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EXISTENCE OF EQUILIBRIA IN COMPLETE METRIC SPACES

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Abstract. In this paper, we establish equilibrium version of Ekeland's variational principle without assuming any kind of semicontinuity of the bifunction involved in the formulation of the principle. By using such principle, we derive some existence results for a solution of equilibrium problems with or without compactness assumption on the underlying set. A coercivity condition is introduced to obtain a solution of an equilibrium problem for noncompact case. Our results extend and improve several known results in the literature.

1. INTRODUCTION

By an equilibrium problem, we understand the problem of finding

$$\text{(EP)} \qquad f(\bar{x}, y) \geq 0, \quad \forall y \in K,$$

where X is a given set, $D \subseteq X$ a nonempty set and $f : D \times D \to \mathbb{R}$ is a given functions. In the recent past, (EP) is among the most interesting and intensively studied classes of problems. It includes fundamental mathematical problems, for instance, optimization problem, Nash equilibrium problem, fixed point problems, variational inequality problems, minimax inequalities and complementarity problems. It is studied in several papers, among others we quote Brézis et al. [6], Bianchi and Pini [3], Bianchi et al. [2] and Kas et al. [8]. The convexity of the underlying set D and generalized convexity and monotonicity together with some weak continuity assumptions on f are the most used assumptions in dealing with equilibrium problems; see, for example, [3, 4, 7] and references therein. Blum and Oettli [5] and Oettli and Théra [9] proved the existence of a solution of (EP) by using Ekeland's variational principle but without any kind of convexity assumption. Recently, Bianchi et al. [2] extended Ekeland's variational principle for equilibrium-type bifunctions and obtained some existence results for equilibria on compact sets or noncompact case together with a coercivity condition in reflexive Banach spaces.

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In section 2, we establish equilibrium version of Ekeland's variational principle without assuming any kind of semicontinuity of the bifunction involved in the formulation of the principle. We also provide an example which shows that our result is an improvement of Theorem 2.1 in [2]. By using such principle, we derive some existence results for a solution of equilibrium problem under compactness assumption on the underlying set. In section 3, we introduce a coercivity condition and obtain an existence result for a solution of equilibrium problem for noncompact case. Our results extend and improve several known results in the literature.

2. Existence Results for Equilibria on Compact Sets

The Ekeland's variational principle has been widely used in nonlinear analysis since it entails the existence of approximate solutions of a minimization problem for lower semicontinuous functions on a complete metric spaces. Bianchi et al. [2] extended Ekeland's theorem to the setting of equilibrium problem involving a bifunction f defined on $D \times D$ where D is a closed subset of a reflexive Banach space. By modifying the technique of [2], we establish the following equilibrium version of Ekeland's variational principle without assuming lower semicontinuity of the function $f(x, \cdot)$ for each fixed $x \in D$.

Theorem 2.1. *Let D be a nonempty closed subset of a complete metric space (M, d) and $f : D \times D \to \mathbb{R}$ be a bifunction. Assume that $\varepsilon > 0$ and the following assumptions are satisfied:*

(i) *lev*$_{\leq 0} f(x, .) + \varepsilon d(x, .) := \{y \in D : f(x, y) + \varepsilon d(x, y) \leq 0\}$ *is closed, for all* $x \in D$*;*

(ii) $f(t, t) = 0$*, for all* $t \in D$*;*

(iii) $f(z, x) \leq f(z, y) + f(y, x)$*, for all* $x, y, z \in D$*.*

If $\inf\limits_{y \in D} f(x_0, y) > -\infty$ *for some* $x_0 \in D$*, then there exists* $\bar{x} \in D$ *such that*

$$\begin{cases} f(x_0, \bar{x}) + \epsilon d(x_0, \bar{x}) \leq 0, \\ f(\bar{x}, x) + \epsilon d(\bar{x}, x) > 0, \ \forall x \in D, x \neq \bar{x}. \end{cases}$$

Proof. Without loss of generality, we can restrict the proof to the case $\varepsilon = 1$. We define $F : D \to 2^D$ as follows

$$F(x) = \{y \in D : f(x, y) + d(x, y) \leq 0\}.$$

By (ii), $x \in F(x)$, and by (i), $F(x)$ is closed for every $x \in D$. It is easy to see, from (iii), that if $y \in F(x)$, then $F(y) \subset F(x)$. Now we show that there is a sequence

$\{x_n\}$ of D such that

$$x_{n+1} \in F(x_n), \quad f(x_n, x_{n+1}) < \inf_{z \in F(x_n)} f(x_n, z) + \frac{1}{n+1}, \quad \forall n \geq 0.$$

Indeed, let $v(x_0) = \inf_{z \in F(x_0)} f(x_0, z) > -\infty$. Hence, there exists $x_1 \in F(x_0)$ such that $f(x_0, x_1) < v(x_0) + \frac{1}{1}$. From $x_1 \in F(x_0)$ and (iii), we have

$$\inf_{z \in F(x_1)} f(x_1, z) \geq \inf_{z \in F(x_1)} f(x_0, z) - f(x_0, x_1) \geq v(x_0) - f(x_0, x_1) > -\infty.$$

If we let $v(x_1) = \inf_{z \in F(x_1)} f(x_1, z)$, then there exists $x_2 \in F(x_1)$ such that $f(x_1, x_2) < v(x_1) + \frac{1}{2}$ and so (iii) implies that

$$v(x_2) = \inf_{z \in F(x_2)} f(x_2, z) \geq \inf_{z \in F(x_2)} f(x_1, z) - f(x_1, x_2) \geq v(x_1) - f(x_1, x_2).$$

Continuing this process, we obtain a sequence $\{x_n\}$ such that

$$x_{n+1} \in F(x_n), \quad f(x_n, x_{n+1}) < v(x_n) + \frac{1}{n+1},$$
$$v(x_{n+1}) \geq v(x_n) - f(x_n, x_{n+1}), \quad \forall n \geq 0,$$

which imply that

$$-v(x_n) \leq -f(x_n, x_{n+1}) + \frac{1}{n+1} \leq v(x_{n+1}) - v(x_n) + \frac{1}{n+1}.$$

Consequently, by the above inequality,

$$v(x_{n+1}) + \frac{1}{n+1} \geq 0, \quad \forall n \geq 0. \tag{2.1}$$

Now if $z_1, z_2 \in F(x_n)$, then

$$d(z_1, z_2) \leq d(x_n, z_1) + d(x_n, z_2) \leq -f(x_n, z_1) - f(x_n, z_2) \leq -2v(x_n).$$

Hence, the diameter of $F(x_n)$, diam$F(x_n) \leq -2v(x_n)$. Thus, by (2.1), diam$F(x_n)$ $\to 0$ if $n \to \infty$. This, $F(x_n)$ being closed, and $F(x_{n+1}) \subseteq F(x_n)$ for every $n \geq 0$ imply that

$$\bigcap_{n \geq 0} F(x_n) = \{\bar{x}\}.$$

Therefore $F(\bar{x}) \subset F(x_n)$, for every $n \geq 0$, and hence since $\bar{x} \in F(\bar{x})$ we have $F(\bar{x}) = \{\bar{x}\}$ and $F(\bar{x}) \subseteq F(x_0)$ and so the proof is completed. ∎

Remark 2.1. If for each fixed $x \in D$, the function $f(x, .)$, as a mapping on D, is lower semicontinuous, then condition (i) of Theorem 2.1, for all $\epsilon > 0$, holds. This follows from the fact that the sum of two lower semicontinuous function is lower semicontinuous function again.

The following example shows that the converse of Remark 2.1 need not be true in general. Therefore, Theorem 2.1 improves Theorem 2.1 in [2].

Example 2.1. Let $D = [0, \infty)$ and $f : D \times D \to \mathbb{R}$ defined as follows

$$f(x,y) = \begin{cases} 0, & \text{if } (x,y) \in \{0\} \times D \text{ or } x = y \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to check that f satisfies all of the assumptions of Theorem 2.1 for $\varepsilon > 0$, but $f(x, .)$ is not lower semicontinuous at $x = 0$.

The following example shows that condition (iii) of Theorem 2.1 cannot be omitted.

Example 2.2. Let $D = [0, 1]$ and $f : D \times D \to \mathbb{R}$ defined by $f(x, y) = -\frac{1}{3}\sqrt{|x - y|}$. If in Theorem 2.1 we let $\epsilon = \frac{1}{2}$, then f satisfies all the assumptions of Theorem 2.1 except (iii), but the conclusion of Theorem 2.1 does not hold.

As applications of Theorem 2.1, we derive the following corollaries.

Corollary 2.1. *If the following condition*

(iv) *For every $x \in K$ with $\inf\limits_{y \in D} f(x, y) < 0$ there exists $z \in D$ such that $z \neq x$ and $f(x, z) + \varepsilon d(x, z) \leq 0$,*

together with the assumptions of Theorem 2.1 are satisfied, then the solution set of (EP) is nonempty, that is, there exists $\bar{x} \in D$ such that $f(\bar{x}, y) \geq 0$, for all $y \in D$.

Proof. By Theorem 2.1, there exists $\bar{x} \in D$ such that

$$f(\bar{x}, y) + \varepsilon d(\bar{x}, y) > 0, \ \forall y \in D, \ y \neq \bar{x}. \tag{2.2}$$

We claim that $\bar{x}$ is a solution of (EP). Otherwise, there exists $y \in D$ such that $f(\bar{x}, y) < 0$. From assumption (iv), we obtain $z \in D$ with $z \neq \bar{x}$ and $f(x, z) + \varepsilon d(x, z) \leq 0$ which contradicts (2.2). ■

The following corollary guarantees the existence of a solution of the minimization problem for a lower bounded function on a closed set.

Corollary 2.2. *Let D be a nonempty closed subset of a complete metric space (M, d), $\phi : D \to \mathbb{R}$ be a lower bounded real function and the set $\{y \in D : \phi(y) + \varepsilon d(x, y) \le \phi(x)\}$ be closed for all $x \in D$ and for some $\varepsilon > 0$. If for every $x \in D$ with $\inf_{y \in D}(\phi(y) - \phi(x)) < 0$, there exists $z \in D$ such that $z \neq x$ and $\phi(z) + \varepsilon d(x, z) \le \phi(x)$, then there exists $\bar{x} \in D$ such that $\phi(\bar{x}) \le \phi(y)$ for all $y \in D$.*

Proof. Define $f : D \times D \to \mathbb{R}$ as

$$f(x, y) = \phi(y) - \phi(x), \ \forall x, y \in D.$$

Then f satisfies all of the assumptions of Corollary 2.1. So, there exists $\bar{x} \in D$ such that $f(\bar{x}, y) \ge 0$, for all $y \in D$. ∎

Remark 2.2. Let D be a nonempty set and $f : D \times D \to \mathbb{R}$ be a real function which satisfies assumptions (ii) and (iii) of Theorem 2.1. If for $x_0 \in D$, $\inf_{y \in D} f(x_0, y) \ge 0$, then for every $\varepsilon > 0$, there exists $x_\varepsilon \in D$ such that

$$\begin{cases} f(x_0, x_\varepsilon) + \epsilon\delta(x_0, x_\varepsilon) \le 0, \\ f(x_\varepsilon, x) + \epsilon \ge 0, \ \forall x \in D, \end{cases}$$

where $\delta(x, y) = 0$, if $x = y$; otherwise $\delta(x, y) = 1$.

This fact (slightly) leads us to consider equilibrium problems on arbitrary sets together with properties on functions instead of topologically or linearity properties on the domains of functions.

Remark 2.3. Let $\{t_n\}$ be a sequence of positive real numbers which is bounded below by some positive numbers. If conditions of Theorem 2.1, for $\varepsilon = t_n$, for all n, hold, then the sequence of approximate solutions $\{x_n\}$ obtained by Theorem 2.1 corresponding t_n is bounded. Indeed, putting $\alpha = \inf_{y \in D} f(x_0, y)$ and $\beta = \inf_{n \in \mathbb{N}} t_n$ and using Theorem 2.1 with $\varepsilon = t_n$ for every n, we have

$$d(x_0, x_n) \le \frac{1}{t_n} - f(x_0, x_n) \le \frac{1}{t_n} - \inf_{y \in D} f(x_0, y) \le \frac{1}{\beta} - \alpha.$$

We need the following definition for the next result.

Definition 2.1. Given $f : D \times D \to \mathbb{R}$ and $\epsilon > 0$, $\overline{x}$ is said to be an $\epsilon-$equilibrium point of f if

$$f(\bar{x}, y) \ge -\epsilon d(\bar{x}, y), \quad \forall y \in D.$$

By applying Theorem 2.1, we establish the following result which improves Proposition 3.2 in [4].

Theorem 2.2. *Let D be a nonempty compact subset of a metric space (M, d), $f : D \times D \to \mathbb{R}$, and $\{t_n\}$ be a decreasing sequence of positive real numbers such that $t_n \to 0$. Assume that*

- (i) *lev$_{\leq 0} f(x, .) + t_n d(x, .)$ is closed for every $x \in D$ and for all positive integer n;*
- (ii) *$f(t, t) = 0$, for all $t \in D$;*
- (iii) *$f(z, x) \leq f(z, y) + f(y, x)$, for all $x, y, z \in D$;*
- (iv) *lev$_{\geq 0} f(., x) + t_n d(., x)$ is closed for every $x \in D$ and for all positive integer n.*

If $\inf_{y \in D} f(x_0, y) > -\infty$ for some $x_0 \in D$, then the set of solutions of (EP) is nonempty.

Proof. By Theorem 2.1, for each $n \in \mathbb{N}$, there exists t_n-equilibrium point of f, that is, there exists $x_n \in D$ such that

$$f(x_n, y) \geq -t_n d(x_n, y), \quad \forall y \in D. \tag{2.3}$$

Since D is compact, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$. By (2.3) and $\{t_n\}$ is decreasing sequence, for every fixed positive integer k_0, we have

$$x_{n_k} \in \text{lev}_{\geq 0} f(., y) + t_{n_{k_0}} d(., y), \quad \forall\ k \geq k_0, \quad \forall y \in D.$$

By (iv) and $x_{n_k} \to \bar{x}$, we deduce that

$$f(\bar{x}, y) + t_{n_{k_0}} d(\bar{x}, y) \geq 0, \quad \forall y \in D. \tag{2.4}$$

Since n_{k_0} is arbitrary and t_{n_k} approach to zero if $k \to +\infty$, (2.4) implies that $\bar{x}$ is a solution of (EP). ■

Corollary 2.3. [4, Proposition 3.2]. *Let D be a compact subset of a complete metric space and $f : D \times D \to \mathbb{R}$ be a function satisfying the following conditions.*

- (i) *$f(x, .)$ is lower semicontinuous, for every $x \in D$;*
- (ii) *$f(x, x) = 0$, for every $x \in K$;*
- (iii) *$f(z, x) \leq f(z, y) + f(y, x)$, for every $x, y, z \in K$;*
- (iv) *$f(., y)$ is upper semicontinuous, for every $y \in K$.*

Then the set of solutions of (EP) *is nonempty and compact.*

Remark 2.4. Conditions (i) and (ii) of Corollary 2.3 imply assumptions (i) and (ii) of Theorem 2.2, respectively but the converse dose not hold. Since, for instance, if we consider Example 2.1 with $D = [0, 1]$ and $\{t_n = \frac{1}{n}\}_{n\in\mathbb{N}}$, then it satisfies all of the assumptions of Theorem 2.2 but it does not satisfy all the conditions of Corollary 2.3 because $f(0, .)$ is not lower semicontinuous.

3. Equilibria on Noncompact Sets

We consider now the noncompact case. Throughout this section we assume that (M, d) is a metric space with Heine-Borel property, that is, each closed bounded subset of M is compact. For instance, If X is a reflexive Banach space with sparable dual then X equipped with metrizable weak topology has Heine-Borel property. Let D be a closed subset of M and $f : D \times D \to \mathbb{R}$ be a given function.

Consider the following coercivity condition:

$$(C) \qquad \exists B(c, r) \ : \ \forall x \in D\backslash K_r \ \exists \ y \in D, \ d(y, c) < d(x, c); f(x, y) \leq 0,$$

where $K_r = D \cap B(c, r)$ and $B(c, r) = \{y \in M : d(c, y) \leq r\}$.

If we set $X = \mathbb{R}^n$ Euclidean space and $c = 0$, then the above coercivity condition reduces to the coercivity introduced in [4].

The following theorem improves Theorem 4.1 in [4]. However its proof is similar to Theorem 4.1 in [4].

Theorem 3.1. *Let* D *be a nonempty closed subset of* (M, d) *and* $\{t_n\}$ *be a decreasing sequence of the positive real numbers such that* $t_n \to 0$. *Suppose that* $f : D \times D \to \mathbb{R}$ *satisfies conditions (i)–(iv) of Theorem 2.2. If* $\inf_{y\in D} f(x_0, y) > -\infty$ *for some* $x_0 \in D$ *and coercivity condition* (C) *hold, then the set of solutions of (EP) is nonempty.*

Proof. For each $x \in D$, consider the nonempty set

$$S(x) = \{y \in D : d(y, c) \leq d(x, c), f(x, y) \leq 0\}.$$

Observe that for every $x, y \in K_r$, $y \in S(x)$ implies that $S(y) \subseteq S(x)$. Indeed, for $z \in S(y)$ we have $d(z, c) \leq d(y, c) \leq d(x, c)$ and by (iii) $f(x, z) \leq f(x, y) + f(y, z) \leq 0$. Since K_r is nonempty and compact, by Theorem 2.2 there exists $x_r \in K_r$ such that

$$f(x_r, y) \geq 0, \quad \forall y \in K_r. \tag{3.1}$$

Suppose that there exists $x \in D$ with $f(x_r, x) < 0$ and put

$$a = \min_{y \in S(x)} d(y, c).$$

We distinguish two cases.

Case 1. $a \leq r$. Let $y_0 \in S(x)$ such that $d(y_0, c) = a \leq r$. Then, we have $f(x, y_0) \leq 0$. Since $f(x_r, x) < 0$, it follows by (iii) that

$$f(x_r, y_0) \leq f(x_r, x) + f(x, y_0) < 0,$$

which contradicted by (3.1).

Case 2. $a > r$. Let again $y_0 \in S(x)$ such that $d(y_0, c) = a > r$. Then by (C) we can choose an element $y_1 \in D$ with $d(y_1, c) < d(y_0, c) = a$ such that $f(y_0, y_1) \leq 0$. Thus, $y_1 \in S(y_0) \subseteq S(x)$. Hence

$$d(y_1, c) < a = \min_{y \in S(x)} d(y, c),$$

which is a contradiction. Therefore, there is no $x \in D$ such that $f(x_r, x) < 0$, that is, x_r is a solution of (EP) on D. ■

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