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GRADED MORITA THEORY FOR GROUP CORING AND GRADED MORITA-TAKEUCHI THEORY

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Abstract. A Graded Morita context is constructed for any comodule of a group coring. For any right G-$\underline{\mathcal{C}}$-comodule $\underline{M}$ with dual graded ring R, we define a graded ring $T = HOM^{G,\underline{\mathcal{C}}}(M,M) = \bigoplus_{g\in G} HOM^{G,\underline{\mathcal{C}}}(M,M)_g$, and a G-graded R-T bimodule $Q = \bigoplus_{g\in G} Q^g$, where Q^g is a family of right A-linear maps $q^g_\alpha;\ M_\alpha \to R_{g\alpha}$ in $\mathcal{M}_A$. We construct a graded Morita context $M = (T,\ R,\ \bigoplus_{\alpha\in G} M_\alpha,\ Q,\ \tau,\ \mu)$ with connecting homomorphisms $\tau:\ {}_T(\bigoplus_{\alpha\in G} M_\alpha)\otimes_R Q_T \to T,\ m\otimes q \mapsto mq(-),\ \mu:\ {}_RQ\otimes_T(\bigoplus_{\alpha\in G} M_\alpha)_R \to R,\ q\otimes m\mapsto q(m)$, which generalized the Morita context in [3, 5-7, 10, 13].

Meanwhile, we prove the graded Morita-Takeuchi theory as a generalization of Morita-Takeuchi theory which characterize the equivalence of comodule over field.

1. INTRODUCTION

Graded Morita theory for group ring has been introduced by Dade [11, 12] since 1980. Boisen[4] introduced the definition of graded Morita context for all group graded rings. Graded Morita theory can be thought of as a generalization of Morita theory in the sense that when the grading group is trivial the two theories coincide. It can also be viewed as a refinement of Morita theory, since two rings with graded structure which are graded equivalent are necessarily Morita equivalent as rings.

Morita theory associating to comodule algebras for a Hopf algebra H was first introduced by Cohen, Fishman and Montgomery [6], in that paper a Morita context was constructed under the assumption that H is a finite dimensional Hopf algebra over a field (or a Frobenius algebra over a commutative ring). Doi in [13] extended the Morita theory to arbitrary Hopf algebra H. Caenepeel et al. [5, 8] constructed a Morita context for coring comodule which is finitely generated and projective as an A-module. Bohm and Vercruysse [3] generalize their construction, they construct

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a Morita context for an arbitrary comodule M of an A-coring $\mathcal{C}$ which connects the algebra of $\mathcal{C}$-comodule endomorphism of M and the A-dual algebra of $\mathcal{C}$.

Group coring was introduced by Caenepeel et al. [7], which generalized coring, group coalgebras and Hopf group coalgebras. In section 2, we give a graded Morita context for any group coring comodule connects the dual graded ring of a group coring and the graded endomorphism ring of any group coring comodule, which generalized the Morita context in [3, 5-7, 10, 13]. Let G be a finite group with unit e, A a ring with unit, a G-A-coring $\underline{\mathcal{C}}$, and R be the left dual graded ring, for any right G-$\underline{\mathcal{C}}$-comodule $\underline{M}$, we define $T = HOM^{G,\underline{\mathcal{C}}}(\underline{M}, \underline{M}) = \bigoplus_{g\in G} HOM^{G,\underline{\mathcal{C}}}(\underline{M}, \underline{M})_g$, where $(f_\alpha^g)_{\alpha\in G} \in HOM^{G,\underline{\mathcal{C}}}(\underline{M}, \underline{M})_g$ is a family of right A-linear maps $f_\alpha^g : M_\alpha \to M_{g\alpha}$ which are comodule maps. Then, we give a G-graded R-T bimodule $Q = \bigoplus_{g\in G} Q^g$, where Q^g is a family of right A-linear maps $(q_\alpha^g)_{\alpha\in G}$; $M_\alpha \to R_{g\alpha}$ in $\mathcal{M}_A$. By these definition, we construct a graded Morita context

$$M = (T,\ \ R,\ \ \bigoplus_{\alpha\in G} M_\alpha,\ \ Q,\ \ \tau,\ \ \mu)$$

with connecting homomorphisms

$$\tau :\ {}_T(\bigoplus_{\alpha\in G} M_\alpha) \otimes Q_T \to T,\ m\otimes q \mapsto mq(-)$$

$$\mu :\ {}_RQ \otimes (\bigoplus_{\alpha\in G} M_\alpha)_R \to R,\ q\otimes m \mapsto q(m).$$

Takeuchi [18] introduced the Morita-Takeuchi theory that characterizes equivalences of comodule categories over fields, dualizing Morita results on equivalences of module categories. Associated with Morita-Takeuchi context it is possible, using the functors cotensor and co-hom to establish the equivalences of comodule categories. The general concepts of graded Morita-Takeuchi context for graded coalgebras over arbitrary groups are introduced [2, 10, 19].

In section 3, we recall the definition of graded Morita-Takeuchi context and prove the theorem titled graded Morita-Takeuchi theorem following the treatment of Takeuchi given in [18]. In other words, we show that the well know Morita-Takeuchi theorem on equivalence of category of graded modules holds true for category of graded comodules over all field k. We go parallel with Boisen's [4] graded Morita theory.

Throughout this paper, k will be a field. For a general theory of Hopf algebras, we refer to the standard books [17, 20]. We use Sweedler's [20] "sigma" notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for an element c in a coalgebra (C, Δ, ε), and $\rho(m) = m_{[0]} \otimes m_{[1]}$ for an element m in a right C-comodule $\left(M, \rho^C\right)$. If M and N are C-comodules, a comodule map from M and N is a k-map $f : M \to N$ such that $(f\otimes 1)\rho_M = \rho_N f$. The k-space of all comodule maps from a right C-comodule M to a right C-comodule

N is denoted by $Com_{-C}(M, N)$. Let $\mathcal{M}^C$ and ${}^C\mathcal{M}$ denote the categories of right and left C-comodules, respectively.

2. Graded Morita Theory for Group Coring

2.1. Group coring

Let G be a group, and A an associative unital algebra over a fixed field k. The unit element of G will be denoted by e. A G-group A-coring (or shortly a G-A-coring) $\underline{\mathcal{C}}$ is a family $(\mathcal{C}_\alpha)_{\alpha\in G}$ of A-bimodule together with a family of bimodule maps

$$\triangle_{\alpha,\beta} : \mathcal{C}_{\alpha\beta} \to \mathcal{C}_\alpha \otimes_A \mathcal{C}_\beta; \;\; \epsilon : \mathcal{C}_e \to A,$$

such that

$$(\triangle_{\alpha,\beta} \otimes_A \mathcal{C}_\gamma) \circ \triangle_{\alpha\beta,\gamma} = (\mathcal{C}_\alpha \otimes_A \triangle_{\beta,\gamma}) \circ \triangle_{\alpha,\beta\gamma}$$

and

$$(\mathcal{C}_\alpha \otimes_A \epsilon) \circ \triangle_{\alpha,e} = \mathcal{C}_\alpha = (\epsilon \otimes_A \mathcal{C}_\alpha) \circ \triangle_{e,\alpha}$$

for all $\alpha, \beta, \gamma \in G$. We use the following Sweedler-type notation for the comultiplication maps $\triangle_{\alpha,\beta}$:

$$\triangle_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes_A c_{(2,\beta)}$$

for all $c \in \mathcal{C}_{\alpha\beta}$. Then the above equations take the form

$$c_{(1,\alpha)}\epsilon(c_{(2,e)}) = c = \epsilon(c_{(1,e)})c_{(2,\alpha)} \text{ for all } c \in C_\alpha$$

$$((\triangle_{\alpha,\beta} \otimes_A \mathcal{C}_\gamma) \circ \triangle_{\alpha\beta,\gamma})(c) = ((\mathcal{C}_\alpha \otimes_A \triangle_{\beta,\gamma}) \circ \triangle_{\alpha,\beta\gamma})(c) = c_{(1,\alpha)} \otimes_A c_{(2,\beta)} \otimes_A c_{(3,\gamma)}$$

for all $c \in \mathcal{C}_{\alpha\beta\gamma}$.

A morphism between two G-A-corings $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ consists of a family of A-bimodule maps

$(f_\alpha)_{\alpha\in G}$, $f_\alpha : \mathcal{C}_\alpha \to \mathcal{D}_\alpha$ such that

$$(f_\alpha \otimes_A f_\beta) \circ \triangle_{\alpha,\beta} = \triangle_{\alpha,\beta} \circ f_{\alpha\beta} \text{ and } \epsilon \circ f_e = \epsilon.$$

A right G-$\underline{\mathcal{C}}$- comodule $\underline{M}$ is a family of right A-modules $(M_\alpha)_{\alpha\in G}$, for every $\alpha \in G$, M_α is a k-linear space, and a family of right A-linear maps

$$\rho_{\alpha,\beta} : M_{\alpha\beta} \to M_\alpha \otimes_A \mathcal{C}_\beta$$

such that

$$(M_\alpha \otimes_A \triangle_{\beta,\gamma}) \circ \rho_{\alpha,\beta\gamma} = (\rho_{\alpha,\beta} \otimes_A \mathcal{C}_\gamma) \circ \rho_{\alpha\beta,\gamma}$$

and

$$(M_\alpha \otimes_A \epsilon) \circ \rho_{\alpha,e} = M_\alpha$$

for $m \in M_{\alpha\beta}$. We also use the Sweedler-type notation:

$$\rho_{\alpha,\beta}(m) = m_{[0,\alpha]} \otimes_a m_{[1,\beta]},$$

so that, above equations justify the notation

$$m_{[0,\alpha]}\epsilon(m_{[1,e]}) = m \quad \text{for all } m \in M_\alpha$$

$$((M_\alpha \otimes_A \triangle_{\beta,\gamma}) \circ \rho_{\alpha,\beta\gamma})(m) = (\rho_{\alpha,\beta} \otimes_A \mathcal{C}_\gamma \circ \rho_{\alpha\beta,\gamma})(m) = m_{[0,\alpha]} \otimes_A m_{[1,\beta]} \otimes_A m_{[2,\gamma]}$$

for all $m \in M_{\alpha\beta\gamma}$.

A morphism between two right G-$\underline{\mathcal{C}}$-comodules $\underline{M}$ and $\underline{N}$ is a family of right A-linear maps $f_\alpha : M_\alpha \to N_\alpha$ such that

$$(f_\alpha \otimes_A \mathcal{C}_\beta) \circ \rho_{\alpha,\beta} = \rho_{\alpha,\beta} \circ f_{\alpha\beta}.$$

The category of right G-$\underline{\mathcal{C}}$-comodules will be denoted by $\mathcal{M}^{G,\underline{\mathcal{C}}}$.

Let $\underline{\mathcal{C}}$ be a G-A-coring. For every $\alpha \in G$, $R_\alpha =^* \mathcal{C}_{\alpha^{-1}} =_A Hom(\mathcal{C}_{\alpha^{-1}}, A)$ is an A-bimodule, with

$$(a \cdot f \cdot b)(c) = f(ca)b$$

for all $f_\alpha \in R_\alpha, g_\beta \in R_\beta$ and define $f_\alpha \sharp g_\beta \in R_{\alpha\beta}$ as

$$(f_\alpha \sharp g_\beta)(c) = g_\beta(c_{(1,\beta^{-1})} f_\alpha(c_{(2,\alpha^{-1})}))$$

for all $c \in \mathcal{C}_{(\alpha\beta)^{-1}}$. This defines maps $m_{\alpha,\beta} : R_\alpha \otimes_A R_\beta \to R_{\alpha\beta}$, which makes $R = \bigoplus_{\alpha\in G} R_\alpha$ into a G-graded A-ring, called the left dual graded ring of the group coring $\underline{\mathcal{C}}$. We will also write ${}^*\underline{\mathcal{C}} = R$. In [7], the authors gave the following proposition:

Proposition 1. [7, Proposition 4.1]. *Let $\underline{\mathcal{C}}$ be a G-A-coring, with left dual graded ring R. We have a functor $F_3 : \mathcal{M}^{G,\underline{\mathcal{C}}} \to \mathcal{M}_R^G$, which is an isomorphism of categories if $\underline{\mathcal{C}}$ is left homogeneously finite.*

2.2. Graded Morita theory

Now, we recall the definition of graded Morita theory for graded ring [4][16]. Let $R = \bigoplus_{\alpha\in G} R_\alpha$ and $S = \bigoplus_{\alpha\in G} S_\alpha$ be two G-graded rings, where G is a group. A graded Morita context is a datum $(R, S, {}_R M_S, {}_S N_R, \phi, \psi)$, where M is a R-S-bimodule which is graded, i.e. $R_g M_h S_f \subseteq M_{ghf}$ and N is a S-R-bimodule which is also graded. Moreover, $\phi : M \otimes_S N \to R$ is an R-R-bimodule homomorphism which is graded in the sense that $\phi(M_g \otimes_s N_h) \subseteq R_{gh}$ and $\psi : N \otimes_R M \to S$ is an S-S-bimodule homomorphism which is also graded. Lastly, ϕ and ψ satisfy the following two relations:

$$\phi(m \otimes n)m' = m\psi(n \otimes m')$$

$$\psi(n \otimes m)n' = n\phi(m \otimes n').$$

Given a subset X of G, the symbol R_X denotes $\sum_{\sigma\in X} R_\sigma$. Let H be a subgroup of G, $R_H = \sum_{\sigma\in H} R_\sigma$, $M_H = \sum_{\sigma\in H} M_\sigma$. There is a natural R_H-R_S bimodule map from $N_H \otimes_{S_H} M_H$ to $N \otimes_S M$ given by $a \otimes_{S_H} b \mapsto a \otimes_S b$. Let τ_H denote the composition of this map followed by (the restriction of) the map from $N_H \otimes_{S_H} M_H$ to R_H. Define μ_H similarly. Then $(R_H, S_H, M_H, N_H, \tau, \mu)$ is a Morita context as per [14, Definition 3.11]. Thus a G-graded Morita context is in a sense a collection of Morita context indexed by the subgroups of G.

Theorem 2. [4, Theorem 3.2]. *Let (R, S, M, N, τ, μ) be a G-graded Morita context in which τ_e and μ_e are surjective. Let H be a subgroup of G and let G_R (G_S) denote the largest subgroup of G such that R_{G_R} (resp., S_{G_R}) is fully graded. Then*

(1) $G_R = G_S$;

(2) *M is projective in $GrMod$-R in such a way that it is a direct summand of a direct sum of copies of R. M_e is a generator in mod-R_e and in S_e-mod. For every subgroup H of G, M_H is a progenerator in mod-R_H and in S_H-mod. Similar statements hold for N;*

(3) *τ_H and μ_H are isomorphisms for every subgroup H of G;*

(4) *given $n \in N_H$, define $\iota(n) : M \to R_H$ to be the map $m \mapsto \tau(n \otimes m)$. The map $n \mapsto \iota(n)$ is a graded bimodule isomorphism of the R_H, S_H-module N_H onto $(M_H)^*$;*

(5) *Given $s \in S_H$, let $\lambda(s) \in End_{R_H}(M_H)$ be the map $m \mapsto sm$. The map $\lambda : S_H \to End_{R_H}(M_H)$ is an isomorphism of H-graded rings. Given $r \in R_H$, let $\rho(s) \in End_{S_H}(M_H)$ be the map $m \mapsto mr$. ρ is a graded anti-isomorphism from R_H to the H^{op} graded ring $End_{S_H}(M_H)$;*

(6) *The pair of functors $-\otimes_{R_H} N_H$ and $-\otimes_{S_H} M_H$ form an equivalence of the categories right R_H-module and S_H-module.*

2.3. Graded Morita theory for group coring

Generalizing constructions in [1, 5, 8, 9, 13], the authors in [3] associated Morita context for comodule of an A-coring $\mathcal{C}$, for any right $\mathcal{C}$-comodule $\underline{M}$ they constructed a Morita context connecting the k-modules Q, $T = End^{\mathcal{C}}(\mathcal{M})$, ${}^*\mathcal{C} = Hom(\mathcal{C}, A)$. In this section, we generalize the Morita context[3] to group coring, and construct the following graded Morita context associated a G-A-coring $\underline{\mathcal{C}}$ and $\underline{M} \in \mathcal{M}^{G,\underline{\mathcal{C}}}$.

Let G be a finite group with unit e, A a ring with unit and $\underline{\mathcal{C}}$ a G-A-coring, R be the left dual graded ring. For any right G-$\underline{\mathcal{C}}$ comodule $\underline{M}$, we define $T = HOM^{G,\underline{\mathcal{C}}}(\underline{M}, \underline{M}) = \bigoplus_{g\in G} HOM^{G,\underline{\mathcal{C}}}(\underline{M}, \underline{M})_g$, where $(f^g_\alpha)_{\alpha\in G}$ is the family of right A-linear maps $f^g_\alpha : M_\alpha \to M_{g\alpha}$ which are comodule maps. We also define a G-graded R-T bimodule $Q = \bigoplus_{g\in G} Q^g$, where Q^g is a family of right A-linear maps

$(q^g_\alpha)_{\alpha\in G}: M_\alpha \to R_{g\alpha}$ in $\mathcal{M}_A$. By the above definition, we construct a graded Morita context

$$M = (T,\ R,\ \bigoplus_{\alpha\in G} M_\alpha,\ Q,\ \tau\ \mu)$$

with connecting homomorphisms

$$\tau:\ {}_T(\bigoplus_{\alpha\in G} M_\alpha)\otimes Q_T \to T,\ m\otimes q \mapsto mq(-)$$

and

$$\mu:\ {}_RQ\otimes(\bigoplus_{\alpha\in G} M_\alpha)_R \to R,\ q\otimes m\mapsto q(m),$$

which generalized the Morita context in [3, 5, 6, 7, 8, 13].

Now, we give the specific construction processes. First, we define G-graded k-modules:

$$Q = \bigoplus_{g\in G} Q^g$$

where Q^g is the family of right A-linear maps $q^g_\alpha : M_\alpha \to R_{g\alpha}$ in $\mathcal{M}_A$, such that for all
$m_\alpha \in M_\alpha, c_{\beta g^{-1}} \in \mathcal{C}_{\beta g^{-1}}$, q^g satisfies:

(1) $$q^g_{\beta^{-1}}(m_{[0,\beta^{-1}]})(c_{\beta g^{-1}})m_{[1,\beta\alpha]} = c_{(1,\beta\alpha)}q^g_\alpha(m_\alpha)(c_{(2,\alpha^{-1}g^{-1})}).$$

Especially, we have
$Q^e = \{\underline{q}^e := (q^e_\alpha)_{\alpha\in G} : \underline{M} \to R, q^e_\alpha : M_\alpha \to R_\alpha\}$ in $\mathcal{M}_A$, and

$$q^e_{\beta^{-1}}(m_{[0,\beta^{-1}]})(c_\beta)m_{[1,\beta\alpha]} = c_{[1,\beta\alpha]}q^e_\alpha(m_\alpha)(c_{[2,\alpha^{-1}]}).$$

Now, for $\underline{M}\in\mathcal{M}^{G,\underline{\mathcal{C}}}$, define

$$T_e = HOM^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M})_e = Hom^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M}),\ T_g = HOM^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M})_g$$

$$T = \bigoplus_{g\in G} T_g = \bigoplus_{g\in G} HOM^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M})_g = HOM^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M}) = END^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M})$$

where $f^g \in HOM^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M})_g$ is a family of right A-linear maps $f^g_\alpha : M_\alpha \to M_{g\alpha}$ such that

(2) $$(f^g_\alpha \otimes_A \mathcal{C}_\beta)\circ\rho_{\alpha,\beta}(m_{\alpha\beta}) = \rho_{g\alpha,\beta}f^g_{\alpha\beta}(m_{\alpha\beta}).$$

A straightforward calculation shows that $T_gT_h \in T_{gh}$, and $T = END^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M})$ is a G-graded ring.

Lemma 3. *Let T, Q defined as above, then, Q is a G-graded R-T bimodule, with actions*

$$(f_g \rightharpoonup q^h)(m) = f_g\sharp q^h(m), \text{ for all } f_g \in R_g, q^h \in Q^h,\ m \in \underline{M},$$
$$(q^h \leftharpoonup t^g)(m_\alpha) = q^h(t^g(m_\alpha)), \text{ for all } t^g \in T^g, q^h \in Q^h, m_\alpha \in M_\alpha.$$

Proof.

(1) First, we check $f \rightharpoonup q$ satisfies (1), for all $f_\gamma \in R_\gamma$, $q^g \in G^g$:

$$\begin{aligned}
&(f_\gamma \rightharpoonup q^g_{\beta^{-1}})(m_{[0,\beta^{-1}]})(c_{\beta g^{-1}\gamma^{-1}})m_{[1,\beta\alpha]} \\
=&(f_\gamma\sharp q^g_{\beta^{-1}}(m_{[0,\beta^{-1}]}))(c_{\beta g^{-1}\gamma^{-1}})m_{[1,\beta\alpha]} \\
=&q^g_{\beta^{-1}}(m_{[0,\beta^{-1}]})(c_{(1,\beta g^{-1})}f_\gamma(c_{(2,\gamma^{-1})})m_{[1,\beta\alpha]} \\
=&(c_{(1,\beta\alpha)})q^g_\alpha(m_\alpha)(c_{(2,\alpha^{-1}g^{-1})}f_\gamma(c_{(3,\gamma^{-1})})) && \text{by (1)} \\
=&(c_{(1,\beta\alpha)})(f_\gamma \rightharpoonup q^g_\alpha)(m_\alpha)(c_{(2,\alpha^{-1}g^{-1}\gamma^{-1})}).
\end{aligned}$$

Meanwhile, it's obvious that $f \rightharpoonup q$ is an element of $Hom_A(\underline{M}, R)$, and

$$(f_g \rightharpoonup q^h_\beta)(m_\beta) = f_g\sharp(q^h_\beta(m_\beta)) \in R_{gh\beta}.$$

Thus, $R_g \rightharpoonup Q_h \subseteq Q_{gh}$. We have proved Q is a left R-graded module.

(2) Define the right T-graded module action on Q by $(q \leftharpoonup t)(m) = q(t(m))$. For all $m_\alpha \in M_\alpha, q^g \in Q^g, t^h \in T^h$, we have

$$\begin{aligned}
&(q^g \leftharpoonup t^h)(m_{[0,\beta^{-1}]})(c_{\beta h^{-1}g^{-1}})m_{[1,\beta\alpha]} \\
=&q^g_{h\beta^{-1}}(t^h(m_{[0,\beta^{-1}]}))(c_{\beta h^{-1}g^{-1}})m_{[1,\beta\alpha]} \\
=&q^g_{h\beta^{-1}}(t^h(m_\alpha)_{[0,h\beta^{-1}]})(c_{\beta h^{-1}g^{-1}})(t^h(m_\alpha)_{[1,\beta\alpha]} && \text{by (2)} \\
=&c_{(1,\beta\alpha)}q^g_{h\alpha}(t^h(m_\alpha))(c_{(2,\alpha^{-1}h^{-1}g^{-1})}) && \text{by (1)} \\
=&c_{(1,\beta\alpha)}(q^g \leftharpoonup t^h)(m_\alpha)(c_{(2,\alpha^{-1}h^{-1}g^{-1})}).
\end{aligned}$$

Hence, $q \leftharpoonup t$ satisfies (1). At the same time, the action is a right A-linear map as q and t are right A-linear, and

$$(q^h \leftharpoonup t^g)(m_\alpha) = q^h(t^g(m_\alpha)) \in R_{hg\alpha}.$$

(3) Finally,

$$\begin{aligned}
((f^g \rightharpoonup q^h) \leftharpoonup t^l)(m) =&(f^g \rightharpoonup q^h)(t^l(m)) \\
=&f^g\sharp(q^h(t^l(m))) \\
=&f^g\sharp(q^h \leftharpoonup t^l(m))) \\
=&(f^g \rightharpoonup (q^h \leftharpoonup t^l))(m).
\end{aligned}$$

Thus, we have proved Q is a G-graded R-T bimodule. ■

Lemma 4. *(1) Q is a k-graded submodule of $HOM_R(\underline{M}, R)$;*

(2) M, N be two right G-$\underline{\mathcal{C}}$-comodules, then for every $n_\alpha \in N_\alpha$, there is a k-linear map $Q \to HOM^{G,\underline{\mathcal{C}}}(\underline{M}, \underline{N})$ given by $q^g \mapsto (n_\alpha \leftharpoonup q^g(-))$.

Proof. (1) For all $q^g \in Q^g,\ m_\alpha \in M_\alpha,\ f_\beta \in R_\beta,\ c_{\beta^{-1}\alpha^{-1}g^{-1}} \in C_{\beta^{-1}\alpha^{-1}g^{-1}}$, we have

$$\begin{aligned}
&q^g(m_\alpha \leftharpoonup f_\beta)(c_{\beta^{-1}\alpha^{-1}g^{-1}})\\
=&q^g(m_{[0,\alpha\beta]}f(m_{[1,\beta^{-1}]}))(c_{\beta^{-1}\alpha^{-1}g^{-1}})\\
=&q^g(m_{[0,\alpha\beta]})(f(m_{[1,\beta^{-1}]}))(c_{\beta^{-1}\alpha^{-1}g^{-1}}) && \text{right A-linearity of Q}\\
=&q^g(m_{[0,\alpha\beta]})(c_{\beta^{-1}\alpha^{-1}g^{-1}})(f(m_{[1,\beta^{-1}]})) && \text{R is A-bimodule}\\
=&f(q^g(m_{[0,\alpha\beta]})(c_{\beta^{-1}\alpha^{-1}g^{-1}})(m_{[1,\beta^{-1}]})) && \text{R is left A-linear}\\
=&f(c_{(1,\beta^{-1})}q^g(m_\alpha)(c_{(2,\alpha^{-1}g^{-1})})) && \text{by (1)}\\
=&(q^g(m_\alpha)\sharp f_\beta)(c_{\beta^{-1}\alpha^{-1}g^{-1}}).
\end{aligned}$$

Hence, we have

$$q^g(m_\alpha \leftharpoonup f_\beta) = q^g(m_\alpha)\sharp f_\beta \quad (*).$$

(2) Since R is a G-graded A-ring, and the elements of Q are right A-linear, the map $m_\beta \rightharpoonup n_\alpha \leftharpoonup q^g(m_\beta)$ is right A-linear.

$$\begin{aligned}
&(n_\alpha \leftharpoonup q^g(m_\beta))_0 \otimes (n_\alpha \leftharpoonup q^g(m_\beta))_1\\
=&(n_{[0,\alpha\beta g]}q^g(m_\beta)(n_{[1,\beta^{-1}g^{-1}]}))_{[0,\alpha\beta g\gamma]} \otimes (n_{[0,\alpha\beta g]}q^g(m_\beta)(n_{[1,\beta^{-1}g^{-1}]}))_{\gamma^{-1}}\\
=&n_{[0,\alpha\beta g\gamma]} \otimes n_{[1,\gamma^{-1}]}q^g(m_\beta)(n_{[2,\beta^{-1}g^{-1}]})\\
=&n_{[0,\alpha\beta g\gamma]} \otimes q^g_{\beta\gamma}(m_{[0,\beta\gamma]})(n_{[1,\gamma^{-1}\beta^{-1}g^{-1}]})m_{[1,\gamma^{-1}]} && \text{by (1)}\\
=&n_{[0,\alpha\beta g\gamma]}q^g_{\beta\gamma}(m_{[0,\beta\gamma]})(n_{[1,\gamma^{-1}\beta^{-1}g^{-1}]}) \otimes m_{[1,\gamma^{-1}]}\\
=&n_\alpha \leftharpoonup q^g(m_{[0,\beta\gamma]}) \otimes m_{[1,\gamma^{-1}]}.
\end{aligned}$$

Especially, for every $n_e \in N_e$, there is a k-linear map $Q^e \to Hom^{G,\mathcal{C}}(\underline{M}, \underline{N}),\ m_\alpha \mapsto n_e q^e_\beta(m_\alpha)$. ■

Lemma 5. *Define left T action on $\bigoplus_{\alpha\in G} M_\alpha$ by $t^l \rightharpoonup m_\alpha = t^l(m_\alpha) \in M_{l\alpha}$, right R action on $\bigoplus_{\alpha\in G} M_\alpha$ by $m_\alpha \leftharpoonup f_\beta = m_{[0,\alpha\beta]}f(m_{[1,\beta^{-1}]}) \in M_{\alpha\beta}$. Then, $\bigoplus_{\alpha\in G} M_\alpha$ is a G-graded T-R bimodule.*

Proof. Obviously, $\bigoplus_{\alpha\in G} M_\alpha$ is a left graded T-module. By [7, Proposition 4.1] $\bigoplus_{\alpha\in G} M_\alpha$ is a right G-graded R-module and

$$
\begin{aligned}
& t^l \rightharpoonup (m_\alpha \leftharpoonup f_\beta) \\
=& t^l(m_{[0,\alpha\beta]} f(m_{[1,\beta^{-1}]})) \\
=& t^l(m_{[0,\alpha\beta]}) f(m_{[1,\beta^{-1}]}) && \text{right A-linearity of T} \\
=& (t^l(m_\alpha)_{[0,\alpha\beta]} f(t^l(m_\alpha)_{[1,\beta^{-1}]} && \text{by (2)} \\
=& t^l(m_\alpha) \leftharpoonup f_\beta \\
=& (t^l \rightharpoonup m_\alpha) \leftharpoonup f_\beta .
\end{aligned}
$$

Hence, $\bigoplus_{\alpha\in G} M_\alpha$ is a G-graded T-R bimodule. ■

Theorem 6. *Let $\underline{\mathcal{C}}$ be a G-A-coring, R be the left dual graded ring. $T = HOM^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M}) = \bigoplus_{g\in G} HOM^{G,\underline{\mathcal{C}}}(\underline{M},\underline{M})_g$, $\underline{M} \in \mathcal{M}^{G,\underline{\mathcal{C}}}$, $Q = \bigoplus_{g\in G} Q^g$. We have the following graded Morita context*

$$M = (T,\ R,\ \bigoplus_{\alpha\in G} M_\alpha,\ Q,\ \tau, \mu)$$

with connecting homomorphisms

$$\tau:\ {}_T(\bigoplus_{\alpha\in G} M_\alpha)\otimes Q_T \to T,\ m\otimes q \mapsto (m \leftharpoonup q(-))$$

$$\mu:\ {}_R Q\otimes (\bigoplus_{\alpha\in G} M_\alpha)_R \to R,\ q\otimes m \mapsto q(m).$$

Proof. First, we check τ is a T-T bimodule map, for all $t^\alpha \in T_\alpha$, $m_g \in M_g$, $q^h \in Q^h$, $m_\beta \in M_\beta$:

$$
\begin{aligned}
(t^\alpha \rightharpoonup \tau(m_g\otimes q^h))(m_\beta) &= t^\alpha(\tau(m_g \otimes q^h(m_\beta)) \\
&= t^\alpha(m_g q^h(m_\beta)) \\
&= t^\alpha(m_{[0,gh\beta]}) q^h(m_\beta)(m_{[1,\beta^{-1}h^{-1}]}) && \text{T is right A-linear} \\
&= t^\alpha(m_g)_{[0,\alpha gh\beta]} q^h(m_\beta)(t^\alpha(m_g)_{[1,\beta^{-1}h^{-1}]}) && \text{by (2)} \\
&= t^\alpha(m_g) \leftharpoonup q^h(m_\beta) \\
&= \tau(t^\alpha(m_g)\otimes q^h)(m_\beta) \\
&= \tau(t^\alpha \rightharpoonup m_g \otimes q^h)(m_\beta),
\end{aligned}
$$

and

$$\tau(m_g \otimes q^h) \leftharpoondown t^\alpha = m_g q^h(-) \leftharpoondown t^\alpha = m_g q^h(t^\alpha(-))$$
$$= \tau(m_g \otimes q^h(t^\alpha(-))) = \tau(m_g \otimes q^h \leftharpoondown t^\alpha).$$

Next, we have:

$$\begin{aligned}(\mu(q^h \otimes m_\alpha) \leftharpoondown f^g) &= q^h(m_\alpha) \leftharpoondown f^g \\ &= q^h(m_\alpha) \sharp f^g \\ &= q^h(m_\alpha \leftharpoondown f^g) \qquad \text{by (*)} \\ &= \mu(q^h \otimes m_\alpha \leftharpoondown f^g),\end{aligned}$$

and

$$f^g \rightharpoonup \mu(q^h \otimes m_\alpha) = f^g \rightharpoonup (q^h(m_\alpha)) = f^g \sharp (q^h(m_\alpha))$$
$$= (f^g \rightharpoonup q^h)(m_\alpha) = \mu(f^g \rightharpoonup q^h \otimes m_\alpha).$$

Finally, we check $\mu(q \otimes m)p = q\tau(m \otimes p),\ m\mu(q \otimes m') = \tau(m \otimes q)m'$, for all $q^g \in Q^g$,
$m_\alpha \in M_\alpha$, $p^h \in Q^h$, we have:

$$\begin{aligned}&(\mu(q^g \otimes m_\alpha) \leftharpoondown p^h(m_\beta) \\ =&\mu(q^g \otimes m_\alpha) \sharp p^h(m_\beta) \\ =&q^g(m_\alpha) \sharp p^h(m_\beta) \qquad \text{by (*)} \\ =&q^g(m_\alpha \leftharpoondown p^h(m_\beta)) \\ =&q^g(\tau(m_\alpha \otimes p^h(m_\beta)) \\ =&(q^g \leftharpoondown \tau(m_\alpha \otimes p^h))(m_\beta).\end{aligned}$$

and for all $m, m' \in M$, $q \in Q$

$$m\mu(q \otimes m') = mq(m') = (mq(-))m' = \tau(m \otimes q)m'. \qquad \blacksquare$$

We say that a G-A-coring $\underline{\mathcal{C}}$ is left homogeneously finite if every $\mathcal{C}_\alpha$ is finitely generated and projective as a left A-module.

Remark 7. In the case when $\underline{\mathcal{C}}$ is left homogeneously finite, Q has a particularly simple characterization as $Q = HOM_R(\underline{M}, R)$. By the above Lemma, $Q \subseteq HOM_R(\underline{M}, R)$. The converse inclusion is proven as follows. For every $c_\alpha \in \mathcal{C}_\alpha$, by the left homogeneously finite of $\underline{\mathcal{C}}$, we have the finite dual basis $f_i^{(\alpha)} \otimes e_i^{(\alpha)} \in R_{\alpha^{-1}} \otimes_A \mathcal{C}_\alpha$ of C_α as a left A-module, such that $c_\alpha = \sum f_i^{(\alpha)}(c_\alpha)e_i^{(\alpha)}$. Then, for all $q^\sigma \in HOM_R(\underline{M}, R)_\sigma$, we have

$$\begin{aligned} q^\sigma(m_{[0,\beta]})(c_{\beta^{-1}\sigma^{-1}})m_{[1,\alpha]} &= q^\sigma(m_{[0,\beta]})(c_{\beta^{-1}\sigma^{-1}})f_i^{(\alpha)}(m_{[1,\alpha]})e_i^{(\alpha)} \\ &= q^\sigma(m_{[0,\beta]}f_i^{(\alpha)}(m_{[1,\alpha]}))(c_{\beta^{-1}\sigma^{-1}})e_i^{(\alpha)} \\ &= q^\sigma(m_{\beta\alpha} \leftharpoonup f_i^{(\alpha)})(c_{\beta^{-1}\sigma^{-1}})e_i^{(\alpha)} \\ &= (q^\sigma(m_{\beta\alpha}) \leftharpoonup f_i^{(\alpha)})(c_{\beta^{-1}\sigma^{-1}})e_i^{(\alpha)} \qquad \text{Q is right R-linear} \\ &= (q^\sigma(m_{\beta\alpha})\sharp f_i^{(\alpha)})(c_{\beta^{-1}\sigma^{-1}})e_i^{(\alpha)} \\ &= f_i^{(\alpha)}(c_{(1,\alpha)}q^\sigma(m_{\beta\alpha})(c_{(2,\alpha^{-1}\beta^{-1}\sigma^{-1})}))e_i^{(\alpha)} \\ &= c_{(1,\alpha)}q^\sigma(m_{\beta\alpha})(c_{(2,\alpha^{-1}\beta^{-1}\sigma^{-1})}). \end{aligned}$$

This shows that q^σ belongs to the k-module Q.

Proposition 8. *Assume $\underline{\mathcal{C}}$ is left homogeneously finite G-A-coring, we have $HOM_R(\underline{M}, R) = HOM^{G,\mathcal{C}}(\underline{M}, R)$.*

Proof. For every $T^\sigma \in HOM^{G,\mathcal{C}}(\underline{M}, R)$, we have

$$\begin{aligned} T^\sigma(m_\alpha \leftharpoonup f_\beta) &= T^\sigma(m_{[0,\alpha\beta]}f(m_{[1,\beta^{-1}]})) = T^\sigma(m_{[0,\alpha\beta]})f(m_{[1,\beta^{-1}]}) \\ &= (T^\sigma(m_\alpha))_{[0,\sigma\alpha\beta]}f_\beta(T^\sigma(m_\alpha))_{[1,\beta^{-1}]} \\ &= T^\sigma(m_\alpha) \leftharpoonup f_\beta. \end{aligned}$$

Thus $T^\sigma \in HOM_R(\underline{M}, R)$, and the proposition is completed. ■

Since any right G-$\mathcal{C}$-comodule $\underline{M}$ has also a right R-module structure, we can associate a further Morita context with it, namely,

$$N = (END_R(\underline{M}),\ \bigoplus_{\alpha\in G} M_\alpha,\ HOM_R(\underline{M}, R),\ \tau,\ \mu)$$

with connecting maps

$$\tau:\ HOM_R(\underline{M}, R) \otimes (\bigoplus_{\alpha\in G} M_\alpha) \to R, \quad q\otimes m \mapsto q(m)$$

$$\mu:\ (\bigoplus_{\alpha\in G} M_\alpha) \otimes HOM_R(\underline{M}, R) \to (END_R(\underline{M}), \quad m\otimes q \mapsto m \leftharpoonup q(-).$$

Remark 9. Let $\underline{\mathcal{C}}$ be a G-A-coring, and $\underline{M}$ a right G-$\underline{\mathcal{C}}$-comodule. There exists a morphism of Morita context $M \to N$ which becomes an isomorphism if $\underline{\mathcal{C}}$ is left homogeneously finite.

Lemma 10. *Let $\underline{\mathcal{C}}$ be a G-A-coring, and $\underline{M}$ a right G-$\underline{\mathcal{C}}$-comodule. Consider the Morita context $M = (T, R, \bigoplus_{\alpha\in G} M_\alpha, Q, \tau, \mu)$,*

(1) *If μ is surjective, then $\mathcal{C}_e$ is finitely generated projective left A-module.*

(2) *If τ is surjective, then M_e is a finitely generated projective right A-module.*

Proof. The proof is similar to [3, Lemma 2.5] ∎

At last, we apply [4, Theorem 1] to the graded Morita context.

Theorem 11. *Let $\underline{\mathcal{C}}$ be a G-A-coring, R be the left dual graded ring. $T = HOM^{G,\underline{\mathcal{C}}}(\underline{M}, \underline{M}) = \bigoplus_{g\in G} HOM^{G,\underline{\mathcal{C}}}(\underline{M}, \underline{M})_g$, $Q = \bigoplus_{g\in G} Q^g$, consider the graded Morita context*

$$M = (T,\ R,\ \bigoplus_{\alpha\in G} M_\alpha\ Q,\ \tau, \mu),$$

and suppose $\tau_e,\ \mu_e$ are surjective. Then

(1) *τ and μ are isomorphism.*

(2) *$\bigoplus_{\alpha\in G} M_\alpha$ is projective in GrMod-R in such a way that $\forall\, \alpha \in G$, M_α is a direct sum of R, Q_e is a generator in Mod_{R_e}, and in ${}_{T_e}Mod$.*

(3) *The pair of functors $- \otimes_R Q$ and $- \otimes_{T_e} (\bigoplus_{\alpha\in G} M_\alpha)$ form an equivalence of the categories ${}_R Mod$ and ${}_T Mod$. The equivalence preserves the structure of graded modules.*

3. Graded Morita-takeuchi Theory for Graded Comodule

Associated with Morita context it is possible to establish several equivalences between some subcategories of modules. Equally the (graded) Morita-Takeuchi context plays an important role in the study of (graded) equivalences between (graded) coalgebras. In this section, we recall the definition of (graded) Morita-Takeuchi context, prove the graded Morita-Takeuchi theory for comodule category on coalgebras over field.

Let C, D be coalgebras, $M \in {}^C\mathcal{M}^D$ be a C-D bicomodule. The cotensor product $\Box_C$ determines a k-linear functor $N \mapsto N\Box_C M$ from $\mathcal{M}^C$ to $\mathcal{M}^D$. We call M is quasi-finite if $Com_{-C}(M, M')$ is finite dimensional for all finite dimensional right C-comodule M'.

A right C-comodule M is finitely cogenerated, if it is isomorphic to a subcomodule of $W \otimes M$ for some finite dimensional vector space W. Finitely cogenerated comodules are quasi-finite. The left adjoint of $W \to W \otimes M$ is written as $M' \mapsto h_{-C}(M, M')$

from right C-comodule category to the category of finite dimensional vector spaces. Takeuchi has proved [18] the co-hom $h_{-C}(M, M')$ is a cotra-variant functor of M and a covariant functor of N, and the co-end $e_{-C}(M) = h_{-C}(M, M)$ has a coalgebra structure.

For a bicomodule ${}^{C}N^{D}$, N is quasi-finite if and only if the functor $\mathcal{M}^C \to \mathcal{M}^D$, $M \mapsto M\square_C N$ has the left adjoint. In this case the left adjoint of $M \to M\square_C N$ is given by $M' \mapsto h_{-C}(M, M')$. We refer the reader to [18] for a general theory of Morita-Takeuchi theory.

Definition 12. [18] A Morita-Takeuchi context $= (C,\ D,\ {}^{C}M^{D},\ {}^{D}N^{C},\ \tau,\ \mu)$ consists of coalgebras C and D, bicomodules ${}^{C}M^{D}$ and ${}^{D}N^{C}$, and bicolinear maps $\tau : C \to M\square_D N$ and $\mu : D \to N\square_C M$ making the following diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\cong} & M\square_D D \\ \cong\downarrow & & \downarrow id\square\mu \\ C\square_C M & \xrightarrow[\tau\square id]{} & M\square_D N\square_C M \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\cong} & N\square_C C \\ \cong\downarrow & & \downarrow id\square\tau \\ D\square_D N & \xrightarrow[\mu\square id]{} & N\square_C M\square_D N \end{array}$$

The context is said to be strict if both τ and μ are injections (equivalently, isomorphisms). In this case, the categories $\mathcal{M}^C$ and $\mathcal{M}^D$ are equivalent and we say that C is Morita-Takeuchi equivalent to D.

Now, we begin to develop a graded version of Morita-Takeuchi theory. The presentation here is modeled on that given in [18]. We first recall some definition on graded coalgebras and graded comodules, give the graded Morita-Takeuchi context and prove a theorem which we titled graded Morita-Takeuchi theory, which characterizes equivalences of comodule categories over fields.

A coalgebra C is called G-graded coalgebra if C is a direct sum $C = \oplus_{\sigma\in G} C_\sigma$ of k-space and verifies:

(1) $\triangle(C_\sigma) \subseteq \sum_{\lambda\mu=\sigma} C_\lambda \otimes C_\mu$ for any $\sigma \in G$;

(2) $\epsilon(C_\sigma) = 0$ for any $\sigma \neq e$.

A coalgebra $C = \oplus_{\sigma\in G} C\sigma$ is said to be of finite type if, for all $\sigma \in G$, C_σ is finite dimensional over k. Note that it does not mean that $C = \oplus_{\sigma\in G} C_\sigma$ is finite dimensional (unless $C_\alpha = 0$ for all but a finite number of $\alpha \in G$).

Let M be a right C-comodule, M is called a G-graded comodule over C if M admits a decomposition as a direct sum $M = \oplus_{\sigma\in G} M_\sigma$ of k-space, and $\rho_M(M_\sigma) \subseteq \sum_{\lambda\mu=\sigma} M_\lambda \otimes C_\mu$ for any $\sigma \in G$.

If $M = \oplus_{\sigma\in G} M_\sigma$ is a graded right C-comodule and $\sigma, \tau \in G$, we denote by $\pi_\sigma : M \to M_\sigma$ the canonical projection and by $\rho_{\sigma,\tau} : M_{\sigma\tau} \to M_\sigma \otimes C_\tau$ the unique k-morphism. Then we may define a coalgebra structure $(C_1, \triangle_1 = \rho_{e,e}, \epsilon)$ on C_e and $\pi_1 : C \to C_1$ is a morphism of coalgebras. Moreover, for any $\sigma \in G$, M_σ is a right C_1-comodule via the canonical map $\rho : M_\sigma \to M_\sigma \otimes C_e$, i.e $\rho_{\sigma,e}(m) = \sum m_{[0]} \otimes \pi_1(m_{[1]})$ for any $m \in M_\sigma$.

Definition 13. A G-graded Morita-Takeuchi context is a set $(C,\ D,\ {}^CM^D,\ {}^DN^C,\ \tau,\ \mu)$ of objects which we now define. C and D are G-graded coalgebras. ${}^CM^D$ is a C-D-bicomodule which is G-graded i.e., $({}^C\rho \otimes 1)\rho^D(M_\sigma) \subseteq \sum_{\alpha\beta\gamma=\sigma} C_\alpha \otimes M_\beta \otimes D_\gamma$. N is a D-C-bicomodule which is also graded. $\tau : C \to M\Box_D N$ is a C-bicolinear homomorphism which is graded in the sense that $\tau(C_\sigma) \subseteq \sum_{\alpha\beta=\sigma} M_\alpha \otimes N_\beta$, $\mu : D \to N\Box_C M$ is a D-bicolinear homomorphism which is also graded. Lastly, τ and μ make the following diagrams commute:

$$\begin{array}{ccc} M & \xrightarrow{\ \cong\ } & M\Box_D D \\ \Big\downarrow \cong & & \Big\downarrow id\Box\mu \\ C\Box_C M & \xrightarrow[\tau\Box id]{} & M\Box_D N\Box_C M \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\ \cong\ } & N\Box_C C \\ \Big\downarrow \cong & & \Big\downarrow id\Box\tau \\ D\Box_D N & \xrightarrow[\mu\Box id]{} & N\Box_C M\Box_D N \end{array}$$

The context is said to be strict if τ and μ are graded bicolinear isomorphisms.

We now explain the use of the word "graded" in the phrase "graded Morita context". Let $\sigma \in G$, there is a natural C_e-D_e-bicomodule action on M_σ and D_e-C_e-bicomodule action on $N_{\sigma^{-1}}$. Let $\tau_{\sigma,\sigma^{-1}}$ denote the map obtained as by the restriction of the map τ from C_e to $M_\sigma\Box_{D_e}N_{\sigma^{-1}}$ and $\mu_{\sigma^{-1},\sigma} : D_e \to N_{\sigma^{-1}}\Box_{C_e}M_\sigma$. Then $(C_e,\ D_e,\ {}^{C_e}M_\sigma^{D_e},\ {}^{D_e}N_{\sigma^{-1}}^{C_e},\ \tau_{\sigma,\sigma^{-1}},\ \mu_{\sigma^{-1},\sigma})$ is a Morita-Takeuchi context.

Consider the definition of a graded Morita-Takeuchi context in the special case $G = e$. This is exactly the definition of a Morita-Takeuchi context.

Theorem 14. (graded Morita-Takeuchi theory). *Let* $(C,\ D,\ {}^CM^D,\ {}^DN^C,\ \tau,\ \mu)$ *be a graded Morita-Takeuchi context in which* $\tau_{e,e}$, $\mu_{e,e}$ *are injective. Then*

(1) τ *and* μ *are isomorphisms;*

(2) M^D *is quasi-finite in right* D*-comodule category,* M *is a cogenerator in left* G*-graded* C*-comodule category.* M_e *is a cogenerator in right* D_e*-comodule and left* C_e*-comodule category. Similar statements hold for* N*;*

(3) μ *induces a graded bicomodules isomorphism* $\iota : h_{-D}(M, D) \mapsto {}^DN^C$ *and* $h_{D-}(N, D) \mapsto {}^CM^D$ *We also have similar graded bicomodule isomorphisms of* $h_{-C}(N, C)$ *with* M *and* $h_{C-}(M, C)$ *with* N*;*

(4) *The bicomodule structures of M and N induce graded coalgebra isomorphisms $\lambda : e_{-D}(M) \simeq C$ and $e_{D-}(N) \simeq C$. Similarly, we have $e_{C-}(M) \simeq D$ and $e_{-C}(N) \simeq D$.*

(5) *The pair of functors $F = -\Box_D N$ and $G = -\Box_C M$ form a graded equivalence of the graded right D-comodule and graded right C-comodule categories. The functors $S = N\Box_C-$ and $T = M\Box_D-$ form a graded equivalence of the graded left C-comodule and graded left D-comodule categories.*

Proof.

(1) First, we have $\tau_{\sigma,\sigma^{-1}}$ are injective since $\tau_{e,e}$ is injective. Part (1) follows from the application of Morita-Takeuchi Theory to the graded context $(C_e,\ D_e,\ {}^{C_e}M_\sigma^{D_e},\ {}^{D_e}N_{\sigma^{-1}}^{C_e},\ \tau_{\sigma,\sigma^{-1}},\ \mu_{\sigma^{-1},\sigma})$.

(2) Since τ and μ are isomorphisms, then, $\tau^{-1} : FG \to Id$, $\mu : Id \to TS$ give an adjoint relation. Hence, M is quasi-finite as right D-comodule. Since F is exact, G preserves injective, $M^D = G(C)$ is injective, thus, M is a cogenerator follows from $C \simeq M\Box_D N \hookrightarrow M \otimes N$. M_e is a cogenerator in right D_e-comodule category follows from the Morita-Takeuchi Theory.

(3) The map ι is a bicomodule isomorphism by Morita-Takeuchi Theory. We show that ι respects the graded structure of comodules. Let $\sigma \in G$, and $f_\sigma \in h_{-D}(M, D)$, then $\mu(f_\sigma(M_\alpha)) \subseteq \mu(D_{\sigma\alpha}) \subseteq N_\sigma \otimes M_\alpha$, hence, $\iota(f_\sigma) \subseteq N_\sigma$.

(4) Applying Morita-Takeuchi theory once again, we see that the map λ in (4) is a coalgebra isomorphism. We show that λ respects the graded structure of coalgebras. Let $f_\sigma \in e_{-D}(M)$, then, $f_\sigma(M_\alpha) \subseteq M_{\sigma\alpha} \xrightarrow{\rho} C_\sigma \otimes M_\alpha$, thus, we have $\lambda(f_\sigma) \subseteq C_\sigma$.

(5) Let U be a graded right C-comodule. The corresponding comodule in right D-comodule category we get is $U' = U\Box_C M$. If we pass back to the right C-comodule category we get $U'' = U\Box_C M\Box_D N$. U'' is isomorphic to U via the map $\theta : u\otimes m\otimes n \mapsto u\tau(m\otimes n)$, for all $u \in U$, $m \in M$, $n \in N$. Our vague statement that the equivalence preserves graded comodules means that there is a natural graded structure on U' and U'', and θ is a graded isomorphism. Now, we give the grading of U' and U''. The σ-component of U'_σ is generated as additive group by the set of all $u\Box_C m$, for any $u \in U_\sigma$ and $m \in M_e$, it is straightforward to check that the induced comodule action preserves the grading. Imitate this construction for U'', $U''_\sigma = \{u \otimes m \otimes n,\ u \in U_\sigma, m \in M_e, n \in N_e\}$. The image of U''_σ is the set $\{u\tau(m \otimes n)\} \subseteq U_\sigma$, thus θ is a graded map. ∎

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