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BLOW-UP RATE FOR NON-NEGATIVE SOLUTIONS OF A NON-LINEAR PARABOLIC EQUATION

Chi-Cheung Poon

Abstract. We study solutions of the equation

$$u_t = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial}{\partial x_i}u^m\right) + hu^p \qquad \text{on} \quad \Omega \times (0,T)$$
$$u = 0 \qquad \text{on} \quad \partial\Omega \times (0,T),$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, and $a^{ij} = a^{ij}(x)$ is uniformly positive definite and $h = h(x) > 0$ on Ω. When

$$0 < m < 1 < p < m + \frac{2}{n+1} \qquad \text{or} \qquad 1 < m < p \le m + \frac{2}{n+1},$$

we will show that if u is a non-negative solution and blows up at T, then

$$u(x,t) \le C|T-t|^{-1/(p-1)}.$$

The proof relies on rescaling arguments and some, old and new, Fujita-type results.

1. INTRODUCTION

In this paper, we study solutions of the equation

$$(1.1) \qquad u_t = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial}{\partial x_i}u^m\right) + hu^p \qquad \text{on} \quad \Omega \times (0,T)$$
$$u = 0 \qquad \text{on} \quad \partial\Omega \times (0,T).$$

Here, $m > 0$ and $p > 1$, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary, and $a^{ij} = a^{ij}(x)$ and $h = h(x)$ are smooth functions defined on $\bar{\Omega}$. We also assume that a^{ij} is uniformly positive definite and $h > 0$ on $\bar{\Omega}$. Suppose that $u(x,t)$ is a non-negative solution of (1.1) and blows up at T. Our goal is to show that there is a constant $C > 0$ such that for all $(x,t) \in \Omega \times (0,T)$, we have

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$$u(x,t) \le C|T-t|^{-1/(p-1)}. \tag{1.2}$$

For the semilinear heat equation

$$\begin{aligned} &u_t = \Delta u + u^p \qquad \text{on} \quad \Omega \times (0,T) \\ &u = 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \end{aligned} \tag{1.3}$$

Friedman and McLeod, [4], proved that if Ω is a convex, bounded domain in $\mathbb{R}^n$, $p > 1$, and $u(x,t)$ is non-negative solution of (1.3), and is non-decreasing in time, then (1.2) holds. Giga and Kohn, [10], among other results, proved that if Ω is a convex, and u is a non-negative solution of (1.3) for

$$1 < p < \frac{n+2}{n-2} \quad \text{when} \quad n \ge 3 \qquad \text{or} \quad p > 1 \quad \text{when} \quad n \le 2,$$

then (1.2) holds. Fila and Souplet, [3], proved the same result for solutions defined on domains which may not be convex, but only for

$$1 < p < 1 + \frac{2}{n+1}.$$

Let $u(x,t)$ be a non-negative solution of (1.3). Let

$$M(t) = \max\left\{u(x,t) : (x,t) \quad \text{in} \quad \Omega \times [0,t]\right\}.$$

For any $t_0 \in (0,T)$, we define t_0^+ by

$$t_0^+ = \max\{t \in (t_0,T) : \ M(t) = 2M(t_0)\}.$$

In [3], Fila and Souplet showed that, if u blows up at T, then there is a constant K such that for all $t_0 \in (T/2,T)$, we have

$$M^{p-1}(t_0)(t_0^+ - t_0) \le K. \tag{1.4}$$

We briefly describe their arguments: if (1.4) is not true, there is a sequence of the rescaled solutions converges to a non-trivial bounded global non-negative solution of the equation

$$v_t = \Delta v + v^p \tag{1.5}$$

defined on $\mathbb{R}^n \times (-\infty,\infty)$, or to a non-trivial bounded global non-negative solution, v, of (1.5) which is defined on $\mathbb{R}^n_+ \times (-\infty,\infty)$ and $v = 0$ whenever $x_1 = 0$. Here, we use the notation

$$\mathbb{R}^n_+ = \{x : x_1 > 0\}.$$

However, Fujita, [5], proved that there is no non-trivial global non-negative solution of (1.5) defined on $\mathbb{R}^n \times (0, \infty)$, and Meier, [15], proved that there is no non-trivial global non-negative solution v of (1.5) which is defined on $\mathbb{R}^n_+ \times (0, \infty)$ and $v = 0$ whenever $x_1 = 0$. Thus, (1.4) has to be true. Using an argument of Hu, [11], from (1.4), we obtain (1.2) immediately. The result of Fila and Souplet was improved to the case

$$1 < p < \frac{n(n+1)}{(n-1)^2} \quad \text{if} \quad n \geq 2, \qquad p > 1 \quad \text{if} \quad n = 1.$$

See [18] for more information concerning solutions of the semilinear equation (1.5).

We will use Fila and Souplet's method to study solutions of equation (1.1). In our situations, we need some non-existence results for solutions of

$$v_t = \Delta v^m + v^p \tag{1.6}$$

defined either on $\mathbb{R}^n \times (0, \infty)$ or $\mathbb{R}^n_+ \times (0, \infty)$. Galaktionov, [6] [7], proved that when $1 < m < p < m + 2/n$, then any global solutions of (1.6) in $\mathbb{R}^n \times (0, \infty)$ is identically zero. Kawanago, [12], Mochizuki and Suzuki, [17], proved the case when $m > 1$ and $p = m + 2/n$. If v is a solution of (1.6) in $\mathbb{R}^n_+ \times (0, \infty)$, $v = 0$ when $x_1 = 0$, Lian and Liu, [14], proved that when

$$1 < m < p \leq m + \frac{1}{n+1}, \tag{1.7}$$

then v vanishes identically. See also [13]. In [20], Qi proved that when $0 < m < 1 < p < m + 2/n$, there is no non-trivial global non-negative solution of (1.6) defined on $\mathbb{R}^n \times (0, \infty)$. When $1 < p = m + 2/n$ and $\max\{0, 1 - 2/n\} < m < 1$, Mochizuki and Mukai, [16], proved the same result. Following Qi's idea, we will prove that when

$$0 < m < 1 < p < m + \frac{2}{n+1}, \tag{1.8}$$

there is no non-trivial global non-negative solution, v, of the equation (1.6) which is defined on the half-space $\mathbb{R}^n_+ \times (0, \infty)$ and $v = 0$ whenever $x_1 = 0$. However, when

$$1 < p = m + \frac{2}{n+1} \qquad \text{and} \qquad \max\left\{0, 1 - \frac{2}{n+1}\right\} < m < 1, \tag{1.9}$$

we are not able to prove the same result as in [16].

Suppose that u is a solution of (1.1) and u blows up at T. In cases (1.7) and (1.8), we will show that (1.2) holds by proving that (1.4) is true. If (1.4) is not true, we show that there is a sequence of rescaled solutions which converges to a solution v of the equation (1.6) on $\mathbb{R}^n \times (-\infty, \infty)$ or on $\mathbb{R}^n_+ \times (-\infty, \infty)$ and $v = 0$ whenever $x_1 = 0$. This contradicts the non-existence results described in the above. We note that, as in [3], the domain Ω need not be convex.

Using the same scheme, we also consider solutions of (1.1) which are non-decreasing in time. Suppose that $0 < m < 1$ and

$$1 < p \quad \text{when} \quad n \leq 2, \qquad 1 < p < m\left(\frac{n+2}{n-2}\right) \quad \text{when} \quad n \geq 3;$$

or, $m > 1$ and

$$m < p \quad \text{when} \quad n \leq 2, \qquad m < p < m\left(\frac{n+2}{n-2}\right) \quad \text{when} \quad n \geq 3. \tag{1.10}$$

Let u be a non-negative solution of (1.1) and is non-decreasing in time. If u blows up at T, then we show that (1.2) holds. This technique was used to treat time non-decreasing solutions of the semilinear equation (1.3). See [18], Remarks 4.3(b) and 5.3(b).

In a recent paper, Souplet, [22], proved that, when $1 < m < p$ and p satisfies (1.10), then the equation (1.6) has no non-trivial bounded radial non-negative solution defined on $\mathbb{R}^n \times (-\infty, \infty)$. This implies that, if $1 < m < p$ and (1.10) is satisfied, then (1.2) holds for radial solutions of (1.1) defined on symmetric domains. See [1].

Suppose that in (1.1), we have $a^{ij}(x) = \delta^{ij}$ and $h(x) = 1$ for all $x \in \Omega$. When $m > 0$, by taking $U = u^m$, $M = (m-1)/m$ and $P = p/m$, we see that (1.1) is equivalent to

$$\begin{aligned} &U_t = U^M(\Delta U + U^P) \qquad \text{on} \quad \Omega \times (0, T) \\ &U = 0 \qquad \text{on} \quad \partial\Omega \times (0, T). \end{aligned} \tag{1.11}$$

When $0 < M < 2$ and $P = 1$, Winkler, [23], proved that if U is a non-negative solution of (1.11), then there is a constant $C > 0$ such that

$$U(x,t) \leq C|T-t|^{-1/(P+M-1)}.$$

Also, when $M \geq 2$ and $P = 1$, Winkler, [24], proved that if U is a non-negative solution of (1.11), then $|T-t|^{1/(P+M-1)}u(x,t)$ becomes unbounded as $t \to T$. Furthermore, there are solutions of (1.11) with $M \geq 2$ and $P > 1$, for which $|T-t|^{1/(P+M-1)}u(x,t)$ becomes unbounded as $t \to T$. See [19].

2. Regularity Theorems

We consider the weak solutions of the equation

$$\begin{aligned} &u_t = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial}{\partial x_i}u^m\right) + hu^p \qquad \text{on} \quad \Omega \times (t_1, t_2) \\ &u = 0 \qquad \text{on} \quad \partial\Omega \times (t_1, t_2). \end{aligned} \tag{2.1}$$

Here, $m > 0$ and $p > 1$, and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. We assume that $a^{ij} = a^{ij}(x)$ and $h = h(x)$ are continuous functions defined in $\bar{\Omega}$ and there are positive constant c_0 and c_1 such that for any vector $\xi \in \mathbb{R}^n$ and $x \in \Omega$, we have

$$c_0|\xi|^2 \leq \sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \leq c_1|\xi|^2 \qquad \text{and} \qquad c_0 \leq h(x) \leq c_1. \tag{2.2}$$

Let u be a function so that $u^m \in L^\infty_{loc}(t_1, t_2; H^1_0(\Omega))$ and $u(x,t) \geq 0$ for all $(x,t) \in \Omega \times (t_1, t_2)$. We say u is a weak solution of (2.1) if for any $\eta \in C^\infty_0(\Omega \times (t_1, t_2))$, we have

$$\int\int \left(u\eta_t + u^m \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial \eta}{\partial x_i}\right) + hu^p\eta \right) dx\ dt = 0.$$

We have the following regularity theorems by DiBenedetto and Sacks. See [2] and [21]. We use the notaions

$$B(x_0, r) = \{x : |x - x_0| < r\} \qquad \text{and} \qquad B(r) = B(0, r) = \{x : |x| < r\}.$$

Theorem 2.1. *Let $u(x,t)$ be a bounded weak solution of the equation*

$$u_t = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial(u^m)}{\partial x_i}\right) + h(x)u^p \tag{2.3}$$

with $m > 0$ and $p > 1$, in $B(2r) \times (-2r, 0)$. Then u is continuous on $B(r) \times (-r, 0)$ and the modulus of continuity of u depends only on $\sup u$, c_0, c_1, m and p only. Here, c_0 and c_1 are the constants in (2.2).

Let Ω be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary. Let $x_0 \in \partial\Omega$. Suppose that there is a constant $\theta \in (0,1)$ and $r_0 > 0$ such that

$$\text{meas}(\Omega \cap B(x_0, r)) \leq (1 - \theta)\text{meas}(B(x_0, r)). \tag{2.4}$$

Theorem 2.2. *Let $u(x,t)$ be a bounded weak solution of the equation (2.3) with $m > 0$ and $p > 1$ in $(B(x_0, 2r) \cap \Omega) \times (-2r, 0)$, and $u = 0$ on $\partial\Omega$. Then u is continuous in $(B(x_0, r) \cap \Omega) \times (-r, 0)$ and the modulus of continuity of u in $(B(x_0, r) \cap \Omega) \times (-r, 0)$ depends only on $\sup u$, c_0, c_1, m, p, r_0 and θ only.*

If we assume that the domain Ω is smooth, then there are constant θ and r_0 such that for all $x_0 \in \partial\Omega$ and $0 < r < r_0$, so that condition (2.4) holds.

3. Non-existence of Global Solutions

Let Ω be a domain in $\mathbb{R}^n$. Let $v(x,t)$ be a non-negative function defined on

$\Omega \times (0, \infty)$. Suppose that $v(x,t)$ is bounded. We say v is a weak solution of the equation

$$v_t = \Delta v^m + v^p \qquad \text{on} \quad \Omega \times (0,\infty)$$
$$v = 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty)$$

with $m > 0,\ p > 1$, if for any function $\eta(x,t) \in C_0^\infty(\Omega \times (0,\infty))$, we have

(3.1) $$\int\int_\Omega (v\eta_t + v^m \Delta\eta + v^p \eta)\ dx\ dt = 0.$$

We state some known results concerning the non-existence of weak solutions. When $m > 1$, and the domain is the whole space, we have

Theorem 3.1. ([6, 7, 12, 17]). *Let v be a bounded non-negative continuous weak solution of the equation*

$$v_t = \Delta(v^m) + v^p \qquad \text{in} \qquad \mathbb{R}^n \times (0,\infty).$$

If

$$1 < m < p \le m + \frac{2}{n},$$

then v is identically zero on $\mathbb{R}^n \times (0,\infty)$.

When $m > 1$, and the domain is the half space, we have

Theorem 3.2. ([13, 14]) *Let v be a bounded non-negative continuous weak solution of the equation*

$$v_t = \Delta(v^m) + v^p \qquad \text{in} \qquad \mathbb{R}^n_+ \times (0,\infty),$$

$$v = 0 \qquad \text{in} \qquad \{x_1 = 0\} \times (0,\infty)$$

If

$$1 < m < p \le m + \frac{2}{n+1},$$

then v is identically zero on $\mathbb{R}^n_+ \times (0,\infty)$.

When, $0 < m < 1$, the non-existence result is of the form:

Theorem 3.3. ([20, 16]) *Let v be a bounded non-negative continuous weak solution of the equation*

$$v_t = \Delta(v^m) + v^p \qquad \text{in} \qquad \mathbb{R}^n \times (0,\infty).$$

If

$$0 < m < 1 \qquad \text{and} \qquad 1 < p < m + \frac{2}{n},$$

or

$$1 < p = m + \frac{2}{n} \qquad \text{and} \qquad \max\left\{0, 1 - \frac{2}{n}\right\} < m < 1,$$

then v is identically zero on $\mathbb{R}^n \times (0,\infty)$.

If the solution is defined on $\mathbb{R}^n_+ \times (0,\infty)$, where $\mathbb{R}^n_+ = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : x_1 > 0\}$, we prove

Theorem 3.4. *Let u be a bounded non-negative continuous weak solution of the equation*

$$v_t = \Delta(v^m) + v^p \qquad \text{in} \qquad \mathbb{R}^n_+ \times (0,\infty),$$
$$v = 0 \qquad \text{in} \qquad \{x_1 = 0\} \times (0,\infty)$$

If

$$0 < m < 1 \qquad \text{and} \qquad 1 < p < m + \frac{2}{n+1},$$

then v is identically zero on $\mathbb{R}^n_+ \times (0,\infty)$.

We will need the following technical lemma.

Lemma 3.5. *Let $w(s)$ be an absolutely continuous positive function which satisfies the inequality*

$$w' \geq -\lambda w^m + w^p \qquad \text{for} \qquad s > 0, \tag{3.2}$$

where $m > 0$, and $p > \max\{m, 1\}$. If $w^{p-m}(0) > \lambda$, then w blows up in finite time.

Proof. Suppose that z is a solution of the equation

$$z' = -\lambda z^m + z^p, \qquad 0 < z(0) < w(0) \qquad \text{and} \qquad z^p(0) > \lambda z^m(0).$$

We first claim that $z'(s) > 0$ whenever $z(s)$ is defined. Assume that there is $s_1 > 0$ such that $z'(s) > 0$ in $(0, s_1)$ and $z'(s_1) = 0$. Then, $z(s) > 0$ and $z^p(s) - \lambda z^m(s) > 0$ in $(0, s_1)$ and $z^p(s_1) - \lambda^m(s_1) = 0$. It is easy to see that

$$\begin{aligned} \frac{d}{ds}\left(z^p(s) - \lambda z^m(s)\right) =& z'(s)\left(\frac{p}{z(s)} z^p(s) - \lambda \frac{m}{z(s)} z^m(s)\right) \\ \geq& \frac{mz'(s)}{z(s)}\left(z^p(s) - \lambda z^m(s)\right). \end{aligned} \tag{3.3}$$

Thus, the function $z^p(s) - \lambda z^m(s)$ is increasing on $(0, s_1)$. If $z^p(s) - \lambda z^m(s) > 0$ for $s \in (0, s_1)$, then $z'(s_1) = z^p(s_1) - \lambda z^m(s_1) > 0$. Thus, $z(s)$ is increasing whenever $z(s)$ is defined. Furthermore, by (3.3), the function $z^p(s) - \lambda z^m(s)$ is also increasing whenever $z(s)$ is defined. This implies that $z'(s)$ is increasing, and $z'(s) \geq z'(0) > 0$, and

$$z(s) \geq z(0) + sz'(0) \tag{3.4}$$

whenever $z(s)$ is defined. When s is large enough, by (3.4), we have $z^m(s) > 2\lambda$ and

$$z^p(s) - 2\lambda z^m(s) = z^{p-m}(z^m(s) - 2\lambda) > 0.$$

Therefore, when s is large enough,

$$z'(s) = z^p(s) - \lambda z^m(s) \geq \frac{1}{2}\, z^p(s).$$

By solving the ODE, we see that, if $p > 1$, the solution z has to blow up in finite time. If w satisfies the inequality (3.2), then $w(s) \geq z(s)$ whenever $z(s)$ and $w(s)$ are defined. Thus, the function w also blows up in finite time. ■

Proof. [Proof of Theorem 3.4] For $x \in \mathbb{R}^n_+$, let

$$\phi(x) = x_1^\gamma \exp(-k^2|x|^2),$$

where $k > 0$ and $\gamma > 1$ are constants to be determined. By straightforward computations, we have

$$\begin{aligned}&\Delta\phi + \lambda\phi \\ &= \left(\lambda - 4k^2\gamma + \gamma(\gamma-1)x_1^{-2} + 4k^4|x|^2 - 2nk^2\right)\phi.\end{aligned}$$

Thus, we see that if $\lambda \geq 2k^2(2\gamma + n)$ then $\Delta\phi + \lambda\phi \geq 0$ in $\mathbb{R}^n_+$. We let $1 < \gamma \leq 2$ and choose

$$\lambda = \lambda(k) = 2(n+4)k^2. \tag{3.5}$$

Then $\Delta\phi + 2\lambda\phi \geq 0$ in $\mathbb{R}^n_+$.

One can compute that

$$\int_{\mathbb{R}^n_+} \phi(x)\ dx = \frac{C(n)\Gamma(\gamma)}{k^{n+\gamma}} \tag{3.6}$$

where

$$\Gamma(\gamma) = \int_0^\infty\ x^\gamma \exp(-x^2)\ dx = \frac{1}{2}\int_0^\infty\ u^{(\gamma-1)/2}e^{-u}\ du,$$

and $C(n)$ is a positive constant depending on n only. It is not difficult to see that

$$\sup_{1<\gamma\leq 2} F(\gamma) = \sup_{1<\gamma\leq 2}\ \frac{1}{2}\int_0^\infty\ u^{(\gamma-1)/2}e^{-u}\ du < \infty. \tag{3.7}$$

We also note that ϕ is in $C^{1,\sigma}(\mathbb{R}^n_+)$, with $\sigma = \gamma - 1$, $\phi(x) = 0$, $D\phi(x) = 0$ whenever $x_1 = 0$. Furthermore, $D\phi$ and $D^2\phi$ are in $L^1(\mathbb{R}^n_+)$. Therefore, we may find a sequence of functions φ_j which are smooth with compact support in $\mathbb{R}^n_+$ and $\varphi_j \to \phi$, $D\varphi_j \to D\phi$ and $D^2\varphi_j \to D^2\phi$ in $L^1(\mathbb{R}^n_+)$. Thus, by (3.1), we have

$$\begin{aligned}&\int_{\mathbb{R}^n_+}\ v(x,t_2)\phi(x)\ \ dx\ - \int_{\mathbb{R}^n_+}\ v(x,t_1)\phi(x)\ \ dx \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^n_+}\ (v^m(x,s)\Delta\phi(x) + v^p(x,s)\phi(x))\ dx\ dt \\ &\geq \int_{t_1}^{t_2}\int_{\mathbb{R}^n_+}\ (-\lambda v^m(x,s)\phi(x) + v^p(x,s)\phi(x))\ dx\ dt.\end{aligned} \tag{3.8}$$

Let

$$F(s) = \frac{k^{n+\gamma}}{C(n)\Gamma(\gamma)} \int_{\mathbb{R}^n_+} v(x,s)\phi(x)\, dx.$$

By (3.6), if $0 < m < 1$ and $p > 1$, we have

$$\frac{k^{n+\gamma}}{C(n)\Gamma(\gamma)} \int_{\mathbb{R}^n_+} v^m(x,s)\phi(x)\, dx \le F^m(s) \tag{3.9}$$

and

$$\frac{k^{n+\gamma}}{C(n)\Gamma(\gamma)} \int_{\mathbb{R}^n_+} v^p(x,s)\phi(x)\, dx \ge F^p(s).$$

Thus, from (3.8), we obtain,

$$F'(s) \ge -\lambda F^m(s) + F^p(s).$$

By Lemma 3.5, if v is a global solution, then for any $s > 0$, we have

$$F(s) = \frac{k^{n+\gamma}}{C(n)\Gamma(\gamma)} \int_{\mathbb{R}^n_+} v(x,s)\phi(x)\, dx \le \lambda^{1/(p-m)}. \tag{3.10}$$

It implies that, using (3.5), for any $0 < k < 1$,

$$\begin{aligned} & k^{n+\gamma-2/(p-m)} \int_{\mathbb{R}^n_+} v(x,s) x_1^\gamma \exp(-|x|^2)\, dx \\ \le & k^{n+\gamma-2/(p-m)} \int_{\mathbb{R}^n_+} v(x,s)\phi(x)\, dx \\ \le & C(n)\Gamma(\gamma)\left(2(n+4)\right)^{1/(p-m)}. \end{aligned}$$

We fix γ so that

$$\gamma > 1 \qquad \text{and} \qquad n + \gamma < \frac{2}{p-m}.$$

Using (3.7), we see that if k is chosen small enough, then it is impossible, unless v is identically zero. ■

4. Blowup Rate

We consider the non-negative solutions of the equation

$$\begin{aligned} & u_t = \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial}{\partial x_i} u^m\right) + hu^p \qquad \text{on} \quad \Omega \times (0,T) \\ & u = 0 \qquad \text{on} \quad \partial\Omega \times (0,T). \end{aligned} \tag{4.1}$$

Here, $m > 0$ and $p > 1$, and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. We assume that $a^{ij} = a^{ij}(x)$ is symmetric and $a^{ij} \in C^1(\bar{\Omega})$, $h = h(x) > 0$ and $h \in C(\bar{\Omega})$. Also, we assume that a^{ij} and h satisfy (2.2). Let u be a function so that $u^m \in L^\infty_{loc}(0, T; H^1_0(\Omega))$ and $u(x,t) \geq 0$ for all $(x,t) \in \Omega \times (0,T)$. We say u is a weak solution of (4.1) in $\Omega \times (0,T)$ if for any $\eta \in C_0^\infty(\Omega \times (0,T))$, we have

$$\int_0^T \int_\Omega \left(u\eta_t + u^m \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a^{ij} \frac{\partial \eta}{\partial x_i}\right) + hu^p \eta \right) dx\ dt = 0.$$

Let

$$M(t) = \max\{u(x,t) : (x,t) \quad \text{in} \quad \Omega \times [0,t]\}. \tag{4.2}$$

We assume that $M(t) < \infty$, for each $t \in (0,T)$. We say u blows up at T if $M(t) \to \infty$ as $t \to T$. Given any $t_0 \in (0,T)$, we define t_0^+ by

$$t_0^+ = \max\{t \in (t_0, T) :\ M(t) = 2M(t_0)\}. \tag{4.3}$$

We need the following lemma which is due to Hu, [11] p895.

Lemma 4.1. *Suppose that there is a constant $K > 0$ such that is true for all $t_0 \in (T/2, T)$, we have*

$$M^{p-1}(t_0)(t_0^+ - t_0) \leq K. \tag{4.4}$$

Then there is a constant $C > 0$ such that

$$M(t) \leq C|T-t|^{-1/(p-1)}.$$

Proof. We pick any $t_0 \in (T/2, T)$. Using (4.3), for each $k \geq 0$, we let $t_{k+1} = t_k^+$. By our assumption (4.4), we obtain

$$t_{k+1} - t_k \leq \frac{K}{(M(t_k))^{p-1}} = \frac{K}{(2^k M(t_0))^{p-1}}.$$

Then, it follows that

$$T - t_0 = \sum_{k=0}^\infty (t_{k+1} - t_k) \leq \sum_{k=0}^\infty \frac{K}{(2^k M(t_0))^{p-1}} = \frac{C}{(M(t_0))^{p-1}}.$$

Hence, the Lemma is true. ∎

Theorem 4.2. *Let* u *be a non-negative weak solution of (4.1) in* $\Omega \times (0,T)$ *with*

$$0 < m < 1, \qquad \text{and} \qquad 1 < p < m + \frac{2}{n+1}.$$

If u *blows up at* T*, then there is a constant* $C > 0$ *such that*

$$\max_x u(x,t) \le C|T-t|^{-1/(p-1)}.$$

Proof. We follow the arguments in [3]. By Lemma 4.1, we only need to prove (4.4). Suppose that (4.4) is not true. There exists a sequence t_k, such that $t_k \to T$ and

$$M^{p-1}(t_k)(t_k^+ - t_k) \to \infty \qquad \text{as} \qquad k \to \infty. \tag{4.5}$$

For each $k = 1, 2, 3...$, there is $\hat{x}_k \in \Omega$ and $\hat{t}_k \in (0, t_k]$ such that

$$M(\hat{t}_k) = u(\hat{x}_k, \hat{t}_k) \ge \frac{M(t_k)}{2}. \tag{4.6}$$

Let

$$d_k = \operatorname{dist}(\hat{x}_k, \partial\Omega).$$

Suppose that

$$\limsup_{k\to\infty} \left(d_k M^{(p-m)/2}(t_k)\right) = \infty. \tag{4.7}$$

There are subsequences, also denoted by $\hat{x}_k$, t_k and $\hat{t}_k$, and a point $\hat{x}_0 \in \bar{\Omega}$ such that

$$\hat{x}_k \to \hat{x}_0 \quad \text{and} \quad d_k M^{(p-m)/2}(t_k) \to \infty \quad \text{as} \quad k \to \infty. \tag{4.8}$$

We rescale the solution u about the point $(\hat{x}_k, \hat{t}_k)$ as follows:

$$v_k(y,s) = \frac{1}{M(t_k)}\, u\left(\frac{y}{M^{(p-m)/2}(t_k)} + \hat{x}_k, \frac{s}{M^{p-1}(t_k)} + \hat{t}_k\right). \tag{4.9}$$

The function v_k is defined for

$$y \in B\left(d_k M^{(p-m)/2}(t_k)\right) \quad \text{and} \quad s \in \left(-M^{p-1}(t_k)\hat{t}_k, M^{p-1}(t_k)(\hat{t}_k^+ - \hat{t}_k)\right), \tag{4.10}$$

where we denote $B(r)$ to be the open ball centered at 0 with radius r. Clearly, by (4.6), for each $k = 1, 2, 3...$,

$$v_k(0,0) \ge 1/2 \qquad \text{and} \qquad 0 \le v_k(y,s) \le 2. \tag{4.11}$$

Moreover, v_k is a weak solution of the equation

$$v_{ks} = \sum_{i,j=1}^{n} \frac{\partial}{\partial y_j} \left(a_k^{ij} \frac{\partial}{\partial y_i} v_k^m \right) + h_k v_k^p, \tag{4.12}$$

where

$$a_k^{ij}(y) = a^{ij} \left(\frac{y}{M^{(p-m)/2}(t_k)} + \hat{x}_k \right) \quad \text{and} \quad h_k(y) = h \left(\frac{y}{M^{(p-m)/2}(t_k)} + \hat{x}_k \right).$$

Let $\eta(y,s)$ be a smooth function with compact support in $\mathbb{R}^n \times (-\infty, \infty)$. When k is large enough, we have

$$\int\int \left(v_k \eta_s + v_k^m \sum_{i,j=1}^{n} \frac{\partial}{\partial y_j} \left(a_k^{ij} \frac{\partial \eta}{\partial y_i} \right) + h_k v_k^p \eta \right) \, dy \, ds = 0. \tag{4.13}$$

From (4.5), (4.10) and Theorem 2.1, the functions $\{v_k\}$ are equicontinuous on compact subsets in $\mathbb{R}^n \times (-\infty, \infty)$. Also, by (4.8), $a_k^{ij}(y)$ converges to $a^{ij}(\hat{x}_0)$, $Da_k^{ij}(y)$ converges to 0, and $h_k(y)$ converges to $h(\hat{x}_0)$ uniformly on compact subsets in $\mathbb{R}^n$. Hence, there is a subsequence, which we also denote by $\{v_k\}$, and a continuous function v such that v_k converges to v uniformly on compact subsets in $\mathbb{R}^n \times (-\infty, \infty)$. When letting $k \to \infty$ in (4.13), we have

$$\int\int \left(v \eta_s + v^m \sum_{i,j=1}^{n} \frac{\partial}{\partial y_j} \left(a^{ij}(\hat{x}_0) \frac{\partial \eta}{\partial y_i} \right) + h(\hat{x}_0) v^p \eta \right) \, dy \, ds = 0,$$

i.e., v is a weak solution of the equation

$$v_s = \sum_{i,j=1}^{n} \frac{\partial}{\partial y_j} \left(a^{ij}(\hat{x}_0) \frac{\partial}{\partial y_i} v^m \right) + h(\hat{x}_0) v^p \tag{4.14}$$

in $\mathbb{R}^n \times (-\infty, \infty)$. After a change of variables, we may assume that v is a solution of the equation

$$v_s = \Delta(v^m) + v^p \quad \text{in} \quad \mathbb{R}^n \times (-\infty, \infty).$$

Also, by (4.11) and the fact that v_k converges to v uniformly on compact sets, we have $v(0,0) \geq 1/2$ and $0 \leq v(y,s) \leq 2$. However, by Theorem 3.3, it is impossible.

If (4.7) is not true, we may choose subsequences, also denoted by t_k, $\hat{x}_k$, $\hat{t}_k$, so that

$$\lim_{k \to \infty} \left(d_k M^{(p-m)/2}(t_k) \right) = c \geq 0. \tag{4.15}$$

It follows that $d_k \to 0$ as $k \to \infty$. We may assume that there is $\hat{x}_0 \in \partial\Omega$ so that

$$\hat{x}_k \to \hat{x}_0 \qquad \text{as} \qquad k \to \infty.$$

Also, we may choose $\tilde{x}_k \in \partial\Omega$ so that

$$|\hat{x}_k - \tilde{x}_k| = d_k.$$

Let R_k be an orthonormal transformation in $\mathbb{R}^n$ that maps $(-1, 0, ..., 0)$ onto the outer normal vector to $\tilde{x}_k$. Again, we rescale the solution u about the point $(\hat{x}_k, \hat{t}_k)$. Let

(4.16) $$v_k(y, s) = \frac{1}{M(t_k)}\, u\left(\frac{R_k(y)}{M^{(p-m)/2}(t_k)} + \hat{x}_k, \frac{s}{M^{p-1}(t_k)} + \hat{t}_k\right),$$

for

$$y \in \Omega_k = \left\{y : \frac{R_k(y)}{M^{(p-m)/2}(t_k)} + \hat{x}_k \in \Omega\right\},$$

and

$$s \in \left(-M^{p-1}(t_k)\hat{t}_k, M^{p-1}(t_k)(\hat{t}_k^+ - \hat{t}_k)\right).$$

Then, v_k is a weak solution of (4.12) with

$$a_k^{ij}(y) = a_k^{ij}\left(\frac{R_k(y)}{M^{(p-m)/2}(t_k)} + \hat{x}_k\right), \quad h_k(y) = h\left(\frac{R_k(y)}{M^{(p-m)/2}(t_k)} + \hat{x}_k\right).$$

Also, (4.11) holds. For each $r > 0$, if k is large enough,

$$\begin{aligned} &\Omega_k \cap B(0, r) \\ &= \left\{y : |y| < r, \quad y_1 > -d_k M^{(p-m)/2}(t_k) + M^{(p-m)/2}(t_k) f_k\left(\frac{y'}{M^{(p-m)/2}(t_k)}\right)\right\}, \end{aligned}$$

where $y' = (y_2, ..., y_n)$ and $f_k = f_k(x')$ is a smooth function defined on the set $\{x' : |x'| < r\}$. Also, since we assume that (-1,0,...,0) is the outward normal to $\tilde{x}_k$, we have $D_l f_k(0) = 0$ for each $l = 2, ..., n$. By (4.15), we may assume that when $k \to \infty$, the set Ω_k approaches the halfspace

$$H_c = \{y : y_1 > -c\}.$$

By Theorem 2.2, the functions $\{v_k\}$ are equicontinuous on compact subsets in $\overline{H}_c \times (-\infty, \infty)$. Hence, there is a subsequence, which we also denote by $\{v_k\}$, and a continuous function v such that v_k converges to v uniformly on compact subsets in $\overline{H}_c \times (-\infty, \infty)$. It follows that v is a weak solution of the equation (4.14). By (4.11), we have $v(0, 0) \geq 1/2$ and $0 \leq v(y, s) \leq 2$. However, by Theorem 3.4, it is impossible. This completes the proof of Theorem 4.2. ■

For $m > 1$, using Theorem 3.1 and 3.2, we have

Theorem 4.3. *Let u be a non-negative weak solution of (4.1) in $\Omega \times (0,T)$ with*

$$1 < m < p \leq m + \frac{2}{n+1}.$$

If u blows up at T, then there is a constant $C > 0$ such that

$$\max_x u(x,t) \leq C|T-t|^{-1/(p-1)}.$$

The proof is similar and is left to the reader.

5. Solutions Which Are Non-decreasing in Time

In this section, we consider solutions which are non-decreasing in time. In the semilinear case, a similar result was obtained in [18] Remarks 4.3b and 5.3b.

Theorem 5.1. *Let v be a bounded continuous weak solution of the equation*

$$v_t = \Delta(v^m) + v^p \tag{5.1}$$

in $\mathbb{R}^n \times (0,\infty)$, with $m > 0$,

$$1 < \frac{p}{m} \quad \text{when} \quad n \leq 2, \qquad 1 < \frac{p}{m} < \frac{n+2}{n-2} \quad \text{when} \quad n \geq 3.$$

Suppose that for each $x \in \mathbb{R}^n$, the function $t \to v(x,t)$ is a non-decreasing function, then v is identically zero on $\mathbb{R}^n \times (0,\infty)$.

Proof. Let

$$w(x) = \lim_{t\to\infty} v(x,t).$$

Let $\eta(x,t) = \varphi(x)\xi(t)$ in (3.1), where $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\xi \in C_0^\infty(0,1)$. Then, for each $k = 1,2,3...$,

$$\begin{aligned} &-\int_0^1\int_{\mathbb{R}^n} v(x,t+k)\varphi(x)\xi'(t)\ dx\ dt \\ &= \int_0^1\int_{\mathbb{R}^n} \left(v^m(x,t+k)\Delta\varphi(x)\xi(t) + v^p(x,t+k)\varphi(x)\xi(t)\right)\ dx\ dt. \end{aligned}$$

By letting $k \to \infty$, using the bounded convergence theorem, we obtain

$$\begin{aligned} &-\int_0^1\int_{\mathbb{R}^n} w(x)\varphi(x)\xi'(t)\ dx\ dt \\ &= \int_0^1\int_{\mathbb{R}^n} \left(w^m(x)\Delta\varphi(x)\xi(t) + w^p(x)\varphi(x)\xi(t)\right)\ dx\ dt. \end{aligned}$$

Since ξ has compact support in (0,1), we have

$$\int_0^1 \int_{\mathbb{R}^n} w(x)\varphi(x)\xi'(t)\ dx\ dt = \left(\int_{\mathbb{R}^n} w(x)\varphi(x)\ dx\right)\left(\int_0^1 \xi'(t)\ dt\right) = 0.$$

Thus, for any $\xi \in C_0^\infty(0,1)$ and $\varphi \in C_0^\infty(\mathbb{R}^n$, we obtain

$$\int_0^1 \int_{\mathbb{R}^n} \left(w^m(x)\Delta\varphi(x) + w^p(x)\varphi(x)\right)\xi(t)\ dx\ dt = 0.$$

This implies that, for any $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \left(w^m(x)\Delta\varphi(x) + w^p(x)\varphi(x)\right)\ dx = 0,$$

and w is a non-negative weak solution of the equation $\Delta w^m + w^p = 0$. Let $W = w^m$. Then the function W is a non-negative bounded weak solution of the equation

(5.2) $$\Delta W + W^P = 0 \qquad \text{with} \qquad P = p/m,$$

in $\mathbb{R}^n$. By the elliptic regularity theory, W is smooth in $\mathbb{R}^n$. Gidas and Spruck's result, [8], tells us that if $1 \le P < (n+2)/(n-2)$, then $W(x) = 0$ in $\mathbb{R}^n$. This implies that $w(x) = 0$ in $\mathbb{R}^n$. Since $v(x,t)$ is non-negative, $t \to v(x,t)$ is non-decreasing and

$$\lim_{t\to\infty} v(x,t) = 0,$$

we must have $v(x,t) = 0$ in $\mathbb{R}^n \times (0,\infty)$. ■

The same method also works for solution u of (5.1) in $\mathbb{R}^n_+ \times (0,\infty)$, which vanishes on the plane $\{x_1 = 0\}$.

Theorem 5.2. *Let v be a bounded continuous weak solution of the equation (5.1) in $\mathbb{R}^n_+ \times (0,\infty)$, with $m > 0$,*

$$1 < \frac{p}{m} \quad \text{when} \quad n \le 2, \qquad 1 < \frac{p}{m} < \frac{n+2}{n-2} \quad \text{when} \quad n \ge 3.$$

Suppose that $v(x,t) = 0$ when $x_1 = 0$, and, for each $x \in \mathbb{R}^n_+$, the function $t \to v(x,t)$ is a non-decreasing function, then v is identically zero on $\mathbb{R}^n_+ \times (0,\infty)$.

Proof. By Theorem 2.2, for all $t > 1$, the functions $x \to v(x,t)$ are equicontinuous on compact subset of the set $\{x : x_1 \ge 0\}$. If

$$w(x) = \lim_{t\to\infty} v(x,t),$$

then $w(x) = 0$ when $x_1 = 0$. It follows that the function $W = w^m$ also vanishes on the plane $\{x : x_1 = 0\}$ and is a solution of (5.2) in in $\mathbb{R}^n_+$. Gidas and Spruck, [9], also proved that if W is a non-negative solution of (5.2) in $\mathbb{R}^n_+$ and $W(x) = 0$ on $\{x_1 = 0\}$, then $W(x) = 0$ in $\mathbb{R}^n_+$. We have the same conclusion as in Theorem 5.1. ■

Using Theorem 5.1 and 5.2, we obtain results for solutions which are non-decreasing in time. The conditions for the domain Ω, the coefficients a^{ij} and h, and the solution u are the same as in previous section.

Theorem 5.3. *Let u be a positive solution of (4.1). When $0 < m < 1$, we assume that*

$$p > 1 \quad \text{when} \quad n \leq 2, \qquad 1 < p < m\left(\frac{n+2}{n-2}\right) \quad \text{when} \quad n \geq 3.$$

When $m > 1$, we assume that

$$p > m \quad \text{when} \quad n \leq 2, \qquad m < p < m\left(\frac{n+2}{n-2}\right) \quad \text{when} \quad n \geq 3.$$

If u is non-decreasing in time and u blows up at T, then there is a constant $C > 0$ such that

$$\max_x u(x,t) \leq C|T-t|^{-1/(p-1)}.$$

Proof. Suppose that for each $x \in \Omega$, the function $t \to u(x,t)$ is non-decreasing. For for each $k = 1, 2, 3...$, let v_k be the function in (4.9) or (4.16). Then, for each fixed y in the domain of v_k, the function $s \to v_k(y,s)$ is non-decreasing. By Theorem 2.1 or Theorem 2.2, v_k converges uniformly to a function v in compact subsets in $\mathbb{R}^n \times (-\infty, \infty)$ or $\mathbb{R}^n_+ \times (-\infty, \infty)$. Thus, $s \to v(y,s)$ is also non-decreasing. The rest of the proof is almost the same as in Theorem 4.2. ■

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Chi-Cheung Poon
Department of Mathematics
National Chung Cheng University
Minghsiung, Chiayi 621, Taiwan
E-mail: ccpoon@math.ccu.edu.tw