TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 2, pp. 651-663, April 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

THE INTERLACING OF SPECTRA BETWEEN CONTINUOUS AND DISCONTINUOUS STURM-LIOUVILLE PROBLEMS AND ITS APPLICATION TO INVERSE PROBLEMS

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Abstract. The discontinuous Sturm-Liouville problem defined on $[0, 1]$ with jump conditions at point $d \in (0, 1)$ is considered. The interlacing of the spectra between the discontinuous Sturm-Liouville problem and two Sturm-Liouville problems defined on $[0, d]$ and $[d, 1]$ is provided. As the application of this interlacing to inverse problems, we prove that the potential is determined uniquely by the three spectra generated by the discontinuous Sturm-Liouville problem and two Sturm-Liouville problems defined on $[0, d]$ and $[d, 1]$.

1. Introduction

In this paper, we consider the discontinuous Sturm-Liouville problem (DSLP) consisting of the equation

$$-y'' + qy = \lambda y, \text{ on } [0, 1], \tag{1.1}$$

and the boundary conditions

$$y'(0) - hy(0) = 0, \tag{1.2}$$

$$y'(1) + Hy(1) = 0, \tag{1.3}$$

and the jump conditions

$$y(d + 0) = ay(d - 0), \quad y'(d + 0) = y'(d - 0)/a, \tag{1.4}$$

Received March 1, 2010, accepted December 10, 2010.
Communicated by Jen-Chih Yao.
2010 *Mathematics Subject Classification*: Primary 34A55; Secondary 34L40.
Key words and phrases: Discontinuous Sturm-Liouville problem, Eigenvalue, Weyl-Titchmarsh m-function, Inverse spectral problem.
This research is supported by the NNSF of P. R. China (No. 10771165) and the NSF of Guangdong Province, China (No. 5012285).

where $q \in L^1([0,\pi])$ is real-valued and $h, H \in \mathbb{R}$, $a > 0$ and $d \in (0,1)$. In particular, if $a = 1$ in (1.4), then the DSLP (1.1-4) reduces the classical Sturm-Liouville problem (SLP), which, in our sense, is called the continuous SLP corresponding to the DSLP.

It is well known [4, pp. 234-235] that the DSLP (1.1-4) is self-adjoint in $L^2[0,1]$ and, therefore, its spectrum is real. Further, it is shown [4] that the spectrum consists in simple and bounded lower eigenvalues.

For the study of inverse SLP's, Gesztesy and Simon [5] and Pivovarchik [9, 10], proved that, if the three spectra are pairwise disjoint, then the potential q can be uniquely determined by the three spectra of the problems defined on three intervals $[0,1]$, $[0,d]$ and $[d,1]$ for some $d \in (0,1)$. Furthermore, Gesztesy and Simon [5, p. 90] gave a counterexample to show that the pairwise disjoint condition is necessary. The main goal of this paper is to generalize the result of the above three spectra associated with the SLP's to the DSLP's. Our particular interest will to be examine under what conditions the three spectra generated by the DSLP on $[0,1]$ and two SLP on $[0,d]$ and $[d,1]$ are pairwise disjoint.

The contents of the paper are twofold. First, we will consider the pairwise disjoint problem of the three spectra. This problem can be identified in terms of the interlacing of the spectra between DSLP (1.1-4) and two SLP's on $[0,d]$ and $[d,1]$, which are imposed the boundary conditions (1.2) and (1.3) and the same condition at $x = d$:

$$y'(d) + h_0 y(d) = 0, \tag{1.5}$$

respectively. It should be noted that the interlacing among SLP's has been study by a number of authors (see, for example, [3, 7, 9] references cited therein). However, less works are known to deal with the same problem for the DSLP's. We shall concern the problem and prove the interlacing associated with three spectra, and provide suitable conditions such that they are pairwise disjoint. Second, we shall apply the interlacing to consider the inverse eigenvalue problems for the DSLP. Note that the inverse problems of the DSLP have been considered by a number of authors, see, for example, [6, 4, 1]. The Borg theorem and half-inverse theorem were established (see [4, 6]). We will prove that the potential q can be determined uniquely by the three spectra under the more distinguishable condition (see Theorems 4.2 and 4.3 below).

The method we use is based on the Weyl-Titchmarsh-m-function. The interlacing of the eigenvalues can be identified by the monotone increasing of the m-function on the suitable intervals and the determination of the potential q is distinguished in terms of the asymptotics, poles and residues of the m-function.

The organization of the paper is as follows. In Sections 2 and 3, we will consider the interlacing of the three spectra in two cases of the Dirichlet and non-Dirichlet conditions at the inter point $x = d$. The determination problem of the potential is treated in Section 4.

2. The Interlacing Associated with Dirichlet Condition

In this section, we will mainly consider the interlacing of the eigenvalue sequences between the DSLP (1.1-4) and two SLP's on subintervals $[0, d]$ and $[d, 1]$, which are imposed the Dirichlet boundary condition at d.

Let $v_-(x, \lambda)$ and $v_+(x, \lambda)$ be the fundamental solutions of equation (1.1) satisfying the initial conditions

$$v_-(0) = 1, \ v_-'(0) = h, \tag{2.1}$$

and

$$v_+(1) = 1, \ v_+'(1) = -H, \tag{2.2}$$

respectively. Let

$$\begin{aligned} \Delta(\lambda) &= \begin{vmatrix} v_+(d,\lambda) & av_-(d,\lambda) \\ av_+'(d,\lambda) & v_-'(d,\lambda) \end{vmatrix} \\ &= v_-'(d,\lambda)v_+(d,\lambda) - a^2 v_-(d,\lambda)v_+'(d,\lambda). \end{aligned} \tag{2.3}$$

It is easy to check that $\Delta(\lambda)$ is the characteristic function of the DSLP. Since $v_\pm^{(k)}(d, \lambda)$ $(k = 0, 1)$ are entire functions of order $\frac{1}{2}$, it follows that $\Delta(\lambda)$ is also an entire function of order $\frac{1}{2}$ and, therefore, has at most countable of zeros. All zeros of $\Delta(\lambda)$ are simple (see [4, Theorem 4.4.1]).

We denote the boundary condition at d point by:

$$y(d) = 0. \tag{2.4}$$

Then, equation (1.1) and the boundary conditions (1.2) and (2.4) generate a continuous SLP on $[0, d]$ and the increasing sequence of its eigenvalues is denoted by $\{\mu_n^D | n = 1, 2, \cdots\}$. Similarly, on $[d, 1]$, equation (1.1) and the boundary conditions (2.4) and (1.3) generate a continuous SLP too, and its increasing sequence of the eigenvalues is denoted by $\{\nu_n^D | n = 1, 2, \cdots\}$.

Define the Weyl-Titchmarsh m-functions

$$m_+(\lambda) = \frac{v_+'(d,\lambda)}{v_+(d,\lambda)}, \quad m_-(\lambda) = -\frac{v_-'(d,\lambda)}{v_-(d,\lambda)}. \tag{2.5}$$

It is known [5] that both $m_\pm(\lambda)$ are the Herglotz functions, that is, analytic functions in the upper half-plane $\mathbb{C}^+$, with positive imaginary part.

Lemma 2.1. *λ^* is a zero of $\Delta(\lambda)$ if and only if*

$$m_-(\lambda^*) = -a^2 m_+(\lambda^*), \tag{2.6}$$

where (2.6) implies the case that both sides are ∞, but which don't take $\pm\infty$ apart.

Proof. We note that the equality (2.6) is a simple transformation of $\Delta(\lambda) = 0$ and, therefore, the conclusion is obviously true when $v_-(d, \lambda^*)v_+(d, \lambda^*) \neq 0$. If one of

$$v_-(d, \lambda^*) = 0 \tag{2.7}$$

and

$$v_+(d, \lambda^*) = 0 \tag{2.8}$$

holds, and also (2.6) is satisfied, then we can easily check that the other must hold too. On the other hand, if both sides of (2.6) are ∞, we have (2.7) and (2.8). The proof is complete. ■

Remark 2.2. Lemma 2.1 implies that all of the elements of $\{\mu_n^D\}_{n=1}^{+\infty} \cap \{\nu_n^D\}_{n=1}^{+\infty}$ are eigenvalues of the DSLP. Furthermore, $\lambda^* \in \{\mu_n^D\}_{n=1}^{+\infty} \cap \{\nu_n^D\}_{n=1}^{+\infty}$, if and only if (2.6) yields the case $\infty = \infty$.

The results of Lemma 2.1 construct the relation between the spectrum of the DSLP and m-function. In order to obtain the spectral distribution of the DSLP, we consider a property of m-function as following.

Lemma 2.3. *$m_-(\lambda)$ is a strictly monotone increasing continuous function both on interval $(-\infty, \mu_1^D)$ and on interval (μ_n^D, μ_{n+1}^D) $(n = 1, 2, \cdots)$.*

Proof. Note that the zeros of $v_-(d, \lambda)$ are exactly $\{\mu_n^D\}_{n=1}^{+\infty}$ and, therefore, if $\lambda \in \mathbb{R} \setminus \{\mu_n^D\}_{n=1}^{+\infty}$, then $m_-(\lambda)$ is continuous.

Differentiating both sides of

$$-v_-''(x, \lambda) + q(x)v_-(x, \lambda) = \lambda v_-(x, \lambda), \tag{2.9}$$

with respect to λ, we have

$$-(\frac{\partial v_-(x, \lambda)}{\partial \lambda})'' + q(x)\frac{\partial v_-(x, \lambda)}{\partial \lambda} = \lambda \frac{\partial v_-(x, \lambda)}{\partial \lambda} + v_-(x, \lambda). \tag{2.10}$$

On the condition that without ambiguities, in order to maintain consistency of the sign, where "$'$" denotes still the derivative with respect to x. Integrating the difference of (2.10) multiplying by $v_-(x, \lambda)$ and (2.9) multiplying by $\frac{\partial v_-(x,\lambda)}{\partial \lambda}$, integrating by parts, we infer

$$\Big[\frac{\partial v_-(x,\lambda)}{\partial\lambda}v'_-(x,\lambda)-\big(\frac{\partial v'_-(x,\lambda)}{\partial\lambda}\big)v_-(x,\lambda)\Big]\Big|_0^d=\int_0^d v_-^2(x,\lambda)dx. \tag{2.11}$$

Note that both $v_-(0,\lambda)=1$ and $v'_-(0,\lambda)=h$ are independent of λ and, hence, $\frac{\partial v_-(0,\lambda)}{\partial\lambda}=0$ and $\frac{\partial v'_-(0,\lambda)}{\partial\lambda}=0$. Thus, (2.11) can be rewritten as

$$\frac{\partial v_-(d,\lambda)}{\partial\lambda}v'_-(d,\lambda)-\big(\frac{\partial v'_-(d,\lambda)}{\partial\lambda}\big)v_-(d,\lambda)=\int_0^d v_-^2(x,\lambda)dx. \tag{2.12}$$

From the definition of $m_-(\lambda)$, it then follows that

$$\begin{aligned}\frac{\mathtt{d}}{\mathtt{d}\lambda}m_-(\lambda)&=-\frac{\partial}{\partial\lambda}\frac{v'_-(d,\lambda)}{v_-(d,\lambda)}\\&=\Big[\frac{\partial v_-(d,\lambda)}{\partial\lambda}v'_-(d,\lambda)-\big(\frac{\partial v'_-(d,\lambda)}{\partial\lambda}\big)v_-(d,\lambda)\Big]/v_-^2(d,\lambda).\end{aligned} \tag{2.13}$$

If $\lambda\neq\mu_n^D$ $(n=1,2,\cdots)$, then we have

$$\frac{\mathtt{d}}{\mathtt{d}\lambda}m_-(\lambda)=\frac{\int_0^d v_-^2(x,\lambda)dx}{v_-^2(d,\lambda)}>0. \tag{2.14}$$

This completes the proof. ∎

It can be verified

$$\lim_{\lambda\to-\infty}m_\pm(\lambda)=-\infty. \tag{2.15}$$

In fact, for any $\varepsilon>0$, if $\varepsilon\le\arg\lambda\le 2\pi-\varepsilon$, then $m_\pm(\lambda)$ has the asymptotic behavior (see [8])

$$m_\pm(\lambda)=i\sqrt{\lambda}\Big(1+o\big(\frac{1}{\sqrt{\lambda}}\big)\Big),\quad \mathtt{as}\quad \lambda\to\infty. \tag{2.16}$$

Specially, when $\lambda\to-\infty$, we have

$$m_\pm(\lambda)=-\sqrt{|\lambda|}\Big(1+o\big(\frac{1}{\sqrt{|\lambda|}}\big)\Big)\to-\infty.$$

Hence, from

$$\lim_{\lambda\to\mu_n^D}m_-(\lambda)=\infty,\quad n=1,2,\cdots, \tag{2.17}$$

together with Lemma 2.3, next lemmas may thus be concluded.

Lemma 2.4. *$m_-(\lambda)$ is continuous and strictly increasing from $-\infty$ to $+\infty$ on intervals $(-\infty, \mu_1^D)$ and (μ_n^D, μ_{n+1}^D) $(n = 1, 2, \cdots)$.*

Lemma 2.5. *$m_+(\lambda)$ is continuous and strictly increasing from $-\infty$ to $+\infty$ on intervals $(-\infty, \nu_1^D)$ and (ν_n^D, ν_{n+1}^D) $(n = 1, 2, \cdots)$.*

Theorem 2.6. *(i) Rearrange $\{\mu_n^D\}_{n=1}^{+\infty} \cup \{\nu_n^D\}_{n=1}^{+\infty}$, the new sequence will form a partition of $\mathbb{R}$, then there contains exactly one eigenvalue of the DSLP (1.1-4) on each open subinterval of the partition.*

(ii) If $\lambda^* \in \{\mu_n^D\}_{n=1}^{+\infty} \cap \{\nu_n^D\}_{n=1}^{+\infty}$, then λ^* is the eigenvalue of the DSLP (1.1-4). Instead, if some μ_k^D (resp. ν_k^D) is the eigenvalue of the DSLP, then $\mu_k^D \in \{\nu_n^D\}_{n=1}^{+\infty}$ (resp. $\nu_k^D \in \{\mu_n^D\}_{n=1}^{+\infty}$).

Proof. (i) For convenience, denoting $\mu_0^D = \nu_0^D = -\infty$, we consider the function

$$g(\lambda) = m_-(\lambda) + a^2 m_+(\lambda). \tag{2.18}$$

From Lemmas 2.4 and 2.5, $g(\lambda)$ is strictly monotone increasing and continuous on each open interval of the supposed partition, and

$$\lim_{\lambda \to \mu_n^D + 0} g(\lambda) = -\infty, \quad \lim_{\lambda \to \nu_n^D + 0} g(\lambda) = -\infty,$$

$$\lim_{\lambda \to \mu_n^D - 0} g(\lambda) = +\infty, \quad \lim_{\lambda \to \nu_n^D - 0} g(\lambda) = +\infty,$$

for $n = 0, 1, \cdots$. Thus, $g(\lambda)$ has exactly one zero on each open interval of the partition. Following from Lemma 2.1, there contains exactly one eigenvalue of the DSLP on each open interval above-mentioned.

(ii) Which actually is the retelling of Remark 2.2.

The proof is completed. ■

Remark 2.7. This theorem gives the interlacing of the spectra between the DSLP and two SLP's. Note that $\{\mu_n^D\}_{n=1}^{+\infty} \cup \{\nu_n^D\}_{n=1}^{+\infty}$ are independent of the a. Thus, when $d \in (0, 1)$ is fixed, but a can change in $(0, +\infty)$, since a has no influence on the limit of $g(\lambda)$ at the division points, it follows that every DSLP's, corresponding to a, has exactly one eigenvalue on each open interval of partition mentioned in Theorem 2.6. Moreover, it should be noted that if $a = 1$ then the DSLP is equivalence to a classic SLP. Theorem 2.6 shows the conformity of the spectra of the DSLP's and SLP.

The interlacing of spectra can be phrased as the following corollary.

Corollary 2.8. *Fixed $a > 0$, for every $n > 0$, only one of the following alternative is valid:*

(i) *The interval* (λ_0, λ_n) *contains exactly* n*(counting multiplicities) elements of set* $\{\mu_n^D\}_{n=1}^{+\infty} \cup \{\nu_n^D\}_{n=1}^{+\infty}$*, and* $\lambda_n \notin \{\mu_n^D\}_{n=1}^{+\infty} \cup \{\nu_n^D\}_{n=1}^{+\infty}$.

(ii) *The interval* (λ_0, λ_n) *contains exactly* $n-1$*(counting multiplicities) elements of set* $\{\mu_n^D\}_{n=1}^{+\infty} \cup \{\nu_n^D\}_{n=1}^{+\infty}$*, and* $\lambda_n \in \{\mu_n^D\}_{n=1}^{+\infty} \cup \{\nu_n^D\}_{n=1}^{+\infty}$.

Specially when $a = 1$, Corollary 2.8 is the case of Theorem 1.1 in [9].

Attention is now focused on the result corresponding to symmetric case phrased by H. Hochstadt.

When $d = \frac{1}{2}$, $h = H$ and $q(x)$ is symmetric (that means $q(1-x) = q(x)$), we conclude that $\mu_n^D = \nu_n^D$ $(n = 1, 2, \cdots)$. So all of $\mu_n^D = \nu_n^D$ $(n = 1, 2, \cdots)$ are eigenvalues of the DSLP, and each interval, $(-\infty, \nu_1^D)$ and (ν_n^D, ν_{n+1}^D) $(n = 1, 2, \cdots)$, contains exactly one eigenvalue of the DSLP.

Corollary 2.9. *If* $d = \frac{1}{2}$, $h = H$ *and* $q(x)$ *is symmetric, then* $\lambda_{2n-1} = \mu_n^D = \nu_n^D$ $(n = 1, 2, \cdots)$*, while* $\lambda_{2n} \in (\mu_n^D, \mu_{n+1}^D) = (\nu_n^D, \nu_{n+1}^D)$ $(n = 0, 1, \cdots)$*, where* $\mu_0^D = \nu_0^D := -\infty$.

3. The Interlacing Associated with Non-dirichlet Condition

In this section, we will consider the interlacing of the spectra of the DSLP and two SLP's on two subintervals $[0, d]$ and $[d, 1]$, which are imposed the non-Dirichlet boundary conditions at d, respectively. Since their proofs are similar to the above, we omit all of them.

For every $h_0 \in \mathbb{R}$, the equation (1.1) and boundary conditions (1.2) and

$$y'(d) + a^2 h_0 y(d) = 0, \tag{3.1}$$

lead to an continuous SLP on $[0, d]$. The equation (1.1) and boundary conditions (1.3) and

$$y'(d) + h_0 y(d) = 0, \tag{3.2}$$

also lead to an continuous SLP on $[d, 1]$. Whose increasing sequences of eigenvalues are denoted by $\{\mu_n\}_{n=0}^{+\infty}$ and $\{\nu_n\}_{n=0}^{+\infty}$, respectively. Specially, the two sequence corresponding to $h_0 = 0$, be denoted by $\{\mu_n^N\}_{n=0}^{+\infty}$ and $\{\nu_n^N\}_{n=0}^{+\infty}$.

Define the Weyl-Titchmarsh M-functions

$$M_-(\lambda) = \frac{v_-(d,\lambda)}{v_-'(d,\lambda) + a^2 h_0 v_-(d,\lambda)} = -\frac{1}{m_-(\lambda) - a^2 h_0}, \tag{3.3}$$

$$M_+(\lambda) = -\frac{v_+(d,\lambda)}{v_+'(d,\lambda) + h_0 v_+(d,\lambda)} = -\frac{1}{m_+(\lambda) + h_0}. \tag{3.4}$$

It is easy to check that $M_\pm(\lambda)$ are also the Herglotz functions by the facts that $m_\pm(\lambda)$ are the Herglotz functions.

Lemma 3.1. *λ^* is a zero of $\Delta(\lambda)$ if and only if*

$$M_+(\lambda^*) = -a^2 M_-(\lambda^*) \tag{3.5}$$

where both sides allow ∞, but not take $\pm\infty$ apart.

Lemma 3.2. *(i) $M_-(\lambda)$ is always positive on $(-\infty, \mu_0)$, and continuously monotonically increase from 0 to $+\infty$.*

(ii) While on (μ_n, μ_{n+1}) $(n = 0, 1, \cdots)$, which is continuously monotonically increase from $-\infty$ to $+\infty$.

By Lemma 2.4, $m_-(\lambda) - a^2h_0 \to -\infty$ as $\lambda \to -\infty$, and $m_-(\lambda) - a^2h_0 \to 0$ as $\lambda \to \mu_0$. This means $m_-(\lambda) - a^2h_0 < 0$ when $\lambda \in (-\infty, \mu_0)$ and, therefore, $M_-(\lambda) = -1/(m_-(\lambda) - a^2h_0) > 0$. The result (ii) in Lemma 3.2 can be derived from Lemma 2.4 and (3.3).

Lemma 3.3. *$M_+(\lambda)$ is always positive on $(-\infty, \nu_0)$, and continuously monotonically increase from 0 to $+\infty$. While on (ν_n, ν_{n+1}) $(n = 0, 1, \cdots)$, which is continuously and strictly monotonically increase from $-\infty$ to $+\infty$.*

Theorem 3.4. *(i) Rearrange $\{\mu_n\}_{n=0}^{+\infty} \cup \{\nu_n\}_{n=0}^{+\infty}$, the new sequence will form a partition of $\mathbb{R}$, then each open subinterval of the partition contains exactly one eigenvalue of the DSLP, but $(-\infty, \min\{\mu_0, \nu_0\})$ does not contain any eigenvalue of the DSLP.*

(ii) If $\lambda^* \in \{\mu_n\}_{n=0}^{+\infty} \cap \{\nu_0\}_{n=0}^{+\infty}$, then λ^* is the eigenvalue of the DSLP. Instead, if some μ_k (resp. ν_k) is the eigenvalue of the DSLP, then $\mu_k \in \{\nu_n\}_{n=0}^{+\infty}$ (resp. $\nu_k \in \{\mu_n\}_{n=0}^{+\infty}$).

Consider the function

$$G(\lambda) = M_+(\lambda) + a^2 M_-(\lambda). \tag{3.6}$$

Together with Lemmas 3.1-3.3, similar to the proof of Theorem 2.6, we conclude that except for $(-\infty, \min\{\mu_0, \nu_0\})$, each open subinterval of the partition contains exactly one eigenvalue of the DSLP, and the conclusion (ii) is correct.

With regard to $(-\infty, \min\{\mu_0, \nu_0\})$, following from Lemmas 3.2 and 3.3, we will find that $G(\lambda)$ has no zero on this interval.

Remark 3.5. Different from Theorem 2.6, there is no eigenvalue of the DSLP on interval $(-\infty, \min\{\mu_0, \nu_0\})$, which implies that $\min\{\mu_0, \nu_0\}$ is the lower bound for the eigenvalues of the DSLP's to all $a > 0$ and is also the lower bound for the eigenvalues of the SLP.

Corollary 3.6. *Fixed $a > 0$, for every $n > 0$, one of the following alternative is valid:*

(i) *The interval* $(-\infty, \lambda_n)$ *contains exactly* $n+1$ $($*counting multiplicities*$)$ *elements of set* $\{\mu_n\}_{n=0}^{+\infty} \cup \{\nu_n\}_{n=0}^{+\infty}$, *and* $\lambda_n \notin \{\mu_n\}_{n=0}^{+\infty} \cup \{\nu_n\}_{n=0}^{+\infty}$.

(ii) *The interval* $(-\infty, \lambda_n)$ *contains exactly* n*(counting multiplicities) elements of set* $\{\mu_n\}_{n=0}^{+\infty} \cup \{\nu_n\}_{n=0}^{+\infty}$, *and* $\lambda_n \in \{\mu_n\}_{n=0}^{+\infty} \cup \{\nu_n\}_{n=0}^{+\infty}$.

If $d = \frac{1}{2}$, $h = H$ and $q(x)$ is symmetric, we then have $\mu_n^N = \nu_n^N$ $(n = 0, 1, \cdots)$. So all of $\mu_n^N = \nu_n^N$ $(n = 0, 1, \cdots)$ are the eigenvalues of the DSLP, and each open interval (ν_n^N, ν_{n+1}^N) $(n = 0, 1, \cdots)$, contains exactly one eigenvalue of the DSLP, but there is no eigenvalue of the DSLP on interval $(-\infty, \nu_0^N)$, which can be phrased as the following corollary.

Corollary 3.7. *If* $d = \frac{1}{2}$, $h = H$ *and* $q(x)$ *is symmetric, then we have that* $\lambda_{2n} = \mu_n^N = \nu_n^N$ $(n = 0, 1, \cdots)$, *and* $\lambda_{2n+1} \in (\mu_n^N, \mu_{n+1}^N) = (\nu_n^N, \nu_{n+1}^N)$ $(n = 0, 1, \cdots)$.

Theorem 3.8. *If* $d = \frac{1}{2}$, $h = H$ *and* $q(x)$ *is symmetric, for every* $a > 0$, *all eigenvalues of the DSLP,* $\{\lambda_n\}_{n=0}^{+\infty}$, *satisfy*

$$\lambda_{2n} = \mu_n^N = \nu_n^N, \quad n = 0, 1, \cdots, \tag{3.7}$$

and

$$\lambda_{2n-1} = \mu_n^D = \nu_n^D, \quad n = 1, 2, \cdots. \tag{3.8}$$

Proof. It can be obtained from Corollary 3.7 and Corollary 2.9 together. ■

Remark 3.9. Does that mean, on symmetric case, for every a, the sets of eigenvalues of the DSLP's all are equal to $\{\mu_n^N\}_{n=0}^{\infty} \cup \{\mu_n^D\}_{n=0}^{\infty}$.

4. Determination of the Potential

In this section, we will give three theorems of discontinuous inverse spectral problems as the applications of the interlacing in Sections 2 and 3. The technique which used to prove these theorems is an adaptation of the method discussed by F. Gesztesy and B. Simon in [5]. We need following lemma on asymptotics, poles and residues determining a meromorphic Herglotz function, see Theorem 2.3 in [5].

Lemma 4.1. *Let* $f_1(z)$ *and* $f_2(z)$ *be two meromorphic Herglotz functions with identical sets of poles and residues, respectively. If*

$$f_1(ix) - f_2(ix) \to 0, \quad as \ x \to \infty,$$

then $f_1 = f_2$.

We need still the classic theory of inverse spectral problems, that is, the Borg theorem: two spectra can uniquely determine the potential function of the SLP.

Theorem 4.2. *Let h, H, $d \in (0,1)$ and $a > 0$ be fixed. If $\{\mu_n^D\}_{n=1}^{+\infty}$, $\{\nu_n^D\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$ are pairwise disjoint, then these three spectra determine the potential q uniquely on $[0,1]$.*

Remark 4.3. This result is a extension of [9, 5] to the discontinuous problems with one jump point. For the pairwise disjoint condition, indeed, we need know only $\{\mu_n^D\}_{n=1}^{+\infty}$ and $\{\nu_n^D\}_{n=1}^{\infty}$ are disjoint, see Theorems 2.6(ii) in Section 2.

Proof of Theorem 4.2. Rewrite $g(\lambda)$ defined in (2.18) as

$$g(\lambda) = -\frac{\Delta(\lambda, a)}{v_-(d,\lambda)v_+(d,\lambda)}.$$

It is clear that the set of poles of $g(\lambda)$ is precisely $\{\mu_n^D\}_{n=1}^{+\infty} \cup \{\nu_n^D\}_{n=1}^{\infty}$ and the set of zeros of $g(\lambda)$ is precisely $\{\lambda_n\}_{n=0}^{+\infty}$.

Let $\tilde{q}(x)$ be another potential such that $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n^D\}_{n=1}^{+\infty}$ and $\{\nu_n^D\}_{n=1}^{\infty}$ are the eigenvalues of the DSLP consisting of (1.1-1.4) and the SLP's consisting of (1.1-1.3) and (2.4) with $q(x)$ replaced by $\tilde{q}(x)$, respectively. Define $\tilde{m}_+(\lambda)$, $\tilde{m}_-(\lambda)$ and $\tilde{g}(\lambda)$ in an analogous manner (see (2.5) and (2.18)).

Let

$$F(\lambda) = g(\lambda)/\tilde{g}(\lambda).$$

Then F is an entire function, since g has the same zeros and poles with $\tilde{g}$. Recall the asymptotic behavior of m-functions (2.16) . For any $\varepsilon > 0$,

$$F(\lambda) = g(\lambda)/\tilde{g}(\lambda) = 1 + O(\frac{1}{\sqrt{\lambda}})$$

holds in the sector of $\varepsilon \le \arg\lambda \le 2\pi - \varepsilon$. By Liouville's Theorem, we have

$$F(\lambda) \equiv 1,$$

which concludes

$$g(\lambda) = \tilde{g}(\lambda). \tag{4.1}$$

Note that the poles of $m_-(\lambda)$ and $m_+(\lambda)$ are precisely the points of $\{\mu_n^D\}_{n=1}^{+\infty}$ and $\{\nu_n^D\}_{n=1}^{\infty}$, respectively. By the hypothesis of the theorem that $\{\mu_n^D\}_{n=1}^{+\infty}$ and $\{\nu_n^D\}_{n=1}^{\infty}$ are disjoint. We have

$$\texttt{res}\ m_-(\mu_n^D) = \texttt{res}\ g(\mu_n^D), \quad \texttt{res}\ m_+(\nu_n^D) = \frac{1}{a^2}\texttt{res}\ g(\nu_n^D),$$

for $n = 1, 2, \cdots$. This means

$$\texttt{res}\ m_-(\mu_n^D) = \texttt{res}\ \tilde{m}_-(\mu_n^D), \quad \texttt{res}\ m_+(\nu_n^D) = \texttt{res}\ \tilde{m}_+(\nu_n^D),$$

for all $n \geq 1$. This, together with Lemma 4.1 and the asymptotic behavior (2.16), obtains

$$m_\pm(\lambda) = \tilde{m}_\pm(\lambda).$$

Therefore $q(x) = \tilde{q}(x)$ on $[0, d]$ and $[d, 1]$, respectively, by Borg theorem [8]. Thus, completes the proof of Theorem 4.2. ∎

Theorem 4.4. *Let h, H, $d \in (0,1)$ and $a > 0$ be fixed. If $\{\mu_n\}_{n=0}^{+\infty} \cap \{\nu_n\}_{n=0}^{\infty} = \emptyset$, then these two sets together with $\{\lambda_n\}_{n=0}^{\infty}$ can determine the potential q uniquely on $[0,1]$.*

Proof. Rewrite $G(\lambda)$ defined in (3.6) as

$$G(\lambda) = -\frac{\Delta(\lambda)}{[v_-'(d,\lambda) + h_0 v_-(d,\lambda)][v_+'(d,\lambda) + a^2 h_0 v_+(d,\lambda)]}. \tag{4.2}$$

It is clear that the set of poles of $G(\lambda)$ is precisely $\{\mu_n\}_{n=0}^{+\infty} \cup \{\nu_n\}_{n=0}^{\infty}$ and the set of zeros of $G(\lambda)$ is precisely $\{\lambda_n\}_{n=0}^{+\infty}$.

As in the proof of Theorem 4.2, we can prove that the given three disjoint spectra determine all residues of $M_\pm(\lambda)$.

Since $M_\pm(\lambda)$ have the asymptotic behavior

$$M_\pm(\lambda) = -\frac{1}{m_\pm(\lambda) + const} = -\frac{i}{\sqrt{\lambda}}\Big(1 + o(\frac{1}{\sqrt{\lambda}})\Big), \quad \texttt{as}\ \lambda \to \infty\ \texttt{in}\ \mathbb{C}, \tag{4.3}$$

when $\varepsilon \leq \arg\lambda \leq 2\pi - \varepsilon$ for any $\varepsilon > 0$. By Lemma 4.1, it determines the functions $M_\pm(\lambda)$. Then $M_\pm(\lambda)$ determine the $q(x)$ on $[0, d]$ and $[d, 1]$, respectively, by Borg theorem. Thus, completes the proof of Theorem 4.4. ∎

Theorem 4.5. *If $d = \frac{1}{2}$, $h = H$ and $q(x)$ is symmetric, for every $a > 0$, $\{\lambda_n\}_{n=0}^{\infty}$ determine the potential q uniquely on $[0,1]$.*

Remark 4.6. When $a = 1$, this is the symmetric case of Borg theorem. We will prove it holds also for the DSLP.

Proof. By Corollary 2.9,

$$\mu_n^D = \nu_n^D = \lambda_{2n-1}\ (n = 1, 2, \cdots). \tag{4.4}$$

Consider $g(\lambda)$ and $\tilde{g}(\lambda)$, again. $g(\lambda) = \tilde{g}(\lambda)$ still holds as (4.1) in the proof of Theorem 4.2. By the symmetry hypothesis, we have

$$v_-(x,\lambda) = v_+(1-x,\lambda), \quad v_-'(x,\lambda) = v_+'(1-x,\lambda). \tag{4.5}$$

This means

$$m_-(\lambda) = m_+(\lambda). \tag{4.6}$$

So we have

$$m_-(\lambda) = m_+(\lambda) = \frac{1}{1+a^2} g(\lambda). \tag{4.7}$$

Then $m_\pm(\lambda)$ determine uniquely the $q(x)$ on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively, by Borg theorem. ∎

Remark 4.7. For different $a > 0$, all of $g(\lambda)$ have the same zeros and poles, but $g(\lambda)$ is dependent on a.

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