TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 2, pp. 479-496, April 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

A NOTE ON INTEGRAL INEQUALITIES OF HADAMARD TYPE FOR LOG-CONVEX AND LOG-CONCAVE FUNCTIONS

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Abstract. In this note, we establish new inequalities of Hadamard type involving several log-convex functions and log-concave functions.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:[a,b]\to \mathbf{R}$ is a convex function with $a<b$ is well known in the literature as the *Hadamard inequality* (see $[4]$). A function $f: I \to (0,\infty), I$ is an interval in $\mathbf{R}$, is said to be *log-convex function*, if for all $x, y \in I$ and $t\in[0,1]$ one has the inequality (see $[6,\ p.\ 3]$):

$$f(tx+(1-t)y) \le [f(x)]^t[f(y)]^{1-t}\ ,$$

f is said to be *log-concave* if

$$f(tx+(1-t)y) \ge [f(x)]^t[f(y)]^{1-t}\ .$$

Recall that the extended logarithmic mean L_p of two positive numbers a, b is given for $a=b$ by $L_p(a,a)=a$ and for $a\neq b$ by

$$L_p(a,b)=\begin{cases}\left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p\neq -1,0\\ \dfrac{b-a}{\ln b-\ln a}, & p=-1\\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{(b-a)}}, & p=0\end{cases}$$

Received April 23, 2007, accepted March 3, 2008.
Communicated by Sen-Yen Shaw.
2010 *Mathematics Subject Classification*: 26D15, 26D99.
Key words and phrases: Hadamard type inequalities, Log-convex functions, Log-concave functions, Geometric mean, Extended logarithmic mean.
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where the $L_{-1}(a,b)$ is the logarithmic mean $L(a,b)$. In [2], Dragomir and Mond proved that the following inequalities hold for log-convex function f:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \exp\left[\frac{1}{b-a}\int_a^b \ln[f(x)]dx\right] \\ &\le \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))dx \\ &\le \frac{1}{b-a}\int_a^b f(x)dx \le L(f(a), f(b)) \\ &\le \frac{f(a)+f(b)}{2}, \end{aligned}$$

where

$$G(p,g) = \sqrt{pq},$$

is the geometric mean. For the further refinements of (1.1) for log-convex functions and various other results related to (1.1), see $[1-3]$ and $[5-7]$. In [6] Pachpatte proved the following inequalities involving two log-convex functions:

Theorem 1.1. *Let $f, g : I \to (0,\infty)$ be log-convex functions on I and $a, b \in I$ with $a < b$. Then*

$$\begin{aligned} &\frac{2}{b-a}\int_a^b f(x)g(x)dx \\ &\le \frac{f(a)+f(b)}{2}L(f(a), f(b)) + \frac{g(a)+g(b)}{2}L(g(a), g(b)) \,. \end{aligned} \tag{1.2}$$

Theorem 1.2. *Let $f, g : I \to (0,\infty)$ be differentiable log-convex functions on the interval of real numbers I^0 (the interior of I) and $a, b \in I^0$ with $a < b$. Then*

$$\begin{aligned} &\frac{2}{b-a}\int_a^b f(x)g(x)dx \\ &\ge \frac{1}{b-a} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\ &+\frac{1}{b-a} g\left(\frac{a+b}{2}\right)\int_a^b f(x)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx. \end{aligned} \tag{1.3}$$

The main purpose of this note is to establish some generalizations of Theorems 1.1 and 1.2 as well as some new inequalities involving several log-convex functions and log-concave functions.

2. Main Results

Theorem 2.1. *Let* f, g, a, b *be as in Theorem 1.1 and* $\alpha, \beta > 0$ $with$ $\alpha+\beta = 1$. *Then the following inequality holds:*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \alpha\left[L_{\frac{1}{\alpha}-1}(f(a),f(b))\right]^{\frac{1-\alpha}{\alpha}} L(f(a),f(b)) \\ +\beta\left[L_{\frac{1}{\beta}-1}(g(a),g(b))\right]^{\frac{1-\beta}{\beta}} L(g(a),g(b)).\end{aligned} \tag{2.1}$$

Proof. Since f, g are log-convex functions, we have

$$f(ta+(1-t)b) \le [f(a)]^t[f(b)]^{1-t}, \tag{2.2}$$

$$g(ta+(1-t)b) \le [g(a)]^t[g(b)]^{1-t}, \tag{2.3}$$

for all $t \in [0, 1]$. It is easy to observe that

$$\int_a^b f(x)g(x)dx = (b-a)\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt. \tag{2.4}$$

Using the known inequality $cd \le \alpha c^{\frac{1}{\alpha}} + \beta d^{\frac{1}{\beta}}$ ($\alpha, \beta > 0$ and $\alpha+\beta = 1$), (2.2), (2.3) on the right side of (2.4) and making the change of variable we have

$$\begin{aligned}&\int_a^b f(x)g(x)dx \\ &\le (b-a)\int_0^1 \{\alpha\,[f(ta+(1-t)b)]^{\frac{1}{\alpha}} + \beta\,[g(ta+(1-t)b)]^{\frac{1}{\beta}}\}dt \\ &\le\ (b-a)\int_0^1 \left[\alpha\{[f(a)]^t[f(b)]^{1-t}\}^{\frac{1}{\alpha}} + \beta\{[g(a)]^t[g(b)]^{1-t}\}^{\frac{1}{\beta}}\right]dt \\ &=\ (b-a)\left\{\alpha f^{\frac{1}{\alpha}}(b)\int_0^1\left[\frac{f(a)}{f(b)}\right]^{\frac{t}{\alpha}}dt + \beta g^{\frac{1}{\beta}}(b)\int_0^1\left[\frac{g(a)}{g(b)}\right]^{\frac{t}{\beta}}dt\right\} \\ &= (b-a)\left\{\alpha^2 f^{\frac{1}{\alpha}}(b)\int_0^{\frac{1}{\alpha}}\left[\frac{f(a)}{f(b)}\right]^{\sigma}d\sigma + \beta^2 g^{\frac{1}{\beta}}(b)\int_0^{\frac{1}{\beta}}\left[\frac{g(a)}{g(b)}\right]^{\phi}d\phi\right\} \\ &= (b-a)\left\{\alpha^2 f^{\frac{1}{\alpha}}(b)\left[\frac{\left[\frac{f(a)}{f(b)}\right]^{\sigma}}{\log\frac{f(a)}{f(b)}}\right]_0^{\frac{1}{\alpha}} + \beta^2 g^{\frac{1}{\beta}}(b)\left[\frac{\left[\frac{g(a)}{g(b)}\right]^{\phi}}{\log\frac{g(a)}{g(b)}}\right]_0^{\frac{1}{\beta}}\right\} \\ &= (b-a)\left[\alpha^2\left(\frac{f^{\frac{1}{\alpha}}(a)-f^{\frac{1}{\alpha}}(b)}{\log f(a)-\log f(b)}\right) + \beta^2\left(\frac{g^{\frac{1}{\beta}}(a)-g^{\frac{1}{\beta}}(b)}{\log g(a)-\log g(b)}\right)\right]\end{aligned} \tag{2.5}$$

$$= (b-a)\left\{\alpha^2\left(\frac{f^{\frac{1}{\alpha}}(a)-f^{\frac{1}{\alpha}}(b)}{f(a)-f(b)}\right)L(f(a),f(b))\right.$$
$$\left.+\beta^2\left(\frac{g^{\frac{1}{\beta}}(a)-g^{\frac{1}{\beta}}(b)}{g(a)-g(b)}\right)L(g(a),g(b))\right\}$$
$$= (b-a)\left\{\alpha\left[L_{\frac{1}{\alpha}-1}\left(f(a),f(b)\right)\right]^{\frac{1-\alpha}{\alpha}}L(f(a),f(b))\right.$$
$$\left.+\beta\left[L_{\frac{1}{\beta}-1}(g(a),g(b))\right]^{\frac{1-\beta}{\beta}}L(g(a),g(b))\right\}.$$

Rewriting (2.5) we get the required inequality in (2.1). The proof is completed.

Remark 2.1. For $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, the inequality (2.1) reduces to (1.2).

Theorem 2.2. *Let* $f,g:I\to(0,\infty)$ *be log-concave functions on* I *and* $a,b\in I$ *with* $a<b$. *Further, let* $\alpha>1$ *with* $\alpha+\beta=1$ (*or* $\beta>1$ *with* $\alpha+\beta=1$). *Then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \ge \alpha\left[L_{\frac{1}{\alpha}-1}(f(a),f(b))\right]^{\frac{1-\alpha}{\alpha}}L(f(a),f(b)) + \beta\left[L_{\frac{1}{\beta}-1}(g(a),g(b))\right]^{\frac{1-\beta}{\beta}}L(g(a),g(b)). \tag{2.6}$$

Proof. Since f,g are log-concave functions, we have

$$f(ta+(1-t)b)\ge[f(a)]^t[f(b)]^{1-t}, \tag{2.7}$$

$$g(ta+(1-t)b)\ge[g(a)]^t[g(b)]^{1-t}, \tag{2.8}$$

for all $t\in[0,1]$. Using the known inequality $cd\ge\alpha c^{\frac{1}{\alpha}}+\beta d^{\frac{1}{\beta}}$, (2.7), (2.8) on the right side of (2.4) and making the change of variable we have

$$\begin{aligned}
&\int_a^b f(x)g(x)dx\\
&\ge (b-a)\int_0^1\{\alpha\,[f(ta+(1-t)b)]^{\frac{1}{\alpha}}+\beta\,[g(ta+(1-t)b)]^{\frac{1}{\beta}}\}dt\\
&\ge (b-a)\int_0^1\left[\alpha\{[f(a)]^t[f(b)]^{1-t}\}^{\frac{1}{\alpha}}+\beta\{[g(a)]^t[g(b)]^{1-t}\}^{\frac{1}{\beta}}\right]dt\\
&= (b-a)\left\{\alpha\left[L_{\frac{1}{\alpha}-1}\left(f(a),f(b)\right)\right]^{\frac{1-\alpha}{\alpha}}L(f(a),f(b))\right.\\
&\quad\left.+\beta\left[L_{\frac{1}{\beta}-1}(g(a),g(b))\right]^{\frac{1-\beta}{\beta}}L(g(a),g(b))\right\}.
\end{aligned} \tag{2.9}$$

Rewriting (2.9) we get the required inequality in (2.6). The proof is completed.

Theorem 2.3. *Let f, g, a, b be as in Theorem1.2 and $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Then the following inequality holds:*

$$\begin{aligned} &\int_a^b f(x)g(x)dx \\ &\geq \alpha f\left(\frac{a+b}{2}\right)\int_a^b g(x)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\ &\quad+\beta g\left(\frac{a+b}{2}\right)\int_a^b f(x)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \end{aligned} \tag{2.10}$$

Proof. Since f, g are differentiable and log-convex functions on I^0, we have that

$$\log f(x) - \log f(y) = \log\left(\frac{f(x)}{f(y)}\right) \geq \frac{f'(y)}{f(y)}(x-y), \tag{2.11}$$

$$\log g(x) - \log g(y) = \log\left(\frac{g(x)}{g(y)}\right) \geq \frac{g'(y)}{g(y)}(x-y), \tag{2.12}$$

for all $x, y \in I^0$.That is

$$f(x) \geq f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right], \tag{2.13}$$

$$g(x) \geq g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right], \tag{2.14}$$

Multiplying both sides of (2.13) and (2.14) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities we have

$$\begin{aligned} &f(x)g(x) \\ &\geq \alpha g(x)f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right] + \beta f(x)g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right]. \end{aligned} \tag{2.15}$$

By taking $y = \frac{a+b}{2}$ in (2.15) we have

$$\begin{aligned} f(x)g(x) &\geq \alpha g(x)f\left(\frac{a+b}{2}\right)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right] \\ &\quad+\beta f(x)g\left(\frac{a+b}{2}\right)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]. \end{aligned} \tag{2.16}$$

Integrating both sides of (2.14) with respect to x from a to b, we get the desired inequality (2.10).

Remark 2.2. For $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, the inequality (2.6) reduces to the inequality (1.3).

Theorem 2.4. *Let $f, g : I \to (0,\infty)$ be differentiable log-concave functions on the interval of real numbers I^0 and a, b, α, β be as in Theorem 2.3. Then the following inequality holds:*

$$\begin{aligned}\int_a^b f(x)g(x)dx \leq\ & \alpha f\left(\frac{a+b}{2}\right)\int_a^b g(x)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\ & +\beta g\left(\frac{a+b}{2}\right)\int_a^b f(x)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx.\end{aligned} \tag{2.17}$$

Proof. Since f, g are differentiable and log-concave functions on I^0, we have that

$$\log f(x) - \log f(y) = \log\left(\frac{f(x)}{f(y)}\right) \leq \frac{f'(y)}{f(y)}(x-y), \tag{2.18}$$

$$\log g(x) - \log g(y) = \log\left(\frac{g(x)}{g(y)}\right) \leq \frac{g'(y)}{g(y)}(x-y), \tag{2.19}$$

for all $x, y \in I^0$. That is

$$f(x) \leq f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right], \tag{2.20}$$

$$g(x) \leq g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right]. \tag{2.21}$$

Multiplying both sides of (2.20) and (2.21) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities we have

$$\begin{aligned}& f(x)g(x) \\ & \leq \alpha g(x)f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right] + \beta f(x)g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right].\end{aligned} \tag{2.22}$$

By taking $y = \frac{a+b}{2}$ in (2.22) we have

$$\begin{aligned}f(x)g(x) \leq\ & \alpha g(x) f\left(\frac{a+b}{2}\right)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right] \\ & +\beta f(x) g\left(\frac{a+b}{2}\right)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right].\end{aligned} \tag{2.23}$$

Integrating both sides of (2.23) with respect to x from a to b, we get the desired inequality (2.17).

Theorem 2.5. *Let f, a, b be as in Theorem1.2 and g be as in Theorem 2.4. Further, let $\alpha > 1$ with $\alpha + \beta = 1$, then the following inequality holds:*

$$\begin{aligned} &\int_a^b f(x)g(x)dx \\ &\geq \alpha f\left(\frac{a+b}{2}\right)\int_a^b g(x)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\ &\quad +\beta g\left(\frac{a+b}{2}\right)\int_a^b f(x)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx. \end{aligned} \tag{2.24}$$

Proof. Since f is differentiable and log-convex functions on I^0 and g is differentiable and log-concave functions on I^0, we have that

$$f(x) \geq f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right], \tag{2.25}$$

$$g(x) \leq g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right]. \tag{2.26}$$

Multiplying both sides of (2.25) and (2.26) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities we have

$$\begin{aligned} &f(x)g(x) \\ &\geq \alpha g(x)f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right] + \beta f(x)g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right]. \end{aligned} \tag{2.27}$$

By taking $y = \frac{a+b}{2}$ in (2.27) we have

$$\begin{aligned} f(x)g(x) &\geq \alpha g(x)f\left(\frac{a+b}{2}\right)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right] \\ &\quad +\beta f(x)g\left(\frac{a+b}{2}\right)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]. \end{aligned} \tag{2.28}$$

Integrating both sides of (2.28) with respect to x from a to b, we get the desired inequality (2.24).

Theorem 2.6. *Let g, a, b be as in Theorem 1.2 and f be as in Theorem 2.4.*

Further, let $\alpha > 1$ $with$ $\alpha + \beta = 1$, then the following inequality holds:

$$\begin{aligned}(2.29)\qquad &\int_a^b f(x)g(x)dx \\ &\le \alpha f\left(\frac{a+b}{2}\right)\int_a^b g(x)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\ &\quad +\beta g\left(\frac{a+b}{2}\right)\int_a^b f(x)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx.\end{aligned}$$

Proof. Multiplying both sides of (2.20) and (2.14) by $\alpha g(x)$ and $\beta f(x)$ respectively and adding the resulting inequalities we have

$$\begin{aligned}(2.30)\qquad &f(x)g(x) \\ &\le \alpha g(x)f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right] + \beta f(x)g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right].\end{aligned}$$

By taking $y = \frac{a+b}{2}$ in (2.30) we have

$$\begin{aligned}(2.31)\qquad f(x)g(x) &\le \alpha g(x) f\left(\frac{a+b}{2}\right)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right] \\ &\quad +\beta f(x) g\left(\frac{a+b}{2}\right)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right].\end{aligned}$$

Integrating both sides of (2.31) with respect to x from a to b, we get the desired inequality (2.29).

Theorem 2.7. *Let $f_1, f_{2,\cdots,}f_n : I \to (0,\infty)$ be log-convex functions on I and $a, b \in I$ with $a < b$. Further, let α_1,α_2,$\cdots$,$\alpha_n > 0$ with $\sum_{i=1}^{n}\alpha_i = 1$. Then the following inequality holds:*

$$\begin{aligned}(2.32)\qquad &\frac{1}{b-a}\int_a^b \sum_{i=1}^{n} f_i(x)dx \\ &\le \sum_{i=1}^{n}\left\{\alpha_i\left[L_{\frac{1}{\alpha_i}-1}(f_i(a), f_i(b))\right]^{\frac{1-\alpha_i}{\alpha_i}} L(f_i(a), f_i(b))\right\}.\end{aligned}$$

Proof. Since $f_1, f_2, \cdots, f_n$ are log-convex functions, we have

$$(2.33)\qquad f_i(ta + (1-t)b) \le [f_i(a)]^t[f_i(b)]^{1-t},$$

for all $t \in [0,1]$, $i = 1, 2, \cdots, n$. Since

$$\int_a^b \sum_{i=1}^n f_i(x)dx = (b-a)\int_0^1 \sum_{i=1}^n f_i(ta+(1-t)b)dt. \tag{2.34}$$

Using the inequality $f_1 f_2 \cdots f_n \le \alpha_1 (f_1)^{\frac{1}{\alpha_1}} + \alpha_2 (f_2)^{\frac{1}{\alpha_2}} + \cdots + \alpha_n (f_n)^{\frac{1}{\alpha_n}}$ and (2.33) on the right side of (2.34) and making the change of variable we have

$$\begin{aligned}
&\int_a^b \sum_{i=1}^n f_i(x)dx \\
&\le (b-a)\int_0^1 \left\{\sum_{i=1}^n \alpha_i \left[f_i(ta+(1-t)b)\right]^{\frac{1}{\alpha_i}}\right\} dt \\
&\le (b-a)\int_0^1 \left[\sum_{i=1}^n \alpha_i \{[f_i(a)]^t [f_i(b)]^{1-t}\}^{\frac{1}{\alpha_i}}\right] dt \\
&= (b-a)\sum_{i=1}^n \left\{\alpha_i f_i^{\frac{1}{\alpha_i}}(b)\int_0^1 \left[\frac{f_i(a)}{f_i(b)}\right]^{\frac{t}{\alpha_i}} dt\right\} \\
&= (b-a)\sum_{i=1}^n \left\{(\alpha_i)^2 f_i^{\frac{1}{\alpha_i}}(b)\int_0^{\frac{1}{\alpha_i}} \left[\frac{f_i(a)}{f_i(b)}\right]^{\sigma} d\sigma\right\} \\
&= (b-a)\sum_{i=1}^n \left[(\alpha_i)^2 \left(\frac{f_i^{\frac{1}{\alpha_i}}(a) - f_i^{\frac{1}{\alpha_i}}(b)}{\log f_i(a) - \log f_i(b)}\right)\right] \\
&= (b-a)\sum_{i=1}^n \left[(\alpha_i)^2 \left(\frac{f_i^{\frac{1}{\alpha_i}}(a) - f_i^{\frac{1}{\alpha_i}}(b)}{f_i(a) - f_i(b)}\right) L(f_i(a), f_i(b))\right] \\
&= (b-a)\sum_{i=1}^n \left\{\alpha_i \left[L_{\frac{1}{\alpha_i}-1}(f_i(a), f_i(b))\right]^{\frac{1-\alpha_i}{\alpha_i}} L(f_i(a), f_i(b))\right\}.
\end{aligned} \tag{2.35}$$

Rewriting (2.35) we get the required inequality in (2.32). The proof is completed.

Remark 2.3. For $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$, the inequality (2.32) reduces to

$$\begin{aligned}
&\frac{n}{b-a}\int_a^b \sum_{i=1}^n f_i(x)dx \\
&\le \sum_{i=1}^n \left[L_{n-1}(f_i(a) - f_i(b))\right]^{n-1} L\left(f_i(a) - f_i(b)\right).
\end{aligned} \tag{2.36}$$

Remark 2.4. If we choose $n = 2$ in (2.36), then (2.36) reduces to (1.2).

Theorem 2.8. *Let* $f_1, f_{2,\cdots,} f_n : I \to (0,\infty)$ *be log-concave functions on I and* $a, b \in I$ *with* $a < b$. *Further, let* $\alpha_1 > 1$ *and* $\alpha_2, \alpha_3, \cdots, \alpha_n < 0$ *with* $\sum_{i=1}^{n} \alpha_i = 1$, $and\ let \sum_{i=1}^{j} \alpha_i > 0,\ j = 2, 3, \cdots, n$. *Then the following inequality holds:*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \sum_{i=1}^{n} f_i(x)dx \\ &\geq \sum_{i=1}^{n}\left\{\alpha_i\left[L_{\frac{1}{\alpha_i}-1}(f_i(a), f_i(b))\right]^{\frac{1-\alpha_i}{\alpha_i}} L(f_i(a), f_i(b))\right\}. \end{aligned} \tag{2.37}$$

Proof. Since $f_1, f_2, \cdots, f_n$ are log-concave functions, we have

$$f_i(ta + (1-t)b) \geq [f_i(a)]^t [f_i(b)]^{1-t}, \tag{2.38}$$

for all $t \in [0, 1]$, $i = 1, 2, \cdots, n$. Since

$$\int_a^b \sum_{i=1}^{n} f_i(x)dx = (b-a)\int_0^1 \sum_{i=1}^{n} f_i(ta + (1-t)b)dt. \tag{2.39}$$

Using the inequality $f_1 f_2 \cdots f_n \geq \alpha_1 (f_1)^{\frac{1}{\alpha_1}} + \alpha_2 (f_2)^{\frac{1}{\alpha_2}} + \cdots + \alpha_n (f_n)^{\frac{1}{\alpha_n}}$ and (2.38) on the right side of (2.39) and making the change of variable we have

$$\begin{aligned} &\int_a^b \sum_{i=1}^{n} f_i(x)dx \\ &\geq (b-a)\int_0^1 \left\{\sum_{i=1}^{n} \alpha_i \left[f_i(ta + (1-t)b)\right]^{\frac{1}{\alpha_i}}\right\} dt \\ &\geq (b-a)\int_0^1 \left[\sum_{i=1}^{n} \alpha_i \{[f_i(a)]^t [f_i(b)]^{1-t}\}^{\frac{1}{\alpha_i}}\right] dt \\ &= (b-a)\sum_{i=1}^{n}\left\{\alpha_i\left[L_{\frac{1}{\alpha_i}-1}(f_i(a), f_i(b))\right]^{\frac{1-\alpha_i}{\alpha_i}} L(f_i(a), f_i(b))\right\}. \end{aligned} \tag{2.40}$$

Rewriting (2.40) we get the required inequality in (2.37). The proof is completed.

Theorem 2.9. *Let* f, g *and* $h : I \to (0, \infty)$ *be differentiable log-convex functions on the interval of real numbers* I^0 *and* $a, b \in I^0$ *with* $a < b$. *Then the following* *inequality holds:*

$$\begin{aligned}(2.41)\quad & 3\int_a^b f(x)g(x)h(x)dx \\ & \geq f\left(\frac{a+b}{2}\right)\int_a^b g(x)h(x)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\ & +g\left(\frac{a+b}{2}\right)\int_a^b f(x)h(x)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\ & +h\left(\frac{a+b}{2}\right)\int_a^b f(x)g(x)\exp\left[\frac{h'\left(\frac{a+b}{2}\right)}{h\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx.\end{aligned}$$

Proof. Since f, g and h are differentiable and log-convex functions on I^0, we have that

$$f(x) \geq f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right], \tag{2.42}$$

$$g(x) \geq g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right], \tag{2.43}$$

$$h(x) \geq h(y)\exp\left[\frac{h'(y)}{h(y)}(x-y)\right], \tag{2.44}$$

for all $x, y \in I^0$. Multiplying both sides of (2.42), (2.43) and (2.44) by $g(x)h(x)$, $f(x)h(x)$ and $f(x)g(x)$ respectively and adding the resulting inequalities we have

$$\begin{aligned}(2.45)\quad 3f(x)g(x)h(x) \geq\ & g(x)h(x)f(y)\exp\left[\frac{f'(y)}{f(y)}(x-y)\right] \\ & +f(x)h(x)g(y)\exp\left[\frac{g'(y)}{g(y)}(x-y)\right] \\ & +f(x)g(x)h(y)\exp\left[\frac{h'(y)}{h(y)}(x-y)\right].\end{aligned}$$

Now, if we choose $y = \frac{a+b}{2}$, from (2.45) we obtain

$$\begin{aligned}(2.46)\quad 3f(x)g(x)h(x) \geq\ & g(x)h(x)f\left(\frac{a+b}{2}\right)\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right] \\ & +f(x)h(x)g\left(\frac{a+b}{2}\right)\exp\left[\frac{g'\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right] \\ & +f(x)g(x)h\left(\frac{a+b}{2}\right)\exp\left[\frac{h'\left(\frac{a+b}{2}\right)}{h\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right].\end{aligned}$$

Integrating both sides of (2.46) with respect to x from a to b, we get the desired inequality (2.41). The proof is completed.

Remark 2.5. For $h(x) \equiv 1$, the inequality (2.41) is reduces to (1.3).

Remark 2.6. Since $\frac{e^x - e^{-x}}{2x} > 1$ for $x > 0$, it follows that if we choose $g(x) = h(x) \equiv 1$ in (2.41), we have

$$\begin{aligned}\int_a^b f(x)dx &\geq f\left(\frac{a+b}{2}\right)\int_a^b \exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(x - \frac{a+b}{2}\right)\right]dx \\ &= f\left(\frac{a+b}{2}\right)\frac{\exp\left[\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right] - \exp\left[-\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}\left(\frac{b-a}{2}\right)\right]}{\frac{f'\left(\frac{a+b}{2}\right)}{f\left(\frac{a+b}{2}\right)}(b-a)}(b-a) \\ &> f\left(\frac{a+b}{2}\right)(b-a)\end{aligned}$$

which is the first part of the inequality (1.1).

Theorem 2.10. *Let* $f_1,\ f_2 \cdots,\ f_n : I \to (0, \infty)$ *be differentiable log-convex functions on the interval of real numbers* I^0 *and* $a, b \in I^0$ *with* $a < b$*. Further, let* $\alpha_1, \alpha_2, \cdots, \alpha_n > 0$ *with* $\sum_{i=1}^{n} \alpha_i = 1$*.Then the following inequality holds:*

$$\begin{aligned}&\int_a^b \sum_{i=1}^{n} f_i(x)dx \\ &\geq \alpha_1 f_1\left(\frac{a+b}{2}\right)\int_a^b f_2(x)f_3(x)\cdots f_n(x)\exp\left[\frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)}\left(x - \frac{a+b}{2}\right)\right]dx \\ &\quad + \alpha_2 f_2\left(\frac{a+b}{2}\right)\int_a^b f_1(x)f_3(x)\cdots f_n(x)\exp\left[\frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)}\left(x - \frac{a+b}{2}\right)\right]dx \\ &\quad + \alpha_n f_n\left(\frac{a+b}{2}\right)\int_a^b f_1(x)\cdots f_{n-1}(x)\exp\left[\frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)}\left(x - \frac{a+b}{2}\right)\right]dx.\end{aligned} \tag{2.47}$$

Proof. Since $f_1, f_2, \cdots, f_n$ are differentiable and log-convex functions on I^0, we have

$$f_1(x) \geq f_1(y)\exp\left[\frac{f_1'(y)}{f_1(y)}(x-y)\right], \tag{2.48$-$1}$$

$$f_2(x) \geq f_2(y)\exp\left[\frac{f_2'(y)}{f_2(y)}(x-y)\right], \tag{2.48$-$2}$$

$$\vdots$$

$$f_n(x) \geq f_n(y) \exp\left[\frac{f_n'(y)}{f_n(y)}(x-y)\right], \tag{2.48$-n$}$$

for all $x, y \in I^0$. Multiplying both sides of $(2.48-1)$, $(2.48-2)$, $\cdots$ and $(2.48-n)$ by $\alpha_1 f_2(x) f_3(x) \cdots f_n(x)$, $\alpha_2 f_1(x) f_3(x) \cdots f_n(x)$, $\cdots$and $\alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x)$ respectively and adding the resulting inequalities we have

$$\begin{aligned}\sum_{i=1}^{n} f_i(x) &\geq \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1(y) \exp\left[\frac{f_1'(y)}{f_1(y)}(x-y)\right] \\ &\quad + \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2(y) \exp\left[\frac{f_2'(y)}{f_2(y)}(x-y)\right] \\ &\quad \vdots \\ &\quad + \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n(y) \exp\left[\frac{f_n'(y)}{f_n(y)}(x-y)\right].\end{aligned} \tag{2.49}$$

Now, if we choose $y = \frac{a+b}{2}$, from (2.49) we obtain

$$\begin{aligned}&\sum_{i=1}^{n} f_i(x) \\ &\geq \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1\left(\frac{a+b}{2}\right) \exp\left[\frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)}(x - \frac{a+b}{2})\right] \\ &\quad + \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2\left(\frac{a+b}{2}\right) \exp\left[\frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)}(x - \frac{a+b}{2})\right] \\ &\quad \vdots \\ &\quad + \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n\left(\frac{a+b}{2}\right) \exp\left[\frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)}(x - \frac{a+b}{2})\right].\end{aligned} \tag{2.50}$$

Integrating both sides of (2.50) with respect to x from a to b, we get the desired inequality (2.47). The proof is completed.

Remark 2.7. If $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{n}$, then the inequality (2.47) reduces to

$$\begin{aligned}&n \int_a^b \sum_{i=1}^{n} f_i(x) dx \\ &\geq f_1\left(\frac{a+b}{2}\right) \int_a^b f_2(x) f_3(x) \cdots f_n(x) \exp\left[\frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)}\left(x - \frac{a+b}{2}\right)\right] dx \\ &\quad + f_2\left(\frac{a+b}{2}\right) \int_a^b f_1(x) f_3(x) \cdots f_n(x) \exp\left[\frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)}\left(x - \frac{a+b}{2}\right)\right] dx \\ &\quad \vdots \\ &\quad + f_n\left(\frac{a+b}{2}\right) \int_a^b f_1(x) \cdots f_{n-1}(x) \exp\left[\frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)}\left(x - \frac{a+b}{2}\right)\right] dx.\end{aligned} \tag{2.51}$$

Remark 2.8. We note that the inequality (2.41) is a special case of the inequality (2.51) when $n = 3$.

Theorem 2.11. *Let* $f_1,\, f_2,\, \cdots,\; f_n : I \to (0,\infty)$ *be differentiable log-concave functions on the interval of real numbers* $I^{\,0}$ *and* $a, b \in I^0$ *with* $a < b$*. Further, let* $\alpha_1, \alpha_2, \cdots, \alpha_n > 0$ *with* $\sum_{i=1}^{n} \alpha_i = 1$*. Then the following inequality holds:*

$$
\begin{aligned}
&\int_a^b \sum_{i=1}^{n} f_i(x)dx \\
&\le \alpha_1 f_1\left(\frac{a+b}{2}\right)\int_a^b f_2(x)f_3(x)\cdots f_n(x)\exp\left[\frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\
&\quad +\alpha_2 f_2\left(\frac{a+b}{2}\right)\int_a^b f_1(x)f_3(x)\cdots f_n(x)\exp\left[\frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx \\
&\quad \vdots \\
&\quad +\alpha_n f_n\left(\frac{a+b}{2}\right)\int_a^b f_1(x)\cdots f_{n-1}(x)\exp\left[\frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx.
\end{aligned}
\tag{2.52}
$$

Proof. Since $f_1, f_2, \cdots, f_n$ are differentiable and log-concave functions on I^0, we have

$$f_1(x) \le f_1(y)\exp\left[\frac{f_1'(y)}{f_1(y)}(x-y)\right], \tag{2.53$-$1}$$

$$f_2(x) \le f_2(y)\exp\left[\frac{f_2'(y)}{f_2(y)}(x-y)\right], \tag{2.53$-$2}$$

$$\vdots$$

$$f_n(x) \le f_n(y)\exp\left[\frac{f_n'(y)}{f_n(y)}(x-y)\right], \tag{2.53$-n$}$$

for all $x, y \in I^0$.. Multiplying both sides of $(2.53-1)$, $(2.53-2)$, $\cdots$ and $(2.53-n)$ by $\alpha_1 f_2(x)f_3(x)\cdots f_n(x)$, $\alpha_2 f_1(x)f_3(x)\cdots f_n(x)$, $\cdots$ and $\alpha_n f_1(x)$ $f_2(x)\cdots f_{n-1}(x)$ respectively and adding the resulting inequalities we have

$$
\begin{aligned}
\sum_{i=1}^{n} f_i(x) &\le \alpha_1 f_2(x)f_3(x)\cdots f_n(x)f_1(y)\exp\left[\frac{f_1'(y)}{f_1(y)}(x-y)\right] \\
&\quad +\alpha_2 f_1(x)f_3(x)\cdots f_n(x)f_2(y)\exp\left[\frac{f_2'(y)}{f_2(y)}(x-y)\right] \\
&\quad \vdots
\end{aligned}
\tag{2.54}
$$

$$+\alpha_n f_1(x)f_2(x)\cdots f_{n-1}(x)f_n(y)\exp\left[\frac{f_n'(y)}{f_n(y)}(x-y)\right].$$

Now, if we choose $y=\frac{a+b}{2}$,from (2.54) we obtain

$$\begin{aligned}&\sum_{i=1}^{n}f_i(x)\\&\le\alpha_1 f_2(x)f_3(x)\cdots f_n(x)f_1\left(\frac{a+b}{2}\right)\exp\left[\frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)}(x-\frac{a+b}{2})\right]\\&\quad+\alpha_2 f_1(x)f_3(x)\cdots f_n(x)f_2\left(\frac{a+b}{2}\right)\exp\left[\frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)}(x-\frac{a+b}{2})\right]\\&\quad\vdots\\&\quad+\alpha_n f_1(x)f_2(x)\cdots f_{n-1}(x)f_n\left(\frac{a+b}{2}\right)\exp\left[\frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)}(x-\frac{a+b}{2})\right].\end{aligned}\tag{2.55}$$

Integrating both sides of (2.55) with respect to x from a to b, we get the desired inequality (2.52). The proof is completed.

Theorem 2.12. *Let f_1,a,b be as in Theorem 2.10 and $f_2,f_3,\cdots,f_n$ be as in Theorem 2.11. Further, let $\alpha_1>1,\alpha_j<0,j=2,3,\cdots,n$ with $\sum_{i=1}^{n}\alpha_i=1$, then the following inequality holds:*

$$\begin{aligned}&\int_a^b\sum_{i=1}^{n}f_i(x)dx\\&\ge\alpha_1 f_1\left(\frac{a+b}{2}\right)\int_a^b f_2(x)f_3(x)\cdots f_n(x)\exp\left[\frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx\\&\quad+\alpha_2 f_2\left(\frac{a+b}{2}\right)\int_a^b f_1(x)f_3(x)\cdots f_n(x)\exp\left[\frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx\\&\quad\vdots\\&\quad+\alpha_n f_n\left(\frac{a+b}{2}\right)\int_a^b f_1(x)\cdots f_{n-1}(x)\exp\left[\frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx.\end{aligned}\tag{2.56}$$

Proof. Multiplying both sides of $(2.48-1)$, $(2.53-2)$, $\cdots$ and $(2.53-n)$ by $\alpha_1 f_2(x)f_3(x)\cdots f_n(x)$, $\alpha_2 f_1(x)f_3(x)\cdots f_n(x)$, $\cdots$ and $\alpha_n f_1(x)f_2(x)\cdots f_{n-1}(x)$ respectively and adding the resulting inequalities we have

$$\begin{aligned}\sum_{i=1}^{n}f_i(x)&\ge\alpha_1 f_2(x)f_3(x)\cdots f_n(x)f_1(y)\exp\left[\frac{f_1'(y)}{f_1(y)}(x-y)\right]\\&\quad+\alpha_2 f_1(x)f_3(x)\cdots f_n(x)f_2(y)\exp\left[\frac{f_2'(y)}{f_2(y)}(x-y)\right]\\&\quad\vdots\end{aligned}\tag{2.57}$$

$$+\alpha_n f_1(x)f_2(x)\cdots f_{n-1}(x)f_n(y)\exp\left[\frac{f_n'(y)}{f_n(y)}(x-y)\right].$$

Now, if we choose $y=\frac{a+b}{2}$, from (2.57) we obtain

$$\begin{aligned}(2.58)\qquad &\sum_{i=1}^{n} f_i(x)\\ &\geq \alpha_1 f_2(x)f_3(x)\cdots f_n(x)f_1\left(\frac{a+b}{2}\right)\exp\left[\frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)}(x-\frac{a+b}{2})\right]\\ &\quad+\alpha_2 f_1(x)f_3(x)\cdots f_n(x)f_2\left(\frac{a+b}{2}\right)\exp\left[\frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)}(x-\frac{a+b}{2})\right]\\ &\qquad\vdots\\ &\quad+\alpha_n f_1(x)f_2(x)\cdots f_{n-1}(x)f_n\left(\frac{a+b}{2}\right)\exp\left[\frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)}(x-\frac{a+b}{2})\right].\end{aligned}$$

Integrating both sides of (2.58) with respect to x from a to b, we get the desired inequality (2.56). The proof is completed.

Theorem 2.13. *Let $f_2,\, f_3, \cdots, f_n,\, a,\, b$ be as in Theorem 2.10 and f_1 be as in Theorem 2.11. Further, let $\alpha_1>1, \alpha_j<0, j=2,3,\cdots,n$ with $\sum_{i=1}^{n}\alpha_i=1$, then the following inequality holds:*

$$\begin{aligned}(2.59)\qquad &\int_a^b\sum_{i=1}^{n} f_i(x)dx\\ &\leq \alpha_1 f_1\left(\frac{a+b}{2}\right)\int_a^b f_2(x)f_3(x)\cdots f_n(x)\exp\left[\frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx\\ &\quad+\alpha_2 f_2\left(\frac{a+b}{2}\right)\int_a^b f_1(x)f_3(x)\cdots f_n(x)\exp\left[\frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx\\ &\qquad\vdots\\ &\quad+\alpha_n f_n\left(\frac{a+b}{2}\right)\int_a^b f_1(x)\cdots f_{n-1}(x)\exp\left[\frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)}\left(x-\frac{a+b}{2}\right)\right]dx.\end{aligned}$$

Proof. Multiplying both sides of $(2.53-1)$, $(2.48-2)$, $\cdots$ and $(2.48-n)$ by $\alpha_1 f_2(x)f_3(x)\cdots f_n(x)$,$\alpha_2 f_1(x)f_3(x)\cdots f_n(x)$, $\cdots$ and $\alpha_n f_1(x)f_2(x)\cdots f_{n-1}(x)$ respectively and adding the resulting inequalities we have

$$\begin{aligned}\sum_{i=1}^{n} f_i(x) \leq\ & \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1(y) \exp\left[\frac{f_1'(y)}{f_1(y)}(x-y)\right] \\ & + \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2(y) \exp\left[\frac{f_2'(y)}{f_2(y)}(x-y)\right] \\ & \vdots \\ & + \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n(y) \exp\left[\frac{f_n'(y)}{f_n(y)}(x-y)\right] \end{aligned} \tag{2.60}$$

Now, if we choose $y = \frac{a+b}{2}$, from (2.60) we obtain

$$\begin{aligned} & \sum_{i=1}^{n} f_i(x) \\ & \leq \alpha_1 f_2(x) f_3(x) \cdots f_n(x) f_1\left(\frac{a+b}{2}\right) \exp\left[\frac{f_1'\left(\frac{a+b}{2}\right)}{f_1\left(\frac{a+b}{2}\right)}(x-\frac{a+b}{2})\right] \\ & \quad + \alpha_2 f_1(x) f_3(x) \cdots f_n(x) f_2\left(\frac{a+b}{2}\right) \exp\left[\frac{f_2'\left(\frac{a+b}{2}\right)}{f_2\left(\frac{a+b}{2}\right)}(x-\frac{a+b}{2})\right] \\ & \qquad \vdots \\ & + \alpha_n f_1(x) f_2(x) \cdots f_{n-1}(x) f_n\left(\frac{a+b}{2}\right) \exp\left[\frac{f_n'\left(\frac{a+b}{2}\right)}{f_n\left(\frac{a+b}{2}\right)}(x-\frac{a+b}{2})\right]. \end{aligned} \tag{2.61}$$

Integrating both sides of (2.61) with respect to x from a to b, we get the desired inequality (2.59). The proof is completed.

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