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WEAK AND STRONG CONVERGENCE THEOREMS FOR VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS WITH TSENG'S EXTRAGRADIENT METHOD

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Abstract. The paper is concerned with the problem of finding a common solution of a variational inequality problem governed by Lipschitz continuous monotone mappings and of a fixed point problem of nonexpansive mappings. To solve this problem, we introduce two new iterative algorithms which are based on Tseng's extragradient method. Moreover we prove the weak and strong convergence of these new algorithms to a solution of the above-stated problem.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space and C a nonempty closed convex subset of $\mathcal{H}$. A mapping $S : C \to C$ is called *κ-Lipschitz continuous* if there exists a constant $\kappa > 0$ so that

$$\|Sx - Sy\| \leq \kappa\|x - y\| \quad (\forall x, y \in C).$$

In particular, if $\kappa = 1$, then we say S is a *nonexpansive mapping*. A mapping $A : C \to \mathcal{H}$ is called *monotone*, if

$$\langle Ax - Ay, x - y\rangle \geq 0 \quad (\forall x, y \in C);$$

κ-inverse strongly monotone, if there exists a constant $\kappa > 0$ so that

$$\langle Ax - Ay, x - y\rangle \geq \kappa\|Ax - Ay\|^2 \quad (\forall x, y \in C).$$

A variational inequality problem (VIP) is formulated as a problem of finding a point $x^* \in C$ with the property:

$$\langle Ax^*, z - x^*\rangle \geq 0, \quad \forall z \in C, \tag{1.1}$$

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where $A : C \to \mathcal{H}$ is a single-valued mapping. We will denote the solution set of VIP (1.1) by $\mathrm{VI}(A; C)$. A fixed point problem (FPP) is to find a point $\hat{x}$ with the property:

$$\hat{x} \in C, \quad S\hat{x} = \hat{x}, \tag{1.2}$$

where $S : C \to C$ is a nonlinear mapping. The set of fixed points of S is denoted as $\mathrm{Fix}(S)$. In this article we are interested in finding a common solution of VIP (1.1) and of FPP (1.2). Namely, we seek a point x^* such that

$$x^* \in \mathrm{Fix}(S) \cap \mathrm{VI}(A; C). \tag{1.3}$$

In the case where $A : C \to \mathcal{H}$ is inverse strongly monotone and $S : C \to C$ is nonexpansive, Takahashi and Toyoda [11] considered problem (1.3) and introduced an algorithm which generates a sequence (x_n) by the iterative procedure:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n SP_C(x_n - \lambda_n Ax_n), \quad n \geq 0, \tag{1.4}$$

where P_C is the projection of C onto $\mathcal{H}$. The hybrid version of algorithm (1.4):

$$\begin{cases} z_n = (1 - \alpha_n)x_n + \alpha_n SP_C(x_n - \lambda_n Ax_n), \\ C_n = \{u \in C : \|z_n - u\| \leq \|x_n - u\|\}, \\ Q_n = \{u \in C : \langle x_n - u, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.5}$$

was introduced by Iiduka and Takahashi [5]. In both algorithms (1.4) and (1.5), the sequence (α_n) is chosen from the interval $[0, 1]$. Under certain assumptions, the sequence (x_n) generated by algorithm (1.4) (resp., (1.5)) can be weakly (resp., strongly) convergent to a solution of problem (1.3) (see [11, 4]). For some other algorithms on Halpern iteration, we refer to see [5, 13, 14].

In general, the above algorithm dose not work whenever A is only a κ-Lipschitz-continuous and monotone mapping. In this situation, the following iterative method:

$$\begin{cases} x_0 \in C \\ y_n = P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n SP_C(x_n - \lambda_n Ay_n), \end{cases} \tag{1.6}$$

where $\lambda_n \in (0, 1/\kappa)$ and $\alpha_n \in (0, 1)$, was proposed by Nadezhkina and Takahashi [7] for solving problem (1.3). It is worth nothing that this algorithm is motivated by Korpelevich's extragradient method [6]:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases}$$

where $\lambda \in (0, 1/\kappa)$, for solving monotone varational inequality. Also, they introduced the hybrid version of algorithm (1.6), which, for $x_0 \in C$ and $n \geq 0$, generates a iterative sequence as

$$\text{(1.7)} \qquad \begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = (1-\alpha_n)x_n + \alpha_n S P_C(x_n - \lambda_n A y_n), \\ C_n = \{u \in C : \|z_n - u\| \leq \|x_n - u\|\}, \\ Q_n = \{u \in C : \langle x_n - u, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0). \end{cases}$$

where $\lambda_n \in (0, 1/\kappa)$ and $\alpha_n \in (0, 1)$. Under some mild assumptions, the sequence (x_n) generated by algorithm (1.6) (resp., (1.7)) can be weakly (resp., strongly) convergent to a solution of problem (1.3).

In this paper, we shall propose two new methods for solving (1.3) in the case where the governed mapping is only Lipschitz-continuous and monotone. Our algorithm is mainly based on Tseng's extragradient method [12]:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(y_n - \lambda(A y_n - A x_n)), \end{cases}$$

for finding a solution of problem (1.1). The paper is organized as follows. In the next section, some useful lemmas are given. In Section 3, we prove weak convergence of our first algorithm. In Section 4, we prove strong convergence of another algorithm.

2. Preliminary and Notation

Let $\mathcal{H}$ be a real Hilbert space and C a nonempty closed convex subset of $\mathcal{H}$. We use P_C to denote the projection from $\mathcal{H}$ onto C; namely, for $x \in \mathcal{H}$, $P_C x$ is the unique point in C with the property:

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

It is well-known that $P_C x$ is characterized by the inequality:

$$\text{(2.1)} \qquad P_C x \in C, \quad \langle x - P_C x, z - P_C x\rangle \leq 0, \quad z \in C.$$

The lemma below is referred to as the demiclosedness principle for nonexpansive mappings (see [3]).

Lemma 2.1. [Demiclosedness principle]. *Let C be a nonempty closed convex subset of $\mathcal{H}$ and $T : C \to \mathcal{H}$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If (x_n) is a sequence in C such that $x_n \rightharpoonup x$ and $(I-T)x_n \to y$, then $(I-T)x = y$. In particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Recall that $N_C x = \{w \in \mathcal{H} : \langle x - u, w\rangle \geq 0,\ u \in C\}$ is the normal cone to C at $x \in C$. Now let us define a mapping as

$$Tx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Since A is single-valued monotone, it follows from [10, Theorem 3] that T is maximal monotone (i.e., its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone mapping). The following lemma shows that VIP (1.1) is equivalent to finding a zero of the maximal monotone T.

Lemma 2.2. *Assume that C is a closed convex nonempty subset. Let $T := A + N_C$. Then*

(1) *$G(T)$ is sequentially weakly-strongly closed;*

(2) $T^{-1}(0) = \mathrm{VI}(C; A)$.

Proof. (1) is a basic property for any maximal monotone mapping (see for example [1]). To see (2), it suffices to observe that

$$\begin{aligned} 0 \in Tx &\Leftrightarrow 0 \in Ax + N_C x \\ &\Leftrightarrow -Ax \in N_C x \\ &\Leftrightarrow \langle Ax, x - z\rangle \leq 0, \forall z \in C. \end{aligned}$$

This is the result as desired. ∎

Definition 2.3. Assume that C is a closed convex nonempty subset and (x_n) is a sequence in $\mathcal{H}$. The sequence $\{x_n\}$ is called *Féjer monotone* w.r.t. C, if

$$\|x_{n+1} - z\| \leq \|x_n - z\| \quad (\forall z \in C).$$

Lemma 2.4. *If the sequence (x_n) is Féjer monotone w.r.t. the closed convex subset C, then the following hold.*

(a) *$x_n \rightharpoonup x^* \in C$ if and only if $\omega_w(x_n) \subseteq C$;*

(b) *The sequence $(P_C x_n)$ converges strongly;*

(c) *If $x_n \rightharpoonup x^* \in C$, then $x^* = \lim_{n\to\infty} P_C x_n$.*

Proof. That (a) and (b) are taken from [2, Theorem 2.16]. To show (c), let $\hat{x}$ be the limit of the sequence $\{P_C x_n\}$. It follows from inequality (2.1) that

$$\langle x_n - P_C x_n, x^* - P_C x_n\rangle \leq 0.$$

Letting $n \to \infty$ yields

$$\langle x^* - \hat{x}, x^* - \hat{x}\rangle \leq 0,$$

that is, $x^* = \hat{x}$ and thus the proof is complete. ∎

We shall use the following notation:

- $x_n \to x$: strong convergence of (x_n) to x;
- $x_n \rightharpoonup x$: weak convergence of (x_n) to x;
- $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$;
- $\omega_s(x_n) := \{x : \exists x_{n_j} \to x\}$.

3. Weak Convergence Theorem

We now introduce our first iterative algorithm. Take an initial guess $x_0 \in C$; choose $(\alpha_n) \subseteq (0,1)$ and $(\lambda_n) \subseteq (0, 1/\kappa)$; and define a sequence (x_n) by the iterative procedure:

$$\text{(3.1)} \qquad \begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S P_C(y_n - \lambda_n(Ay_n - Ax_n)). \end{cases}$$

Below is the convergence of this algorithm.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $A : C \to \mathcal{H}$ be monotone and κ-Lipschitz for some $\kappa > 0$ and $S : C \to C$ nonexpansive. Suppose that*

(a) $\Omega := \mathrm{Fix}(S) \cap \mathrm{VI}(A; C) \neq \emptyset$;
(b) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(c) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1/\kappa$.

Then, for the sequence (x_n) generated by (3.1)*, the following assertions hold.*

(i) (x_n) *is Féjer-monotone w.r.t.* Ω;
(ii) $\omega_w(x_n) \subseteq \Omega$;
(iii) $x_n \rightharpoonup x^* := P_\Omega x_n$.

Proof. (i) Let $u_n := P_C(y_n - \lambda_n(Ay_n - Ax_n))$. Taking any $u \in \Omega$, we deduce that

$$\begin{aligned} \|u_n - u\|^2 &= \|P_C(y_n - \lambda_n(Ay_n - Ax_n)) - u\|^2 \\ &\leq \|y_n - u - \lambda_n(Ay_n - Ax_n)\|^2 \\ &= \|y_n - u\|^2 + \lambda_n^2 \|Ay_n - Ax_n\|^2 \\ &\quad - 2\lambda_n \langle y_n - u, Ay_n - Ax_n\rangle \\ &= \|x_n - u\|^2 + \|y_n - x_n\|^2 + \lambda_n^2 \|Ay_n - Ax_n\|^2 \\ &\quad + 2\langle x_n - u, y_n - x_n\rangle - 2\lambda_n \langle y_n - u, Ay_n - Ax_n\rangle \\ &= \|x_n - u\|^2 - \|y_n - x_n\|^2 + \lambda_n^2 \|Ay_n - Ax_n\|^2 \\ &\quad + 2\langle y_n - u, y_n - x_n\rangle - 2\lambda_n \langle y_n - u, Ay_n - Ax_n\rangle. \end{aligned}$$

Noting $y_n = P_C(x_n - \lambda_n A x_n)$ and using 2.1, we have

$$\langle y_n - x_n + \lambda_n A x_n, y_n - u\rangle \le 0,$$

which is equivalent to

$$\langle y_n - u, y_n - x_n\rangle \le -\lambda_n \langle y_n - u, Ax_n\rangle.$$

Substituting this into above yields that

$$\begin{aligned}\|u_n - u\|^2 &\le \|x_n - u\|^2 - (1-\kappa^2\lambda_n^2)\|y_n - x_n\|^2 \\ &\quad -2\lambda_n\langle y_n - u, Ay_n\rangle \\ &= \|x_n - u\|^2 - (1-\kappa^2\lambda_n^2)\|y_n - x_n\|^2 \\ &\quad -2\lambda_n\langle y_n - u, Ay_n - Au\rangle - 2\lambda_n\langle y_n - u, Au\rangle \\ &\le \|x_n - u\|^2 - (1-\kappa^2\lambda_n^2)\|y_n - x_n\|^2,\end{aligned} \tag{3.2}$$

where the first inequality uses the Lipschitz continuity and the last inequality uses the monotonicity and the fact $u \in \mathrm{VI}(C; A)$. Since $u \in \mathrm{Fix}(S)$, we have

$$\begin{aligned}\|x_{n+1} - u\|^2 &\le \|(1-\alpha_n)x_n + \alpha_n S u_n - u\|^2 \\ &\le (1-\alpha_n)\|x_n - u\|^2 + \alpha_n\|Su_n - u\|^2 \\ &\le (1-\alpha_n)\|x_n - u\|^2 + \alpha_n\|u_n - u\|^2.\end{aligned}$$

This together with inequality (3.2) immediately gets

$$\|x_{n+1} - u\|^2 \le \|x_n - u\|^2 - \tau_n\|y_n - x_n\|^2, \tag{3.3}$$

where $\tau_n := \alpha_n(1-\kappa^2\lambda_n^2)$. Since $\alpha_n \in (0,1)$ and $\alpha_n \in (0, 1/\kappa)$, this implies that $\tau_n \ge 0$, and hence (x_n) is Féjer-monotone w.r.t. Ω.

(ii) Since (x_n) is bounded by (i), the set $\omega_w(x_n)$ is nonempty. Thus we can take $x' \in \omega_w(x_n)$ and a subsequence (x_{n_k}) of (x_n) such that $x_{n_k} \rightharpoonup x'$. We first show that $x' \in \mathrm{Fix}(S)$. To see this, we deduce from (3.3) that

$$\sum_{i=0}^{n} \tau_i\|y_i - x_i\|^2 \le \|x_0 - u\|^2,$$

Letting $n \to \infty$ yields that $(\tau_n\|x_n - y_n\|^2)$ is a summable sequence. It follows from

$$\liminf_{n\to\infty} \tau_n = \liminf_{n\to\infty} \alpha_n(1-\kappa^2\lambda_n^2) > 0$$

that $\|x_n - y_n\| \to 0$. Since $x_n \in C$, we deduce that

$$\begin{aligned}\|u_n - x_n\| &= \|P_C(y_n - \lambda_n(Ay_n - Ax_n)) - P_C x_n\| \\ &= \|y_n - x_n - \lambda_n(Ay_n - Ax_n)\| \\ &\le (1+\kappa\lambda_n)\|y_n - x_n\| \\ &\le 2\|y_n - x_n\| \to 0.\end{aligned}$$

In view of the nonexpansiveness of S and inequality (3.2),

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|Su_n-u\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-Su_n\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|u_n-u\|^2\\ &\quad -\alpha_n(1-\alpha_n)\|x_n-Su_n\|^2\\ &\le \|x_n-u\|^2-\alpha_n(1-\alpha_n)\|x_n-Su_n\|^2,\end{aligned}$$

which is the same as

$$\|x_n-Su_n\|^2\le\frac{1}{\alpha_n(1-\alpha_n)}(\|x_n-u\|^2-\|x_{n+1}-u\|^2).$$

Letting $n\to\infty$ and noting $\liminf_{n\to\infty}\alpha_n(1-\alpha_n)>0$, we conclude

$$\|Su_n-x_n\|=0.$$

Altogether, we get

$$\begin{aligned}\|Sx_n-x_n\| &\le \|Su_n-x_n\|+\|Su_n-Sx_n\|\\ &\le \|Su_n-x_n\|+\|u_n-x_n\|\to 0.\end{aligned}$$

Applying demiclosedness principle (Lemma 2.1), we conclude that $x'\in\mathrm{Fix}(S)$.

We next show that $x'\in\mathrm{VI}(C;A)$. Let

$$v_n=\frac{x_n-y_n}{\lambda_n}-(Ax_n-Ay_n).$$

Then it is not hard to check that

$$v_n\in(A+N_C)(y_n):=T(y_n).$$

By the Lipschitz continuity, we have that $\|Ax_n-Ay_n\|\to 0$, and that $v_n\to 0$ from $\liminf_{n\to\infty}\lambda_n>0$. Since $\|y_n-x_n\|\to 0$, we have $y_{n_k}\rightharpoonup x'$. By using the maximal monotonicity of T and Lemma 2.2, we conclude that $x'\in T^{-1}(0)=\mathrm{VI}(C;A)$.

(iii) By Lemma 2.4, this is a direct result of (i) and (ii). ∎

4. Strong Convergence Theorem

We now introduce our second iterative algorithm. Take an initial guess $x_0\in C$; choose $(\alpha_n)\subseteq(0,1)$ and $(\lambda_n)\subseteq(0,1/\kappa)$; and define a sequence (x_n) by the iterative procedure:

$$\text{(4.1)}\qquad\begin{cases} y_n=P_C(x_n-\lambda_nAx_n),\\ z_n=(1-\alpha_n)x_n+\alpha_nSP_C(y_n-\lambda_n(Ay_n-Ax_n)),\\ C_n=\{u\in C:\|z_n-u\|\le\|x_n-u\|\},\\ Q_n=\{u\in C:\langle x_n-u,x_n-x_0\rangle\le 0\},\\ x_{n+1}=P_{C_n\cap Q_n}(x_0).\end{cases}$$

Below is the convergence of this algorithm.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $A : C \to \mathcal{H}$ be monotone and κ-Lipschitz for some $\kappa > 0$ and $S : C \to C$ nonexpansive. Suppose that*

(a) $\Omega := \mathrm{Fix}(S) \cap \mathrm{VI}(A; C) \neq \emptyset$;

(b) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(c) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1/\kappa$.

Then, for the sequence (x_n) generated by (4.1)*, the following assertions hold.*

(i) *(x_n) is well defined;*

(ii) $\|x_n - x\| \leq \|x_{n+1} - x\| \leq \|x_0 - P_\Omega(x_0)\|$*;*

(ii) $\omega_w(x_n) \subseteq \Omega$*;*

(iv) $x_n \to P_\Omega(x_0)$ *as* $n \to \infty$.

Proof. (i) First, we show that, for every $n \in \mathbb{N}$, $C_n \cap Q_n$ is a closed convex set. That C_n is closed and Q_n is closed convex is trivial. To show the convexity of C_n, it suffices to note that

$$C_n = \{z \in C : \langle z_n - u, z_n\rangle \leq \langle z_n - u, x_n\rangle\}.$$

Obviously, C_n, is a halfspace and therefore convex.

We next prove that $C_n \cap Q_n$ is nonempty by showing

$$C_n \cap Q_n \supseteq \Omega \neq \emptyset, n \in \mathbb{N}. \tag{4.2}$$

To this end, let $u \in \Omega$ and let $n \in \mathbb{N}$ be fixed. With a proof similar to Theorem 3.1, we have

$$\|z_n - u\|^2 \leq \|x_n - u\|^2 - \tau_n\|x_n - y_n\|^2, \tag{4.3}$$

where $\tau_n = \alpha_n(1 - \kappa^2\lambda_n^2)$. Since $\tau_n \geq 0$, this implies that $\Omega \subseteq C_n$. It remains to show that $\Omega \subseteq Q_n$. For $n = 0$, we have $Q_0 = \mathcal{H}$, and hence $\Omega \subseteq Q_0$. Suppose that x_k is given and $\Omega \subseteq C_k \cap Q_k$ for some $k \in \mathbb{N}$. There exists a unique element x_{k+1} so that $x_{k+1} = P_{C_k \cap Q_k}(x_0)$. It follows from (2.1) and (4.1) that

$$\langle x_{k+1} - z, x_{k+1} - x_0\rangle \leq 0,$$

which in turn implies $\Omega \subseteq Q_{k+1}$ and hence (4.4). Consequently, the sequence (x_n) is well defined.

(ii) Since $x_{n+1} \in Q_n$, it follows that

$$\|x_n - x_0\| = \|P_{Q_n}(x_0) - x_0\| \leq \|x_{n+1} - x_0\|.$$

Note that $P_\Omega(x_0) \in \Omega \subseteq Q_{n+1}$, and hence

$$\|x_{n+1} - x_0\| = \|P_{Q_{n+1}}(x_0) - x_0\| \leq \|P_\Omega(x_0) - x_0\|.$$

(iii) Compared with the proof of Theorem 3.1, it suffices to show that

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{4.4}$$

Noting $x_{n+1} \in Q_n$ and $x_n = P_{Q_n}(x_0)$, we deduce

$$\begin{aligned}\|x_{n+1} - x_0\|^2 &= \|x_{n+1} - x_n\|^2 + \|x_n - x_0\|^2 \\ &\quad + 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\geq \|x_{n+1} - x_n\|^2 + \|x_n - x_0\|^2,\end{aligned}$$

where the inequality uses (2.1). Hence for any $n \in \mathbb{N}$,

$$\sum_{\ell=0}^{n} \|x_{\ell+1} - x_\ell\| \leq \|x_{n+1} - x_0\| \leq \|P_\Omega(x_0) - x_0\|.$$

Letting $n \to \infty$ shows that the sequence $(\|x_{n+1} - x_n\|)$ is summable and therefore

$$\|x_{n+1} - x_n\| \to 0.$$

Since $x_{n+1} \in C_n$, it follows from (4.1) that

$$\|z_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \to 0.$$

Then according to (4.3), we get

$$\begin{aligned}\|y_n - x_n\| &\leq \frac{1}{\tau_n}(\|z_n - u\|^2 - \|x_n - u\|^2) \\ &\leq \frac{M}{\tau_n}\|z_n - x_n\|,\end{aligned}$$

where $M > 0$ is a suitable constant. Since $\lim_{n\to\infty} \tau_n > 0$, this implies that $\|y_n - x_n\| \to 0$ as $n \to \infty$. To see the last equality in (4.4), let $u_n = P_C(y_n - \lambda_n(Ay_n - Ax_n))$. Then it follows from the formula

$$\|Su_n - x_n\| = \frac{1}{\alpha_n}\|z_n - x_n\|$$

and condition $\liminf_{n\to\infty}\alpha_n > 0$ that $\lim_{n\to\infty}\|Su_n - x_n\| = 0$. On the other hand,

$$\begin{aligned}\|u_n - x_n\| &= \|P_C(y_n - \lambda_n(Ay_n - Ax_n)) - P_C x_n\| \\ &\le \|y_n - x_n - \lambda_n(Ay_n - Ax_n)\| \\ &\le (1 + \kappa\lambda_n)\|y_n - x_n\| \\ &\le 2\|y_n - x_n\| \to 0.\end{aligned}$$

Therefore

$$\begin{aligned}\|Sx_n - x_n\| &\le \|Sx_n - Su_n\| + \|Su_n - x_n\| \\ &\le \|x_n - u_n\| + \|Su_n - x_n\| \to 0.\end{aligned}$$

(iv) Take a subsequence (x_{n_k}) so that $x_{n_k} \rightharpoonup x'$ and hence $x' \in \Omega$ by (iii). It then follows from (ii) that

$$\begin{aligned}\|x_{n_k} - P_\Omega(x_0)\|^2 &= \|x_{n_k} - x_0\|^2 + \|x_0 - P_\Omega(x_0)\|^2 \\ &\quad + 2\langle x_{n_k} - x_0, x_0 - P_\Omega(x_0)\rangle \\ &\le 2\|x_0 - P_\Omega(x_0)\|^2 + 2\langle x_{n_k} - x_0, x_0 - P_\Omega(x_0)\rangle \\ &= 2\langle x_{n_k} - P_\Omega(x_0), x_0 - P_\Omega(x_0)\rangle.\end{aligned}$$

Letting $k \to \infty$ and using inequality (2.1) yield

$$\limsup_{k\to\infty}\|x_{n_k} - P_\Omega(x_0)\|^2 \le \langle x' - P_\Omega(x_0), x_0 - P_\Omega(x_0)\rangle \le 0,$$

which implies that $x_{n_k} \to P_\Omega(x_0)$. On one hand,

$$\{P_\Omega(x_0)\} = \omega_w(x_n)$$

from the uniqueness of the projection; on the other hand,

$$\{P_\Omega(x_0)\} \subseteq \omega_s(x_n) \subseteq \omega_w(x_n).$$

Altogether $\{P_\Omega(x_0)\} = \omega_s(x_n)$, that is, $x_n \to P_\Omega(x_0)$ as $n \to \infty$. ∎

Remark 4.2. As in Section 4 of [8], we can apply our algorithms for finding a common fixed point of Lipschitz pseudocontractive and nonexpansive mappings.

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