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GROWTH PROPERTIES FOR THE SOLUTIONS OF THE STATIONARY SCHRÖDINGER EQUATION IN A CONE

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Abstract. Our aim in this paper is to deal with the growth properties at infinity for the solutions of the stationary Schrödinger equation in an n-dimensional cone. Meanwhile, the geometrical properties of the exceptional sets are also discussed.

1. Introduction and Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n (n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance between two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary and the closure of a set $\mathbf{S}$ in $\mathbf{R}^n$ are denoted by $\partial\mathbf{S}$ and $\overline{\mathbf{S}}$, respectively.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}^{n-1}_+ = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by $\mathbf{T}_n$.

For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at P and radius r in $\mathbf{R}^n$. $S_r = \partial B(O, r)$. By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$. We call it a cone. Then T_n is a special cone obtained by putting $\Omega = \mathbf{S}^{n-1}_+$. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by

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$C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$ which is $\partial C_n(\Omega) - \{O\}$.

We shall say that a set $E \subset C_n(\Omega)$ has a covering $\{r_j, R_j\}$ if there exists a sequence of balls $\{B_j\}$ with centers in $C_n(\Omega)$ such that $E \subset \cup_{j=1}^{\infty} B_j$, where r_j is the radius of B_j and R_j is the distance between the origin and the center of B_j. Furthermore, we denote by dS_r the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_r and by dw the elements of the Euclidean volume in $\mathbf{R}^n$.

Let $\mathcal{A}_a$ denote the class of nonnegative radial potentials $a(P)$, i.e. $0 \le a(P) = a(r)$, $P = (r, \Theta) \in C_n(\Omega)$, such that $a \in L^b_{loc}(C_n(\Omega))$ with some $b > n/2$ if $n \ge 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

This article is devoted to the stationary Schrödinger equation

$$Sch_a u(P) = -\Delta u(P) + a(P)u(P) = 0 \quad \text{for} \quad P \in C_n(\Omega),$$

where Δ is the Laplace operator and $a \in \mathcal{A}_a$. These solutions called a-harmonic functions or generalized harmonic functions associated with the operator Sch_a. Note that they are classical harmonic functions in the classical case $a = 0$. Under these assumptions the operator Sch_a can be extended in the usual way from the space $C_0^{\infty}(C_n(\Omega))$ to an essentially self-adjoint operator on $L^2(C_n(\Omega))$ (see [13]). We will denote it Sch_a as well. This last one has a Green's function $G(\Omega, a)(P, Q)$. Here $G(\Omega, a)(P, Q)$ is positive on $C_n(\Omega)$ and its inner normal derivative $\partial G(\Omega, a)(P, Q)/\partial n_Q \ge 0$. We denote this derivative by $\mathbb{P}(\Omega, a)(P, Q)$, which is called the Poisson a-kernel with respect to $C_n(\Omega)$. We remark that $G(\Omega, 0)(P, Q)$ and $\mathbb{P}(\Omega, 0)(P, Q)$ are the Green's function and Poisson kernel of the Laplacian in $C_n(\Omega)$ respectively.

Let Δ^* be a Laplace-Beltrami operator (spherical part of the Laplace) on $\Omega \subset \mathbf{S}^{n-1}$ and λ_j $(j = 1, 2, 3 \ldots, 0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots)$ be the eigenvalues of the eigenvalue problem for Δ^* on Ω (see, e.g., [14, p. 41])

$$\begin{aligned} \Delta^*\varphi(\Theta) + \lambda\varphi(\Theta) &= 0 \text{ in } \Omega, \\ \varphi(\Theta) &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Corresponding eigenfunctions are denoted by φ_{jv} $(1 \le v \le v_j)$, where v_j is the multiplicity of λ_j. We set $\lambda_0 = 0$, norm the eigenfunctions in $L^2(\Omega)$ and $\varphi_1 = \varphi_{11} > 0$.

In order to ensure the existences of λ_j $(j = 1, 2, 3 \ldots)$. We put a rather strong assumption on Ω: if $n \ge 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [5, p. 88-89] for the definition of $C^{2,\alpha}$-domain). Then $\varphi_{jv} \in C^2(\overline{\Omega})$ $(j = 1, 2, 3, \ldots, 1 \le v \le v_j)$ and $\partial\varphi_1/\partial n > 0$ on $\partial\Omega$ (here and below, $\partial/\partial n$ denotes differentiation along the interior normal).

Hence well-known estimates (see, e.g., [12, p. 14]) imply the following inequality:

$$\sum_{v=1}^{v_j} \varphi_{jv}(\Theta) \frac{\partial \varphi_{jv}(\Phi)}{\partial n_\Phi} \le M(n) j^{2n-1}, \tag{1.1}$$

where the symbol $M(n)$ denotes a constant depending only on n.

Let $V_j(r)$ and $W_j(r)$ stand, respectively, for the increasing and non-increasing, as $r \to +\infty$, solutions of the equation

$$-Q''(r) - \frac{n-1}{r}Q'(r) + \left(\frac{\lambda_j}{r^2} + a(r)\right) Q(r) = 0, \quad 0 < r < \infty, \tag{1.2}$$

normalized under the condition $V_j(1) = W_j(1) = 1$.

We shall also consider the class $\mathcal{B}_a$, consisting of the potentials $a \in \mathcal{A}_a$ such that there exists a finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0,\infty)$, moreover, $r^{-1}|r^2 a(r) - k| \in L(1,\infty)$. If $a \in \mathcal{B}_a$, then the g.h.f.s are continuous (see [16]).

In the rest of paper, we assume that $a \in \mathcal{B}_a$ and we shall suppress this assumption for simplicity. Further, we use the standard notations $u^+ = \max(u,0)$, $u^- = -\min(u,0)$, $[d]$ is the integer part of d and $d = [d] + \{d\}$, where d is a positive real number.

Denote

$$\iota^{\pm}_{j,k} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k+\lambda_j)}}{2} \ (j = 0,1,2,3\ldots).$$

It is known (see [6]) that in the case under consideration the solutions to the equation (1.2) have the asymptotics

$$V_j(r) \sim d_1 r^{\iota^+_{j,k}},\ W_j(r) \sim d_2 r^{\iota^-_{j,k}},\ \text{as } r \to \infty, \tag{1.3}$$

where d_1 and d_2 are some positive constants.

Remark 1. $\iota^+_{j,0} = j$ $(j = 0,1,2,3,\ldots)$ in the case $\Omega = \mathbf{S}^{n-1}_+$.

If $a \in \mathcal{A}_a$, it is known that the following expansion for the Green function $G(\Omega,a)(P,Q)$ (see [3, Ch. 11], [7])

$$G(\Omega,a)(P,Q) = \sum_{j=0}^{\infty} \frac{1}{\chi'(1)} V_j(\min(r,t)) W_j(\max(r,t)) \left(\sum_{v=1}^{v_j} \varphi_{jv}(\Theta)\varphi_{jv}(\Phi)\right), \tag{1.4}$$

where $P = (r,\Theta)$, $Q = (t,\Phi)$, $r \neq t$ and $\chi'(s) = w\,(W_1(r), V_1(r))\,|_{r=s}$ is their Wronskian. The series converges uniformly if either $r \le st$ or $t \le sr$ $(0 < s < 1)$. In the case $a = 0$, this expansion coincides with the well-known result by J. Lelong-Ferrand (see [8]). The expansion (1.4) can also be rewritten in terms of the Gegenbauer polynomials.

For a nonnegative integer m and two points $P = (r,\Theta), Q = (t,\Phi) \in C_n(\Omega)$, we put

$$K(\Omega,a,m)(P,Q) = \begin{cases} 0 & \text{if } \ 0 < t < 1, \\ \widetilde{K}(\Omega,a,m)(P,Q) & \text{if } \ 1 \le t < \infty, \end{cases}$$

where

$$\widetilde{K}(\Omega, a, m)(P, Q) = \sum_{j=0}^{m} \frac{1}{\chi'(1)} V_j(r) W_j(t) \left(\sum_{v=1}^{v_j} \varphi_{jv}(\Theta) \varphi_{jv}(\Phi) \right).$$

We use the following modified kernel function defined by

$$G(\Omega, a, m)(P, Q) = G(\Omega, a)(P, Q) - K(\Omega, a, m)(P, Q)$$

for two points $P = (r, \Theta), Q = (t, \Phi) \in C_n(\Omega)$.

Put

$$U(\Omega, a, m; u)(P) = \int_{S_n(\Omega)} \mathbb{P}(\Omega, a, m)(P, Q) u(Q) d\sigma_Q,$$

where

$$\mathbb{P}(\Omega, a, m)(P, Q) = \frac{\partial G(\Omega, a, m)(P, Q)}{\partial n_Q},\ \mathbb{P}(\Omega, a, 0)(P, Q) = \mathbb{P}(\Omega, a)(P, Q),$$

$u(Q)$ is a continuous function on $\partial C_n(\Omega)$ and $d\sigma_Q$ is the surface area element on $S_n(\Omega)$.

Remark 2. The kernel function $P(\mathbf{S}_+^{n-1}, 0, m)(P, Q)$ coincides with ones in Finkelstein-Scheinberg [4] and Siegel-Talvila [15].

If γ is a real number and $\gamma \geq 0$ $(resp.\ \gamma < 0)$, we assume in addition that $1 \leq p < \infty$,

$$\iota_{[\gamma],k}^+ + \{\gamma\} > (-\iota_{1,k}^+ - n + 2)p + n - 1,$$

$$(resp.\ -\iota_{[-\gamma],k}^+ - \{-\gamma\} > (-\iota_{1,k}^+ - n + 2)p + n - 1,)$$

in case $p > 1$

$$\frac{\iota_{[\gamma],k}^+ + \{\gamma\} - n + 1}{p} < \iota_{m+1,k}^+ < \frac{\iota_{[\gamma],k}^+ + \{\gamma\} - n + 1}{p} + 1;$$

$$\left(resp. \quad \frac{-\iota_{[-\gamma],k}^+ - \{-\gamma\} - n + 1}{p} < \iota_{m+1,k}^+ < \frac{-\iota_{[-\gamma],k}^+ - \{-\gamma\} - n + 1}{p} + 1; \right)$$

and in case $p = 1$

$$\iota_{[\gamma],k}^+ + \{\gamma\} - n + 1 \leq \iota_{m+1,k}^+ < \iota_{[\gamma],k}^+ + \{\gamma\} - n + 2.$$

$$\left(resp. \quad -\iota_{[-\gamma],k}^+ - \{-\gamma\} - n + 1 \leq \iota_{m+1,k}^+ < -\iota_{[-\gamma],k}^+ - \{-\gamma\} - n + 2. \right)$$

If these conditions all hold, we write $\gamma \in \mathcal{C}(k,p,m,n)$ $(resp.\ \gamma \in \mathcal{D}(k,p,m,n))$.

Let $\gamma \in \mathcal{C}(k,p,m,n)$ $(resp.\ \gamma \in \mathcal{D}(k,p,m,n))$ and u be functions on $\partial C_n(\Omega)$ satisfying

$$\int_{S_n(\Omega)} \frac{|u(t,\Phi)|^p}{1+t^{\iota^+_{[\gamma],k}+\{\gamma\}}} d\sigma_Q < \infty. \tag{1.5}$$
$$\left(resp.\ \int_{S_n(\Omega)} |u(t,\Phi)|^p (1+t^{\iota^+_{[-\gamma],k}+\{-\gamma\}}) d\sigma_Q < \infty.\right)$$

For γ and u, we define the positive measure μ (resp. ν) on $\mathbf{R}^n$ by

$$d\mu(Q) = \begin{cases} |u(t,\Phi)|^p t^{-\iota^+_{[\gamma],k}-\{\gamma\}} d\sigma_Q & Q=(t,\Phi) \in S_n(\Omega;(1,+\infty)), \\ 0 & Q \in \mathbf{R}^n - S_n(\Omega;(1,+\infty)). \end{cases}$$

$$\left(resp.\ d\nu(Q) = \begin{cases} |u(t,\Phi)|^p t^{\iota^+_{[-\gamma],k}+\{-\gamma\}} d\sigma_Q & Q=(t,\Phi) \in S_n(\Omega;(1,+\infty)), \\ 0 & Q \in \mathbf{R}^n - S_n(\Omega;(1,+\infty)). \end{cases}\right)$$

We remark that the total mass of μ and ν are finite.

Let $\epsilon > 0$, $\xi \geq 0$ and μ be any positive measure on $\mathbf{R}^n$ having finite mass. For each $P=(r,\Theta) \in \mathbf{R}^n - \{O\}$, as in [10], the maximal function is defined by

$$M(P;\mu,\xi) = \sup_{0<\rho<\frac{r}{2}} \frac{\mu(B(P,\rho))}{\rho^\xi}.$$

The set $(P=(r,\Theta) \in \mathbf{R}^n - \{O\}; M(P;\mu,\xi)r^\xi > \epsilon)$ is denoted by $E(\epsilon;\mu,\xi)$.

Recently, Siegel-Talvila (cf. [15, Corollary 2.1]) proved the following result.

Theorem A. *If u is a continuous function on ∂T_n satisfying*

$$\int_{\partial T_n} \frac{|u(t,\Phi)|}{1+t^{n+m}} dQ < \infty,$$

then the function $U(\mathbf{S}_+^{n-1},0,m;u)(P)$ satisfies

$$U(\mathbf{S}_+^{n-1},0,m;u) \in C^2(T_n) \cap C^0(\overline{T_n}),$$
$$\Delta U(\mathbf{S}_+^{n-1},0,m;u) = 0 \text{ in } T_n,$$
$$U(\mathbf{S}_+^{n-1},0,m;u) = u \text{ on } \partial T_n,$$
$$\lim_{r\to\infty, P=(r,\Theta)\in T_n} U(\mathbf{S}_+^{n-1},0,m;u)(P) = o(r^{m+1}\cos^{1-n}\theta_1).$$

Now we have

Theorem 1. *If* $\epsilon > 0$, $0 \le \zeta \le np$, $\gamma \in \mathcal{C}(k,p,m,n)$ $(resp.\ \gamma \in \mathcal{D}(k,p,m,n))$ *and* u *is a measurable function on* $\partial C_n(\Omega)$ *satisfying (1.5), then there exists a covering* $\{r_j, R_j\}$ *of* $E(\epsilon;\mu,np-\zeta)$ $(resp.\ E(\epsilon;\nu,np-\zeta))$ $(\subset C_n(\Omega))$ *satisfying*

$$\sum_{j=0}^{\infty}(\frac{r_j}{R_j})^{(p-1)n+2-\zeta}V_j(\frac{R_j}{r_j})W_j(\frac{R_j}{r_j}) < \infty$$

such that

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)-E(\epsilon;\mu,np-\zeta)} r^{\frac{-\iota^+_{[\gamma],k}-\{\gamma\}+n-1}{p}}\varphi_1^{\frac{\zeta}{p}-1}(\Theta)U(\Omega,a,m;u)(P)=0.$$

$$\left(\text{resp.} \lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)-E(\epsilon;\nu,np-\zeta)} r^{\frac{\iota^+_{[-\gamma],k}+\{-\gamma\}+n-1}{p}}\varphi_1^{\frac{\zeta}{p}-1}(\Theta)U(\Omega,a,m;u)(P)=0.\right)$$

Let $1 \le p < \infty$, $0 \le \zeta \le np$, $\gamma > -(n-1)(p-1)$ and

$$\frac{\gamma-n+1}{p}-1 < m < \frac{\gamma-n+1}{p} \text{ in case } p>1,$$
$$\gamma-n \le m < \gamma-n+1 \text{ in case } p=1;$$

We assume in addition that u is a measurable function on ∂T_n satisfying

$$\int_{\partial T_n}\frac{|u(t,\Phi)|^p}{1+t^{\gamma}}d\sigma_Q < \infty.$$

For this γ and u, we define

$$d\mu'(Q) = \begin{cases} |u(t,\Phi)|^p t^{-\gamma} d\sigma_Q & Q=(t,\Phi)\in S_n(\mathbf{S}_+^{n-1};(1,+\infty)),\\ 0 & Q\in \mathbf{R}^n - S_n(\mathbf{S}_+^{n-1};(1,+\infty)). \end{cases}$$

Obviously, the total mass of μ' is also finite.

If we take $\Omega = \mathbf{S}_+^{n-1}$ and $a=0$ in Theorem 1, then we immediately have the following growth property based on (1.3) and Remark 1.

Corollary 1. *If* p, ζ, γ, m *and* u *are defined as above, then the function* $U(\mathbf{S}_+^{n-1},0,m;u)(P)$ *is a harmonic function on* T_n *and there exists a covering* $\{r_j,R_j\}$ *of* $E(\epsilon;\mu',np-\zeta)$ $(\subset T_n)$ *satisfying*

$$\sum_{j=0}^{\infty}(\frac{r_j}{R_j})^{np-\zeta} < \infty \tag{1.6}$$

such that

$$\lim_{r\to\infty, P=(r,\Theta)\in T_n-E(\epsilon;\mu',np-\zeta)} r^{\frac{n-\gamma-1}{p}}\cos^{\frac{\zeta}{p}-1}\theta_1 U(\mathbf{S}_+^{n-1},0,m;u)(P)=0. \tag{1.7}$$

Remark 3. In the case that $p = 1$, $\gamma = n + m$ and $\zeta = n$, then (1.6) is a finite sum, the set $E(\epsilon; \mu', 0)$ is a bounded set and (1.7) holds in T_n. This is just the result of Mizuta-Shimomura (see [11, Theorem 1 with $\lambda = n$]).

Remark 4. In the case $\zeta = (1-\beta)p$, we can easily show that $E(\epsilon; \mu', (n-1+\beta)p)$ is $(k_{\beta,\lambda}, p)$-thin at infinity in the sense of [11, p. 335].

As an application of Theorem 1, we give the solutions of the Dirichlet problem for the Schrödinger operator on $C_n(\Omega)$.

Theorem 2. *If u is a continuous function on $\partial C_n(\Omega)$ satisfying*

$$\int_{S_n(\Omega)} \frac{|u(t,\Phi)|}{1 + V_{m+1}(t)t^{n-1}} d\sigma_Q < \infty, \tag{1.8}$$

then the function $U(\Omega, a, m; u)(P)$ satisfies

$$U(\Omega, a, m; u) \in C^2(C_n(\Omega)) \cap C^0(\overline{C_n(\Omega)}),$$

$$Sch_a U(\Omega, a, m; u) = 0 \text{ in } C_n(\Omega),$$

$$U(\Omega, a, m; u) = u \text{ on } \partial C_n(\Omega),$$

$$\lim_{r\to\infty, P=(r,\Theta)\in C_n(\Omega)} r^{-\iota^+_{m+1,k}} \varphi_1^{n-1}(\Theta) U(\Omega, a, m; u)(P) = 0.$$

2. Lemmas

Throughout this paper, Let M denote various constants independent of the variables in questions, which may be different from line to line.

Lemma 1.

(i) $\mathbb{P}(\Omega, a)(P, Q) \le M r^{\iota^-_{1,k}} t^{\iota^+_{1,k}-1} \varphi_1(\Theta)$

(ii) *(resp.* $\mathbb{P}(\Omega, a)(P, Q) \le M r^{\iota^+_{1,k}} t^{\iota^-_{1,k}-1} \varphi_1(\Theta)$*) for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in S_n(\Omega)$ satisfying $0 < \frac{t}{r} \le \frac{4}{5}$ (resp. $0 < \frac{r}{t} \le \frac{4}{5}$);*

(iii) $\mathbb{P}(\Omega, 0)(P, Q) \le M \frac{\varphi_1(\Theta)}{t^{n-1}} + M \frac{r\varphi_1(\Theta)}{|P-Q|^n}$ *for any $P = (r, \Theta) \in C_n(\Omega)$ and any $Q = (t, \Phi) \in S_n(\Omega; (\frac{4}{5}r, \frac{5}{4}r))$.*

Proof. (i) and (ii) are obtained by A. Kheyfits (see [3, Ch. 11]). (iii) follows from V. S. Azarin (see [2, Lemma 4 and Remark]).

Lemma 2 (see [7]). *For a non-negative integer m, we have*

$$|\mathbb{P}(\Omega, a, m)(P, Q)| \le M(n, m, s) V_{m+1}(r) \frac{W_{m+1}(t)}{t} \varphi_1(\Theta) \frac{\partial \varphi_1(\Phi)}{\partial n_\Phi} \tag{2.1}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ *and* $Q=(t,\Phi)\in S_n(\Omega)$ *satisfying* $r\le st$ $(0<s<1)$, *where* $M(n,m,s)$ *is a constant dependent of* n, m *and* s.

The proof of the following Lemma is essentially based on Hayman (see [9, p. 109]) in $\mathbf{R}^2$. We extend this result to $\mathbf{R}^n (n\ge 2)$ and give the proof here for the completeness.

Lemma 3. *Let* $\epsilon>0$, $\xi\ge 0$ *and* μ *be any positive measure on* $\mathbf{R}^n$ *having finite total mass. Then* $E(\epsilon;\mu,\xi)$ *has a covering* $\{r_j,R_j\}$ $(j=1,2,\dots)$ *satisfying*

$$\sum_{j=1}^{\infty}(\frac{r_j}{R_j})^{2-n+\xi}V_j(\frac{R_j}{r_j})W_j(\frac{R_j}{r_j})<\infty.$$

Proof. Set

$$E_j(\epsilon;\mu,\xi)=(P=(r,\Theta)\in E(\epsilon;\mu,\xi):2^j\le r<2^{j+1})\ (j=2,3,4,\dots).$$

If $P=(r,\Theta)\in E_j(\epsilon;\mu,\xi)$, then there exists a positive number $\rho(P)$ such that

$$(\frac{\rho(P)}{r})^{2-n+\xi}V_j(\frac{r}{\rho(P)})W_j(\frac{r}{\rho(P)})\sim(\frac{\rho(P)}{r})^{\xi}\le\frac{\mu(B(P,\rho(P)))}{\epsilon}.$$

Here $E_j(\epsilon;\mu,\xi)$ can be covered by the union of a family of balls $(B(P_{j,i},\rho_{j,i}):P_{j,i}\in E_j(\epsilon;\mu,\xi))$ $(\rho_{j,i}=\rho(P_{j,i}))$. By the Vitali Lemma (see [17]), there exists $\Lambda_j\subset E_j(\epsilon;\mu,\xi)$, which is at most countable, such that $(B(P_{j,i},\rho_{j,i}):P_{j,i}\in\Lambda_j)$ are disjoint and $E_j(\epsilon;\mu,\xi)\subset\cup_{P_{j,i}\in\Lambda_j}B(P_{j,i},5\rho_{j,i})$.

So

$$\cup_{j=2}^{\infty}E_j(\epsilon;\mu,\xi)\subset\cup_{j=2}^{\infty}\cup_{P_{j,i}\in\Lambda_j}B(P_{j,i},5\rho_{j,i}).$$

On the other hand, note that $\cup_{P_{j,i}\in\Lambda_j}B(P_{j,i},\rho_{j,i})\subset(P=(r,\Theta):2^{j-1}\le r<2^{j+2})$, so that

$$\begin{aligned}\sum_{P_{j,i}\in\Lambda_j}(\frac{5\rho_{j,i}}{|P_{j,i}|})^{2-n+\xi}V_j(\frac{|P_{j,i}|}{5\rho_{j,i}})W_j(\frac{|P_{j,i}|}{5\rho_{j,i}})&\sim\sum_{P_{j,i}\in\Lambda_j}(\frac{5\rho_{j,i}}{|P_{j,i}|})^{\xi}\\&\le 5^{\xi}\sum_{P_{j,i}\in\Lambda_j}\frac{\mu(B(P_{j,i},\rho_{j,i}))}{\epsilon}\\&\le\frac{5^{\xi}}{\epsilon}\mu(C_n(\Omega;[2^{j-1},2^{j+2}))).\end{aligned}$$

Hence we obtain

$$\sum_{j=1}^{\infty}\sum_{P_{j,i}\in\Lambda_j}(\frac{\rho_{j,i}}{|P_{j,i}|})^{2-n+\xi}V_j(\frac{|P_{j,i}|}{\rho_{j,i}})W_j(\frac{|P_{j,i}|}{\rho_{j,i}}) \sim \sum_{j=1}^{\infty}\sum_{P_{j,i}\in\Lambda_j}(\frac{\rho_{j,i}}{|P_{j,i}|})^{\xi}$$
$$\leq \sum_{j=1}^{\infty}\frac{\mu(C_n(\Omega;[2^{j-1},2^{j+2})))}{\epsilon}$$
$$\leq \frac{3\mu(\mathbf{R}^n)}{\epsilon}.$$

Since $E(\epsilon;\mu,\xi)\cap\{P=(r,\Theta)\in\mathbf{R}^n;r\geq 4\}=\cup_{j=2}^{\infty}E_j(\epsilon;\mu,\xi)$. Then $E(\epsilon;\mu,\xi)$ is finally covered by a sequence of balls $(B(P_{j,i},\rho_{j,i}),B(P_1,6))$ $(j=2,3,\ldots;i=1,2,\ldots)$ satisfying

$$\sum_{j,i}(\frac{\rho_{j,i}}{|P_{j,i}|})^{2-n+\xi}V_j(\frac{|P_{j,i}|}{\rho_{j,i}})W_j(\frac{|P_{j,i}|}{\rho_{j,i}}) \sim \sum_{j,i}(\frac{\rho_{j,i}}{|P_{j,i}|})^{\xi} \leq \frac{3\mu(\mathbf{R}^n)}{\epsilon}+6^{\xi}<+\infty,$$

where $B(P_1,6)$ $(P_1=(1,0,\ldots,0)\in\mathbf{R}^n)$ is the ball which covers $\{P=(r,\Theta)\in\mathbf{R}^n;r<4\}$.

3. Proof of Theorem 1

We only prove the case $p>1$ and $\gamma\geq 0$, the remaining cases can be proved similarly.

For any $\epsilon>0$, there exists $R_\epsilon>1$ such that

$$\int_{S_n(\Omega;(R_\epsilon,\infty))}\frac{|u(Q)|^p}{1+t^{\iota^+_{[\gamma],k}+\{\gamma\}}}d\sigma_Q<\epsilon. \tag{3.1}$$

The relation $G(\Omega,a)(P,Q)\leq G(\Omega,0)(P,Q)$ implies this inequality (see [1])

$$\mathbb{P}(\Omega,a)(P,Q)\leq\mathbb{P}(\Omega,0)(P,Q). \tag{3.2}$$

For $0<s<\frac{4}{5}$ and any fixed point $P=(r,\Theta)\in C_n(\Omega)-E(\epsilon;\mu,np-\zeta)$ satisfying $r>\frac{5}{4}R_\epsilon$, let $I_1=S_n(\Omega;(0,1))$, $I_2=S_n(\Omega;[1,R_\epsilon])$, $I_3=S_n(\Omega;(R_\epsilon,\frac{4}{5}r])$, $I_4=S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$, $I_5=S_n(\Omega;[\frac{5}{4}r,\frac{r}{s}))$, $I_6=S_n(\Omega;[\frac{r}{s},\infty))$ and $I_7=S_n(\Omega;[1,\frac{r}{s}))$, we write

$$\begin{aligned}
&U(\Omega,a,m;u)(P)\\
&=\sum_{i=1}^{6}\int_{I_i}\mathbb{P}(\Omega,a,m)(P,Q)u(Q)d\sigma_Q\\
&=\sum_{i=1}^{5}\int_{I_i}\mathbb{P}(\Omega,a)(P,Q)u(Q)d\sigma_Q-\int_{I_7}\frac{\partial\widetilde{K}(\Omega,a,m)(P,Q)}{\partial n_Q}u(Q)d\sigma_Q\\
&\quad+\int_{I_6}\mathbb{P}(\Omega,a,m)(P,Q)u(Q)d\sigma_Q,
\end{aligned}$$

which yields that

$$U(\Omega, a, m; u)(P) \leq \sum_{i=1}^{7} U_i(P),$$

where

$$U_i(P) = \int_{I_i} |\mathbb{P}(\Omega, a)(P, Q)||u(Q)|d\sigma_Q \ (i = 1, 2, 3, 4, 5),$$
$$U_6(P) = \int_{I_6} |\mathbb{P}(\Omega, a, m)(P, Q)||u(Q)|d\sigma_Q,$$

and

$$U_7(P) = \int_{I_7} |\frac{\partial \widetilde{K}(\Omega, a, m)(P, Q)}{\partial n_Q}||u(Q)|d\sigma_Q.$$

If $\iota^+_{[\gamma],k} + \{\gamma\} > (-\iota^+_{1,k} - n + 2)p + n - 1$, then $(\iota^+_{1,k} - 1 + \frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q + n - 1 > 0$. By (1.5), (3.1), Lemma 1 (i) and Hölder's inequality, we have the following growth estimates

$$\begin{aligned} U_2(P) &\leq Mr^{\iota^-_{1,k}} \varphi_1(\Theta) \int_{I_2} t^{\iota^+_{1,k}-1} |u(Q)| d\sigma_Q \\ &\leq Mr^{\iota^-_{1,k}} \varphi_1(\Theta) \Big(\int_{I_2} \frac{|u(Q)|^p}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}} d\sigma_Q \Big)^{\frac{1}{p}} \Big(\int_{I_2} t^{(\iota^+_{1,k}-1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q} d\sigma_Q \Big)^{\frac{1}{q}} \\ &\leq Mr^{\iota^-_{1,k}} R_\epsilon^{\iota^+_{1,k}+n-2+\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}} \varphi_1(\Theta). \end{aligned} \tag{3.3}$$

$$U_1(P) \leq Mr^{\iota^-_{1,k}} \varphi_1(\Theta). \tag{3.4}$$

$$U_3(P) \leq M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}} \varphi_1(\Theta). \tag{3.5}$$

If $\iota^+_{m+1,k} > \frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}$, then $(\iota^-_{1,k} - 1 + \frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q + n - 1 < 0$. We obtain by (3.1), Lemma 1 (ii) and Hölder's inequality

$$\begin{aligned} U_5(P) &\leq Mr^{\iota^+_{1,k}} \varphi_1(\Theta) \int_{S_n(\Omega;[\frac{5}{4}r,\infty))} t^{\iota^-_{1,k}-1} |u(Q)| d\sigma_Q \\ &\leq Mr^{\iota^+_{1,k}} \varphi_1(\Theta) \Big(\int_{S_n(\Omega;[\frac{5}{4}r,\infty))} \frac{|u(Q)|^p}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}} d\sigma_Q \Big)^{\frac{1}{p}} \\ &\quad \Big(\int_{S_n(\Omega;[\frac{5}{4}r,\infty))} t^{(\iota^-_{1,k}-1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q} d\sigma_Q \Big)^{\frac{1}{q}} \\ &\leq M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}} \varphi_1(\Theta \end{aligned} \tag{3.6}$$

By (3.2) and Lemma 1 (iii), we consider the inequality

$$U_4(P) \le U_4'(P) + U_4''(P),$$

where

$$U_4'(P) = M\varphi_1(\Theta)\int_{I_4} t^{1-n}|u(Q)|d\sigma_Q,\ U_4''(P) = Mr\varphi_1(\Theta)\int_{I_4} \frac{|u(Q)|}{|P-Q|^n}d\sigma_Q.$$

We first have

$$\begin{aligned} U_4'(P) &= M\varphi_1(\Theta)\int_{I_4} t^{\iota_{1,k}^{+}+\iota_{1,k}^{-}-1}|u(Q)|d\sigma_Q \\ &\le Mr^{\iota_{1,k}^{+}}\varphi_1(\Theta)\int_{S_n(\Omega;(\frac{4}{5}r,\infty))} t^{\iota_{1,k}^{-}-1}|u(Q)|d\sigma_Q \\ &\le M\epsilon r^{\frac{\iota_{[\gamma],k}^{+}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta), \end{aligned} \tag{3.7}$$

which is similar to the estimate of $U_5(P)$.

Next, we shall estimate $U_4''(P)$.

Take a sufficiently small positive number d_3 such that $I_4 \subset B(P, \frac{1}{2}r)$ for any $P = (r, \Theta) \in \Pi(d_3)$, where

$$\Pi(d_3) = \{P = (r,\Theta) \in C_n(\Omega);\ \inf_{z\in\partial\Omega} |(1,\Theta) - (1,z)| < d_3,\ 0 < r < \infty\}.$$

and divide $C_n(\Omega)$ into two sets $\Pi(d_3)$ and $C_n(\Omega) - \Pi(d_3)$.

If $P = (r,\Theta) \in C_n(\Omega) - \Pi(d_3)$, then there exists a positive d_3' such that $|P-Q| \ge d_3' r$ for any $Q \in S_n(\Omega)$, and hence

$$\begin{aligned} U_4''(P) &\le M\varphi_1(\Theta)\int_{I_4} t^{1-n}|u(Q)|d\sigma_Q \\ &\le M\epsilon r^{\frac{\iota_{[\gamma],k}^{+}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta), \end{aligned} \tag{3.8}$$

which is similar to the estimate of $U_4'(P)$.

We shall consider the case $P = (r,\Theta) \in \Pi(d_3)$. Now put

$$H_i(P) = \{Q \in I_4;\ 2^{i-1}\delta(P) \le |P-Q| < 2^i\delta(P)\},$$

where $\delta(P) = \inf_{Q\in\partial C_n(\Omega)} |P-Q|$.

Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n : |P-Q| < \delta(P)\} = \varnothing$, we have

$$U_4''(P) = M\sum_{i=1}^{i(P)}\int_{H_i(P)} r\varphi_1(\Theta)\frac{|u(Q)|}{|P-Q|^n}d\sigma_Q,$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \le \frac{r}{2} < 2^{i(P)}\delta(P)$.

Since $r\varphi_1(\Theta) \le M\delta(P)$ $(P=(r,\Theta)\in C_n(\Omega))$, similar to the estimate of $U_4'(P)$, we obtain

$$\begin{aligned}
&\int_{H_i(P)} r\varphi_1(\Theta)\frac{|u(Q)|}{|P-Q|^n}d\sigma_Q\\
&\le 2^{(1-i)n}\varphi_1(\Theta)\delta(P)^{\frac{\zeta-np}{p}}\int_{H_i(P)}\delta(P)^{\frac{np-\zeta}{p}-n}|u(Q)|d\sigma_Q\\
&\le M\varphi_1^{1-\frac{\zeta}{p}}(\Theta)\delta(P)^{\frac{\zeta-np}{p}}\int_{H_i(P)} r^{1-\frac{\zeta}{p}}|u(Q)|d\sigma_Q\\
&\le Mr^{n-\frac{\zeta}{p}}\varphi_1^{1-\frac{\zeta}{p}}(\Theta)\delta(P)^{\frac{\zeta-np}{p}}\int_{H_i(P)} t^{1-n}|u(Q)|d\sigma_Q\\
&\le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n-\zeta+1}{p}+n}\varphi_1^{1-\frac{\zeta}{p}}(\Theta)\left(\frac{\mu(H_i(P))}{(2^i\delta(P))^{np-\zeta}}\right)^{\frac{1}{p}}
\end{aligned}$$

for $i=0,1,2,\ldots,i(P)$.

Since $P=(r,\Theta)\notin E(\epsilon;\mu,np-\zeta)$, we have

$$\begin{aligned}
&\frac{\mu(H_i(P))}{(2^i\delta(P))^{np-\zeta}} \le \frac{\mu(B(P,2^i\delta(P)))}{(2^i\delta(P))^{np-\zeta}}\\
&\le M(P;\mu,np-\zeta)\le \epsilon r^{\zeta-np}\ (i=0,1,2,\ldots,i(P)-1)
\end{aligned}$$

and

$$\frac{\mu(H_{i(P)}(P))}{(2^i\delta(P))^{np-\zeta}} \le \frac{\mu(B(P,\frac{r}{2}))}{(\frac{r}{2})^{np-\zeta}} \le \epsilon r^{\zeta-np}.$$

So

$$U_4''(P) \le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1^{1-\frac{\zeta}{p}}(\Theta). \tag{3.9}$$

We only consider $U_7(P)$ in the case $m\ge 1$, since $U_7(P)\equiv 0$ for $m=0$. By the definition of $\widetilde{K}(\Omega,a,m)$, (1.1) and Lemma 2, we see

$$U_7(P) \le \frac{M}{\chi'(1)}\sum_{j=0}^{m} j^{2n-1}q_j(r),$$

where

$$q_j(r) = V_j(r)\varphi_1(\Theta)\int_{I_7}\frac{W_j(t)|u(Q)|}{t}d\sigma_Q.$$

To estimate $q_j(r)$, we write

$$q_j(r) \le q_j'(r) + q_j''(r),$$

where

$$q'_j(r) = V_j(r)\varphi_1(\Theta)\int_{I_2}\frac{W_j(t)|u(Q)|}{t}d\sigma_Q,\ q''_j(r) = V_j(r)\varphi_1(\Theta)\int_{S_n(\Omega;(R_\epsilon,\frac{r}{s}))}\frac{W_j(t)|u(Q)|}{t}d\sigma_Q.$$

If $\iota^+_{m+1,k} < \frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}+1$, then $(-\iota^+_{m+1,k}-n+2+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q+n-1>0$. Notice that

$$V_j(r)\frac{V_{m+1}(t)}{V_j(t)t} \le M\frac{V_{m+1}(r)}{r} \le Mr^{\iota^+_{m+1,k}-1}\ (t\ge 1, R_\epsilon < \frac{r}{s}).$$

Thus, by (1.3), (1.5) and Hölder's inequality we conclude

$$\begin{aligned}
q'_j(r) &= V_j(r)\varphi_1(\Theta)\int_{I_2}\frac{|u(Q)|}{V_j(t)t^{n-1}}d\sigma_Q\\
&\le MV_j(r)\varphi_1(\Theta)\int_{I_2}\frac{V_{m+1}(t)}{t^{\iota^+_{m+1,k}}}\frac{|u(Q)|}{V_j(t)t^{n-1}}d\sigma_Q\\
&\le r^{\iota^+_{m+1,k}-1}\varphi_1(\Theta)\Big(\int_{I_2}\frac{|u(Q)|^p}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}}d\sigma_Q\Big)^{\frac{1}{p}}\Big(\int_{I_2}t^{(-\iota^+_{m+1,k}-n+2+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q}d\sigma_Q\Big)^{\frac{1}{q}}\\
&\le Mr^{\iota^+_{m+1,k}-1}R_\epsilon^{-\iota^+_{m+1,k}+1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta).
\end{aligned}$$

Analogous to the estimate of $q'_j(r)$, we have

$$q''_j(r) \le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta).$$

Thus we can conclude that

$$q_j(r) \le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta),$$

which yields

$$U_7(P) \le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta). \tag{3.10}$$

If $\iota^+_{m+1,k} > \frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}$, then $(-\iota^+_{m+1,k}-n+1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q+n-1<0$. By (3.1), Lemma 2 and Hölder's inequality we have

$$
\begin{aligned}
U_6(P) &\le MV_{m+1}(r)\varphi_1(\Theta)\int_{I_6}\frac{|u(Q)|}{V_{m+1}(t)t^{n-1}}d\sigma_Q\\
&\le MV_{m+1}(r)\varphi_1(\Theta)\Big(\int_{I_6}\frac{|u(Q)|^p}{t^{\iota^+_{[\gamma],k}+\{\gamma\}}}d\sigma_Q\Big)^{\frac{1}{p}}\\
&\quad\Big(\int_{I_6}t^{(-\iota^+_{m+1,k}-n+1+\frac{\iota^+_{[\gamma],k}+\{\gamma\}}{p})q}d\sigma_Q\Big)^{\frac{1}{q}}\\
&\le M\epsilon r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1(\Theta).
\end{aligned}\tag{3.11}
$$

Combining (3.3)-(3.11), we obtain that if R_ϵ is sufficiently large and ϵ is sufficiently small, then $U(\Omega,a,m;u)(P)=o(r^{\frac{\iota^+_{[\gamma],k}+\{\gamma\}-n+1}{p}}\varphi_1^{1-\frac{\zeta}{p}}(\Theta))$ as $r\to\infty$, where $P=(r,\Theta)\in C_n(\Omega;(R_\epsilon,+\infty))-E(\epsilon;\mu,np-\zeta)$. Finally, there exists an additional finite ball B_0 covering $C_n(\Omega;(0,R_\epsilon])$, which together with Lemma 3, gives the conclusion of Theorem 1.

4. Proof of Theorem 2

For any fixed $P=(r,\Theta)\in C_n(\Omega)$, take a number satisfying $R>\max(1,\frac{r}{s})$ $(0<s<\frac{4}{5})$.

By (1.8) and Lemma 2, we have

$$
\begin{aligned}
&\int_{S_n(\Omega;(R,\infty))}|\mathbb{P}(\Omega,a,m)(P,Q)||u(Q)|d\sigma_Q\\
&\le V_{m+1}(r)\varphi_1(\Theta)\int_{S_n(\Omega;(R,\infty))}\frac{|u(Q)|}{V_{m+1}(t)t^{n-1}}d\sigma_Q\\
&\le MV_{m+1}(r)\varphi_1(\Theta)\\
&<\infty.
\end{aligned}
$$

Then $U(\Omega,a,m;u)(P)$ is absolutely convergent and finite for any $P\in C_n(\Omega)$. Thus $U(\Omega,a,m;u)(P)$ is a generalized harmonic function on $C_n(\Omega)$.

Now we study the boundary behavior of $U(\Omega,a,m;u)(P)$. Let $Q'=(t',\Phi')\in\partial C_n(\Omega)$ be any fixed point and l be any positive number satisfying $l>\max(t'+1,\frac{4}{5}R)$.

Set $\chi_{S(l)}$ is the characteristic function of $S(l)=\{Q=(t,\Phi)\in\partial C_n(\Omega),t\le l\}$ and write

$$
\begin{aligned}
U(\Omega,a,m;u)(P) &= \Big(\int_{S_n(\Omega;(0,1)}+\int_{S_n(\Omega;[1,\frac{5}{4}l])}+\int_{S_n(\Omega;(\frac{5}{4}l,\infty))}\Big)\\
&\quad \mathbb{P}(\Omega,a,m)(P,Q)u(Q)d\sigma_Q\\
&= U'(P)-U''(P)+U'''(P),
\end{aligned}
$$

where

$$U'(P) = \int_{S_n(\Omega;(0,\frac{5}{4}l])} \mathbb{P}(\Omega,a)(P,Q)u(Q)d\sigma_Q$$

$$U''(P) = \int_{S_n(\Omega;[1,\frac{5}{4}l])} \frac{\partial K(\Omega,a,m)(P,Q)}{\partial n_Q} u(Q)d\sigma_Q$$

and

$$U'''(P) = \int_{S_n(\Omega;(\frac{5}{4}l,\infty))} \mathbb{P}(\Omega,a,m)(P,Q)u(Q)d\sigma_Q.$$

Notice that $U'(P)$ is the Poisson a-integral of $u(Q)\chi_{S(\frac{5}{4}l)}$, we have $\lim_{P\to Q', P\in C_n(\Omega)} U'(P) = u(Q')$. Since $\lim_{\Theta\to\Phi'} \varphi_{jv}(\Theta) = 0$ $(j = 1,2,3\ldots; 1 \le v \le v_j)$ as $P = (r,\Theta) \to Q' = (t',\Phi') \in S_n(\Omega)$, we have $\lim_{P\to Q', P\in C_n(\Omega)} U''(P) = 0$ from the definition of the kernel function $K(\Omega,a,m)(P,Q)$. $U'''(P) = O(V_{m+1}(r)\varphi_1(\Theta))$ and therefore tends to zero.

So the function $U(\Omega,a,m;u)(P)$ can be continuously extended to $\overline{C_n(\Omega)}$ such that

$$\lim_{P\to Q', P\in C_n(\Omega)} U(\Omega,a,m;u)(P) = u(Q')$$

for any $Q' = (t',\Phi') \in \partial C_n(\Omega)$ from the arbitrariness of l, which with Theorem 1 gives the conclusion of Theorem 2.

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