*

$$\delta < \bar{r}/2, \ 2\delta(M_0 + 2 + M_1) < 4^{-1},$$
$$2\delta(M_0 + 2 + M_1) < 2^{-1}\tau_1^*\alpha \tag{2.11}$$

and let $\{x_k\}_{k=0}^{\infty} \subset X$ *satisfy*

$$x_0 \in B(u_*, M_0) \tag{2.12}$$

and for all integers $k \ge 0$,

$$\|x_{k+1} - P_C(x_k - \tau_k f(x_k))\| \le \delta. \tag{2.13}$$

Then

$$\|x_k - u_*\| \le M_0 + 1 \text{ for all integers } k \ge 0.$$

Proof. By (1.1), (2.9)-(2.13) and Proposition 2.1,

$$\|x_1 - u_*\| \le \|u_* - P_C(x_0 - \tau_0 f(x_0))\| + \|P_C(x_0 - \tau_0 f(x_0)) - x_1\|$$
$$\le \|u_* - P_C(x_0 - \tau_0 f(x_0))\| + \delta \le \|u_* - x_0 + \tau_0 f(x_0)\| + \delta$$
$$\le \|u_* - x_0\| + \tau_0\|f(x_0)\| + \delta \le M_0 + \tau_0 M_1 + \delta < M_0 + 1.$$

Thus

$$\|x_1 - u_*\| \le M_0 + 1. \tag{2.14}$$

By induction we show that

$$\|x_k - u_*\| \le M_0 + 1 \tag{2.15}$$

for all integers $k \ge 1$. Clearly, (2.15) holds with $k = 1$ in view of (2.14).

Assume that $k \ge 1$ is an integer and (2.15) holds. By (2.9), (2.10), (2.11), (2.13), (2.15) and Lemma 2.2 applied with $u = x_k$, $v = x_{k+1}$, $\tau = \tau_k$,

$$\|x_{k+1} - u_*\|^2 \le \delta^2 + 2\delta(M_0 + 1 + M_1) + (1 - \alpha\tau_k)\|x_k - u_*\|^2 + \tau_k^2 M_1^2. \tag{2.16}$$

There are two cases:

$$\|x_k - u_*\| \le 1; \tag{2.17}$$

$$\|x_k - u_*\| > 1. \tag{2.18}$$

Assume that (2.17) holds. Then by (2.16), (2.11), (2.17) and (2.10),

$$\|x_{k+1} - u_*\|^2 \le 2\delta(M_0 + 2 + M_1) + 1 + \tau_k^2 M_1^2 < 2$$

and

$$\|x_{k+1} - u_*\| \le 2 < M_0 + 1. \tag{2.19}$$

Assume that (2.18) holds. By (2.16), (2.18), (2.15), (2.10) and (2.11),

$$\begin{aligned}\|x_{k+1} - u_*\|^2 &\le 2\delta(M_0 + 2 + M_1) + \tau_k^2 M_1^2 + \|x_k - u_*\|^2 - \alpha\tau_k \\ &\le (M_0 + 1)^2 + 2\delta(M_0 + 2 + M_1) - \tau_k(\alpha - \tau_k M_1^2) \\ &\le (M_0 + 1)^2 + 2\delta(M_0 + 2 + M_1) - \tau_1^*(\alpha - \tau_0^* M_1^2) \\ &\le (M_0 + 1)^2 + 2\delta(M_0 + 2 + M_1) - \tau_1^*\alpha 2^{-1} \le (M_0 + 1)^2\end{aligned}$$

and

$$||x_{k+1} - u_*|| \le M_0 + 1$$

Then the equation above holds in both cases.

This completes the proof of Lemma 2.3. ∎

As it was pointed out by the referee, Lemma 2.3 can also be deduced from inequality (2.16) and Lemma 7.1.1 of [2].

3. Proof of Theorem 1.1

By Lemma 2.3

$$\|x_i - u_*\| \le M_0 + 1 \text{ for all integers } i \ge 0. \tag{3.1}$$

Assume that an integer $i \ge 1$ and that

$$\|x_i - u_*\| > \epsilon/2. \tag{3.2}$$

By Lemma 2.2 applied with $u = x_i$, $v = x_{i+1}$, $\tau = \tau_i$ and (1.7),

$$\begin{aligned}\|x_{i+1} - u_*\|^2 &\le \delta^2 + 2\delta(M_0 + M_1 + 1) + (1 - \alpha\tau_i)\|x_i - u_*\|^2 + \tau_i^2 M_1^2 \\ &\le 2\delta(M_0 + M_1 + 2) + \|x_i - u_*\|^2 - \tau_i\alpha\|x_i - u_*\|^2 + \tau_i^2 M_1^2.\end{aligned}$$

Combined with (3.2), (1.5), (1.4) and (1.7),

$$\begin{aligned}\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 &\geq \tau_i \alpha(\epsilon/2)^2 - \tau_i^2 M_1^2 - 2\delta(M_0 + M_1 + 2)\\ &\geq \tau_i(\alpha(\epsilon/2)^2 - \tau_i M_1^2) - 2\delta(M_0 + M_1 + 2)\\ &\geq \tau_1^* \alpha(\epsilon/2)^2 2^{-1} - 2\delta(M_0 + M_1 + 2)\\ &\geq \alpha\tau_1^*(\epsilon/4)^2.\end{aligned}$$

Thus we have shown that the following property holds:

(P1) if an integer $i \geq 1$ satisfies $\|x_i - u_*\| > \epsilon/2$, then

$$\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2 \geq \alpha\tau_1^*(\epsilon/4)^2. \tag{3.3}$$

Let $p \geq 1$ be an integer. We show that there exists an integer $i \in \{p, \ldots, p + k_0 - 1\}$ such that

$$\|x_i - u_*\| \leq \epsilon/2.$$

Assume the contrary. Then for each integer $i \in \{p, \ldots, p + k_0 - 1\}$,

$$\|x_i - u_*\| > \epsilon/2$$

and in view of (P1), equation (3.3) holds. By (3.1) and (3.3) which holds for all integers $i \in \{p, \ldots, p + k_0 - 1\}$,

$$\begin{aligned}(M_0 + 1)^2 &\geq \|x_p - u_*\|^2 - \|x_{p+k_0} - u_*\|^2\\ &= \sum_{i=p}^{p+k_0-1} [\|x_i - u_*\|^2 - \|x_{i+1} - u_*\|^2]\\ &\geq k_0\alpha\tau_1^*(\epsilon/4)^2\end{aligned}$$

and

$$k_0 \leq (M_0 + 1)^2 \alpha^{-1} (\tau_1^*)^{-1} 16\epsilon^{-2}.$$

This contradicts (1.6).

The contradiction we have reached proves that there is an integer $j \in \{p, \ldots, p + k_0 - 1\}$ such that

$$\|x_j - u_*\| \leq \epsilon/2.$$

Thus we have shown that the following property holds:

For each integer $p \geq 1$ there is an integer $j \in \{p, \ldots, p + k_0 - 1\}$ such that $||x_j - u_*|| \leq \epsilon/2$.

By the property above there is a natural number $j \leq k_0$ such that

$$\|x_j - u_*\| \leq \epsilon/2. \tag{3.4}$$

We show by induction that for each integer $i \geq j$,

$$\|x_i - u_*\| \leq \epsilon. \tag{3.5}$$

Assume that an integer $i \geq j$ and that (3.5) holds. There are two cases:

$$\|x_i - u_*\| \leq \epsilon/2; \tag{3.6}$$

$$\|x_i - u_*\| > \epsilon/2. \tag{3.7}$$

Assume that (3.6) holds. By (1.9), (1.1), Proposition 1.1, (3.6), (1.4), (3.1), (1.3), (1.7) and (1.4),

$$\begin{aligned} \|x_{i+1} - u_*\| &\leq \|x_{i+1} - P_C(x_i - \tau_i f(x_i))\| + \|P_C(x_i - \tau_i f(x_i)) - u_*\| \\ &\leq \delta + \|x_i - \tau_i f(x_i) - u_*\| \\ &\leq \delta + \|x_i - u_*\| + \tau_i \|f(x_i)\| \leq \delta + \epsilon/2 + \tau_0^* M_1 \leq \epsilon. \end{aligned} \tag{3.8}$$

Assume that (3.7) holds. By (3.7), property (P1) and (3.5),

$$\|x_{i+1} - u_*\| \leq \|x_i - u_*\| \leq \epsilon.$$

Thus in both cases

$$\|x_{i+1} - u_*\| \leq \epsilon.$$

Thus we have shown by induction that

$$\|x_i - u_*\| \leq \epsilon$$

for all integers $i \geq 0$. Theorem 1.1 is proved.

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