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LIMITING BEHAVIOR FOR RANDOM ELEMENTS WITH HEAVY TAIL

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Abstract. We present an accurate description of the limiting behavior for the partial sums and the weighted sums of independent and identically distributed random elements with heavy tail. A version of Chover-type law of the iterated logarithm is deduced.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose all random variables and random elements are defined on this space. Suppose that B is a real separable Banach space with norm $\|\cdot\|$. A B-valued random element X is defined as a Borel measurable function from $(\Omega, \mathcal{F})$ into B with Borel σ-algebra. The expected value of a B-valued random element X is defined by Bochner integral and is denoted by EX.

Chover [10] proved that

$$\limsup_{n\to\infty} \left| n^{-1/\alpha} \sum_{k=1}^{n} X_k \right|^{1/\log\log n} = e^{1/\alpha} \ a.s., \tag{1.1}$$

where $\{X, X_n, n \geq 1\}$ is a sequence of independent, identically distributed and symmetric stable random variables with index α, $0 < \alpha < 2$, i.e., its characteristic function is

$$E\exp(itX) = \exp(-c|t|^{\alpha}), \quad \forall \ \ t \in I\!\!R$$

for some $c > 0$.

We call (1.1) the Chover-type law of the iterated logarithm. This result has been generalized by many authors, see [3-8, 16, 18, 19].

Obviously, (1.1) is equivalent to

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$$(1.2) \qquad P\{|\sum_{k=1}^{n} X_k| > (n\log^{1+\delta} n)^{1/\alpha},\ i.o.\} = 0$$

and

$$(1.3) \qquad P\{|\sum_{k=1}^{n} X_k| > (n\log^{1-\delta} n)^{1/\alpha},\ i.o.\} = 1$$

for every $\delta > 0$, where the symbol "$i.o.$" denotes "infinitely often" as n tends to infinity. Equations (1.2) and (1.3) give a description of upper functions of $\{\sum_{k=1}^{n} X_k, n \geq 1\}$ to a certain degree. In fact, Khintchine (cf. Theorem 8.11 in Mijnheeer [15]) earlier proved a characterization of $\{\sum_{k=1}^{n} X_k, n \geq 1\}$'s upper classes via a more accurate description, i.e.

$$(1.4) \qquad \limsup_{n\to\infty} (f(n))^{-1/\alpha} |\sum_{k=1}^{n} X_k| = 0 \text{ or } \infty \ a.s.$$

according to

$$\int_1^\infty \frac{dx}{f(x)} < \infty \text{ or } = \infty$$

respectively, where $f > 0$ is a nondecreasing function. By Khintchine's result, (1.1) holds immediately.

Since an infinite dimensional Banach space is not local compact, the classical strong laws of random varaibles can not directly extend to random elements in Banach setting under same moment conditions. In order to investigate the strong laws of random elements, we may need to put assumptions on the probability conditions of random elements or geometrical conditions of the Banach space. Some examples can be founded in Ledoux and Talagrand [12] such as the strong law of large numbers and the law of the iterated logarithm. The main purpose of this paper is to find suitable conditions under which the analogous study of (1.1) and (1.4) for the sums and the weighted sums of random elements with heavy tail are invested.

In Section 2, we will introduce the concept of regular varying functions and related properties. Some technical lemmas which will be used to prove our main results are also given. In Section 3, we will discuss the limiting behavior for the partial sums and related applications. The limiting behavior for the weighted sums which do not include the partial sums are given in Section 4.

2. Preliminaries

Throughout the paper, we assume that $\{X_n, n \geq 1\}$, $\{X_n, n \geq 0\}$ and $\{X_i, -\infty < i < \infty\}$ are sequences of independent random elements with same distribution as X which satisfies

$$(2.1) \qquad 0 < \liminf_{x\to\infty} A(x)P\{\|X\| > x\} \leq \limsup_{x\to\infty} A(x)P\{\|X\| > x\} < \infty,$$

where $A(x)$ is a regularly varying function with index α, $\alpha > 0$, i.e.

$$\lim_{x\to\infty} \frac{A(tx)}{A(x)} = t^{\alpha}, \quad \text{for all } t > 0.$$

We say that X has heavy tail with index α if (2.1) holds.

Let $B(x)$ be the generalized inverse function of $A(x)$, i.e.

$$B(x) = \inf\{y : A(y) \geq x\}.$$

From Bingham et al. [2] $B(x)$ is a regularly varying function with index $1/\alpha$ and has the following representation:

$$B(x) = c(x)x^{1/\alpha} \exp\{\int_1^x \frac{b(u)}{u} du\},$$

where $\lim_{x\to\infty} c(x) = c \in (0, \infty)$ and $\lim_{x\to\infty} b(x) = 0$. Moreover

$$(2.2) \qquad 0 < \liminf_{x\to\infty} xP\{\|X\| > B(x)\} \leq \limsup_{x\to\infty} xP\{\|X\| > B(x)\} < \infty.$$

For example, $A(x) = x^{\alpha}$ and $A(x) = x^{\alpha} \log x$ are regular varying functions with index α. Their corresponding generalized inverse functions are $B(x) = x^{1/\alpha}$ and $B(x) \sim (\alpha x)^{1/\alpha} (\log x)^{-1/\alpha}$ as $x \to +\infty$.

Suppose X has heavy tail with index α, $B(x)$ is defined as above, and $I(\cdot)$ denotes the indicator function. Then X and $B(x)$ have the following properties.

Property 1. For any $a \neq 0$, (2.1) holds true for aX.

Property 2. Let X' be an independent copy of X, (2.1) also holds true for $X - X'$.

Property 3. For any $p \in (0, \alpha)$, $E\|X\|^p < \infty$.

Property 4. Let $p > \alpha$, by Feller [11] (see Theorem 1, p.273),

$$(B(n))^{-p} E\|X\|^p I(\|X\| \leq B(n)) \leq cn^{-1}.$$

Property 5. For any $\delta > 0$,

$$\lim_{x\to\infty} \frac{B(x)}{x^{1/\alpha-\delta}} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{B(x)}{x^{1/\alpha+\delta}} = 0.$$

Property 6. For any nondecreasing function $f(x) \to \infty$ as $x \to \infty$, for $\delta > 0$,

$$\lim_{x\to\infty} \frac{B(xf(x))}{B(x)f^{1/\alpha-\delta}(x)} = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{B(xf(x))}{B(x)f^{1/\alpha+\delta}(x)} = 0.$$

In the rest of this paper, we denote c as a generic positive number which may be different at different places.

The following lemma is a version of Lemma 3 of Chow and Lai [9] in the setting of Banach spaces, and will be used to prove the divergence part of the main results.

Lemma 2.1. *Let* $\{Y_n, n \geq 1\}$ *and* $\{Z_n, n \geq 1\}$ *be two sequences of random elements such that* $\{Y_k, 1 \leq k \leq n\}$ *and* Z_n *are independent for each* $n \geq 1$. *Suppose* $Y_n + Z_n \to 0$ *almost surely and* $Z_n \to 0$ *in probability as* $n \to \infty$. *Then* $Y_n \to 0$ *almost surely.*

The second lemma is interesting in itself and will be used to prove Corollary 3.4.

Lemma 2.2. *Let* $0 < \alpha < 2$ *and* $f > 0$ *be a nondecreasing function with* $\int_1^\infty \frac{dx}{xf(x)} < \infty$. *Suppose* $\sup_{n\geq 1}(B(nf(n)))^{-1}\|\sum_{k=1}^n X_k\| < \infty$ $a.s.$ *Then for any* $0 < p < \alpha$

$$E\sup_{n\geq 1}\|(B(nf(n)))^{-1}\sum_{k=1}^n X_k\|^p < \infty. \tag{2.3}$$

Proof. From Corollary 6.12 in Ledoux and Talagrand [12], (2.3) is equivalent to

$$E\sup_{n\geq 1}\|(B(nf(n)))^{-1}X_n\|^p < \infty.$$

From the representation of the regularly varying function (see Bingham et al. [2]), there exists a function $d(u)$ with $d(u) \to 0$ as $u \to +\infty$ (i.e. $|d(u)| < \delta \in (0, \alpha - p)$ for μ large enough) such that

$$A(x) = l(x)x^\alpha \exp\{\int_1^x \frac{d(u)}{u}du\},$$

where $\lim_{x\to\infty} l(x) = l \in (0,\infty)$. Fix $t \geq 1$ and $p < \alpha' = \alpha - \delta < \alpha$. From Eq. (2.1) and that $B(x)$ is the inverse function of $A(x)$, for n sufficiently large, we have

$$\begin{aligned} P\{\|X\| > t^{1/p}B(nf(n))\} &\leq c\{A(t^{1/p}B(nf(n)))\}^{-1} \\ &= c(nf(n))^{-1}\cdot\frac{A(B(nf(n)))}{A(t^{1/p}B(nf(n)))} \\ &\leq c(nf(n))^{-1}t^{-\alpha/p}\exp\{\int_{B(nf(n))}^{t^{1/p}B(nf(n))}\frac{|d(u)|}{u}du\} \\ &= c(nf(n))^{-1}t^{-\alpha/p}\cdot t^{\delta/p} \\ &= ct^{-\alpha'/p}(nf(n))^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} E\sup_{n\ge 1}\|(B(nf(n)))^{-1}X_n\|^p &= \int_0^\infty P\{\sup_{n\ge 1}\|(B(nf(n)))^{-1}X_n\| > t^{1/p}\}dt \\ &\le 1+\int_1^\infty P\{\sup_{n\ge 1}\|(B(nf(n)))^{-1}X_n\|>t^{1/p}\}dt \\ &\le 1+\sum_{n=1}^\infty\int_1^\infty P\{\|X_n\|>t^{1/p}B(nf(n))\}dt \\ &\le 1+c\sum_{n=1}^\infty \frac{1}{nf(n)}\int_1^\infty t^{-\alpha'/p}dt \\ &\le 1+c\int_1^\infty \frac{dx}{xf(x)}\int_1^\infty t^{-\alpha'/p}dt<\infty. \end{aligned}$$

The following lemma is a version of the well-known Hoffmann-Jørgensen's inequality (see Li et al. [13]), and will be used to prove Theorem 4.1.

Lemma 2.3. *Let $\{Y_n, n\ge 1\}$ be a sequence of independent symmetric random elements. Then for each $j\ge 1$, there exist positive numbers C_j and D_j depending only on j such that for all $n\ge 1$ and $t>0$*

$$P\{\|\sum_{k=1}^n Y_k\| > 2jt\} \le C_j P\{\max_{1\le k\le n}\|Y_k\|>t\} + D_j(P\{\|\sum_{k=1}^n Y_k\|>t\})^j. \tag{2.4}$$

3. Main Results Concerning with the Partial Sums

With the preliminaries accounted for, the main result of this section may be stated and proved.

Theorem 3.1 Let $0<\alpha<2$ and $f>0$ be a nondecreasing function. Suppose $\{(B(n))^{-1}\sum_{k=1}^n X_k, n\ge 1\}$ is bounded in probability, i.e. for every $\varepsilon>0$, there exists a constant M such that for every $n\ge 1$

$$P\{(B(n))^{-1}\|\sum_{k=1}^n X_k\|>M\}<\varepsilon.$$

Then

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=1}^n X_k\| = 0 \text{ or } \infty \ a.s. \tag{3.1}$$

according to

$$\int_1^\infty \frac{dx}{xf(x)} < \infty \text{ or } =\infty$$

respectively. In particular

$$\limsup_{n\to\infty} \|(B(n))^{-1}\sum_{k=1}^{n} X_k\|^{1/\log\log n} = e^{1/\alpha} \ a.s. \tag{3.2}$$

Proof. Assume that $\int_1^\infty \frac{dx}{xf(x)} < \infty$. By Lemma 2.1 in Chen [6], there exists a function $g(x)$ such that

$$g(x) \to \infty, g(x) \le f(x), \limsup_{x\to\infty} g(2x)/g(x) < \infty, \text{ and } \int_1^\infty \frac{dx}{xg(x)} < \infty.$$

Therefore, in order to prove the first part of (3.1), we may further assume that $f(x)$ satisfies $\limsup_{x\to\infty} f(2x)/f(x) < \infty$.

It is easy to show that $f(x) \to \infty$ as x tends to infinity, hence $\{(B(n))^{-1}\sum_{k=1}^n X_k, n \ge 1\}$ is bounded in probability implies that $(B(nf(n)))^{-1}\sum_{k=1}^n X_k \to 0$ in probability. By standard argument of symmetrization, without loss of generality to assume that $\{X, X_n, n \ge 1\}$ are symmetric (cf. Lemma 7.1 in Ledoux and Talagrand [12]). We first claim that

$$\sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le m\le n} \|\sum_{k=1}^{m} X_k\| > \varepsilon B(nf(n))\} < \infty, \ \forall\, \varepsilon > 0. \tag{3.3}$$

By Lévy inequality

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le m\le n} \|\sum_{k=1}^{m} X_k\| > \varepsilon B(nf(n))\} \\
&\le 2\sum_{n=1}^{\infty} n^{-1} P\{\|\sum_{k=1}^{n} X_k\| > \varepsilon B(nf(n))\} \\
&\le 2\sum_{n=1}^{\infty} P\{\|X\| > B(nf(n))\} \\
&\quad +2\sum_{n=1}^{\infty} n^{-1} P\{\|\sum_{k=1}^{n} X_k I(\|X_k\| \le B(nf(n)))\| > \varepsilon B(nf(n))\} \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, by (2.2)

$$I_1 \le c\sum_{n=1}^{\infty} \frac{1}{nf(n)} \le c\int_1^\infty \frac{dx}{xf(x)} < \infty.$$

For I_2, we first note that by Lemma 7.2 in Ledoux and Talagrand [12]

$$(B(nf(n)))^{-1} E\|\sum_{k=1}^{n} X_k I(\|X_k\| \le B(nf(n)))\| \to 0 \text{ as } n \to \infty.$$

And from Markov's inequality, Theorem 2.1 in Acosta [1] and Property 4

$$
\begin{aligned}
I_2 &\le \sum_{n=1}^{\infty} n^{-1} P\{|\|\sum_{k=1}^{n} X_k I(\|X_k\| \le B(nf(n)))\| \\
&\quad - E\|\sum_{k=1}^{n} X_k I(\|X_k\| \le B(nf(n)))\|| > \varepsilon B(nf(n))/2\} \\
&\le c \sum_{n=1}^{\infty} n^{-1} (B(nf(n)))^{-2} E|\|\sum_{k=1}^{n} X_k I(\|X_k\| \le B(nf(n)))\| \\
&\quad - E\|\sum_{k=1}^{n} X_k I(\|X_k\| \le B(nf(n)))\||^2 \\
&\le c \sum_{n=1}^{\infty} (B(nf(n)))^{-2} E\|X\|^2 I(\|X\| \le B(nf(n))) \\
&\le c \sum_{n=1}^{\infty} \frac{1}{nf(n)} \le c \int_1^{\infty} \frac{dx}{xf(x)} < \infty.
\end{aligned}
$$

Note that for every $\varepsilon > 0$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\{\max_{1\le m\le n} \|\sum_{k=1}^{m} X_k\| > \varepsilon B(nf(n))\} \\
&= \sum_{i=0}^{\infty} \sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\{\max_{1\le m\le n} \|\sum_{k=1}^{m} X_k\| > \varepsilon B(nf(n))\} \\
&\ge \frac{1}{2} \sum_{i=0}^{\infty} P\{\max_{1\le m\le 2^i} \|\sum_{k=1}^{m} X_k\| > \varepsilon B(2^{i+1} f(2^{i+1}))\}
\end{aligned}
$$

By (3.3) and the Borel-Cantelli Lemma

$$
(B(2^{i+1} f(2^{i+1})))^{-1} \max_{1\le m\le 2^i} \|\sum_{k=1}^{m} X_k\| \to 0 \quad a.s. \quad \text{as } i \to \infty.
$$

By the assumption $\limsup_{x\to\infty} f(2x)/f(x) < \infty$, we have

$$
\limsup_{i\to\infty} B(2^{i+1} f(2^{i+1}))(B(2^i f(2^i)))^{-1} < \infty.
$$

Hence

$$
(B(2^i f(2^i)))^{-1} \max_{1\le m\le 2^i} \|\sum_{k=1}^{m} X_k\| \to 0 \quad a.s. \quad \text{as } i \to \infty,
$$

which implies

$$
\limsup_{n\to\infty} (B(nf(n)))^{-1} \|\textstyle\sum_{k=1}^{n} X_k\| = 0 \ a.s. \tag{3.4}
$$

Now assume that $\int_1^\infty \frac{dx}{xf(x)} = \infty$. Suppose that

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=1}^n X_k\| = \infty \ a.s. \tag{3.5}$$

does not hold, then by Kolmogorov 0-1 law, there exists a constant $M \in [0,\infty)$ such that

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=1}^n X_k\| = M \ a.s.$$

It is easy to show that

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\|X_n\| \le 2M \ a.s.,$$

hence by the Borel-Cantelli Lemma,

$$\sum_{n=1}^\infty P\{\|X_n\| > 2MB(nf(n))\} < \infty.$$

But on the other hand by Property 1 and (2.2),

$$\sum_{n=1}^\infty P\{\|X_n\| > 2MB(nf(n))\} \ge c\sum_{n=1}^\infty \frac{1}{nf(n)} \ge c\int_1^\infty \frac{dx}{xf(x)} = \infty,$$

which leads to a contradiction. Therefore (3.5) holds and (3.1) is proved.

For every $\delta > 0$, by (3.4)

$$\limsup_{n\to\infty}(B(n\log^{1+\delta} n))^{-1}\|\sum_{k=1}^n X_k\| = 0 \ \ a.s.$$

By Property 6, we have for any $\delta' > 0$

$$\limsup_{n\to\infty}(B(n)\log^{1/\alpha+\delta/\alpha+\delta'} n)^{-1}\|\sum_{k=1}^n X_k\| = 0 \ \ a.s.,$$

which implies

$$\limsup_{n\to\infty}\|(B(n))^{-1}\sum_{k=1}^n X_k\|^{1/\log\log n} \le e^{1/\alpha+\delta/\alpha+\delta'} \ \ a.s.$$

Thus we have

$$\limsup_{n\to\infty}\|(B(n))^{-1}\sum_{k=1}^n X_k\|^{1/\log\log n} \le e^{1/\alpha} \ \ a.s.$$

By (3.5) and Property 6, using similar argument, we also have

$$\limsup_{n\to\infty} \|(B(n))^{-1}\sum_{k=1}^{n} X_k\|^{1/\log\log n} \ge e^{1/\alpha} \quad a.s.$$

Hence (3.2) holds and prove Theorem 3.1.

Remark 3.1. Assume that B is of Rademacher type p, $1 \le p \le 2$. It is easy to show that $\{(B(n))^{-1}\sum_{k=1}^{n} X_k, n \ge 1\}$ is bounded in probability if (2.1) holds for any $0 < \alpha < p$ and further assume $EX = 0$ for $1 < \alpha < p$. In particular, all Banach spaces are of Rademacher type 1, hence when $0 < \alpha < 1$, (2.1) always implies that $\{(B(n))^{-1}\sum_{k=1}^{n} X_k, n \ge 1\}$ is bounded in probability. See Ledoux and Talagrand [12] for the details of the Rademacher type Banach space.

The following are corollaries of Theorem 3.1.

Corollary 3.1. *Let $0 < \alpha < 2$, $f > 0$ be a nondecreasing function and $\{m_n, n \ge 1\}$ an integer subsequence with $\sup_{n\ge1} m_n/n < \infty$. Suppose that $\{(B(n))^{-1}\sum_{k=1}^{n} X_k, n \ge 1\}$ is bounded in probability. Then*

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=n}^{n+m_n} X_k\| = 0 \text{ or } \infty \; a.s. \tag{3.6}$$

according to

$$\int_1^\infty \frac{dx}{xf(x)} < \infty \text{ or } = \infty$$

respectively. In particular

$$\limsup_{n\to\infty} \|(B(n))^{-1}\sum_{k=n}^{m_n+n} X_k\|^{1/\log\log n} = e^{1/\alpha} \; a.s. \tag{3.7}$$

and

$$\limsup_{n\to\infty} \|(B(n))^{-1}\sum_{k=n}^{m_n+n} X_k\|^{1/(\log\log n+\log(n/m_n))} = e^{1/\alpha} \; a.s. \tag{3.8}$$

Proof. It is enough to prove (3.6), the proofs of (3.7) and (3.8) are similar to the proof of (3.2). We first prove the convergent part. Assume that $\int_1^\infty \frac{dx}{xf(x)} < \infty$. By Lemma 2.1 in Chen [6], without loss of generality, we assume that $\limsup_{x\to\infty} f(2x)$

$/f(x) < \infty$. Hence $\sup_{n\geq 1} m_n/n < \infty$ implies that $\sup_{n\geq 1} B((n+m_n)f(n+m_n))(B(nf(n))^{-1} < \infty$. By (3.4)

$$\begin{aligned}
&\limsup_{n\to\infty}(B(nf(n)))^{-1}\left\|\sum_{k=n}^{n+m_n} X_k\right\| \\
&\leq \limsup_{n\to\infty} B((n+m_n)f(n+m_n)) \\
&\quad (B(nf(n)))^{-1}(B((n+m_n)f(n+m_n)))^{-1}\left\|\sum_{k=1}^{n+m_n} X_k\right\| \\
&\quad + \limsup_{n\to\infty}(B(nf(n)))^{-1}\left\|\sum_{k=1}^{n-1} X_k\right\| = 0 \quad a.s.
\end{aligned}$$

Now assume that $\int_1^\infty \frac{dx}{xf(x)} = \infty$, by Lemma 2.2 in Chen [3], there exists a nondecreasing function $g(x) > 0$ such that

$$\lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \int_1^\infty \frac{dx}{xf(x)g(x)} = \infty.$$

Suppose that

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=n}^{n+m_n} X_k\| = \infty \ a.s.$$

does not hold. Then By Kolomogorov 0-1 law, there exists a constant $M \in [0,\infty)$ such that

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=n}^{n+m_n} X_k\| = M \ a.s.$$

Hence by Property 6

$$\limsup_{n\to\infty}(B(nf(n)g(n)))^{-1}\|\sum_{k=n}^{n+m_n} X_k\| = 0 \ a.s.$$

Let $\{X', X'_n, n \geq 1\}$ be an independent copy of $\{X, X_n, n \geq 1\}$, then we also have

$$\limsup_{n\to\infty}(B(nf(n)g(n)))^{-1}\|\sum_{k=n}^{n+m_n} X'_k\| = 0 \ a.s.$$

Hence

$$\limsup_{n\to\infty}(B(nf(n)g(n)))^{-1}\|\sum_{k=n}^{n+m_n} (X_k - X'_k)\| = 0 \ a.s.$$

By Lemma 2.1, we obtain

$$\limsup_{n\to\infty}(B(nf(n)g(n)))^{-1}\|X_n - X_n')\| = 0 \ a.s.$$

Therefore, from the Borel-Cantelli Lemma, we have

$$\sum_{n=1}^{\infty} P\{\|X_n - X_n'\| > B(nf(n)g(n))\} < \infty.$$

But on the other hand by Property 2 and (2.2)

$$\sum_{n=1}^{\infty} P\{\|X_n - X_n'\| > B(nf(n)g(n))\} \geq c\sum_{n=1}^{\infty}\frac{1}{nf(n)g(n)} \geq c\int_1^{\infty}\frac{dx}{xf(x)g(x)} = \infty,$$

which leads to a contradiction. This completes the proof.

We only state Corollaries 3.2 and 3.3 without proofs. The proofs of the convergence parts is due to the method of Abel's summation, and the proofs of the divergence parts are similar to the proof of Corollary 3.1.

Corollary 3.2. *Let* $0 < \alpha < 2$ *and* $f > 0$ *be a nondecreasing function. Set* $\overline{\beta} = (1-\beta)^{-1}$ *for* $|\beta| < 1$. *Suppose* $\{(B(n))^{-1}\sum_{k=0}^{n-1} X_k, n \geq 1\}$ *is bounded in probability. Then*

$$\limsup_{\beta\to 1^-}(B(\overline{\beta}f(\overline{\beta})))^{-1}\|\sum_{k=0}^{\infty}\beta^k X_k\| = 0 \text{ or } \infty \ a.s. \tag{3.9}$$

according to

$$\int_1^{\infty}\frac{dx}{xf(x)} < \infty \text{ or } = \infty$$

respectively. In particular

$$\limsup_{\beta\to 1^-}\|(B(\overline{\beta}))^{-1}\sum_{k=0}^{\infty}\beta^k X_k\|^{1/\log\log\overline{\beta}} = e^{1/\alpha} \ a.s. \tag{3.10}$$

Corollary 3.3. *Let* $0 < \alpha < 2$, $f > 0$ *be a nondecreasing function and* $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ *be an array of real constants such that*

- (a) $\sup_{n\geq 1}(\sum_{k=1}^{n-1}|a_{nk} - a_{n,k-1}| + |a_{nn}|) < \infty$ *if* $1 \leq \alpha < 2$ *and* $\sup_{n\geq 1}\max_{1\leq k\leq n}|a_{nk}| < \infty$ *if* $0 < \alpha < 1$,
- (b) *there exist two strictly increasing sequences* $\{n(k), k \geq 1\}$ *and* $\{m(k), k \geq 1\}$ *such that*

$$(3.11)\qquad \sup_{k\geq 1}(n(k+1)-n(k))<\infty \text{ and } \liminf_{k\geq 1}|a_{n(k),m(k)}|>0.$$

Suppose $\{(B(n))^{-1}\sum_{k=1}^{n}X_k, n\geq 1\}$ *is bounded in probability. Then*

$$(3.12)\qquad \limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}a_{nk}X_k\|=0 \text{ or } \infty \ a.s.$$

according to

$$\int_1^\infty \frac{dx}{xf(x)}<\infty \text{ or } =\infty$$

respectively. In particular

$$(3.13)\qquad \limsup_{n\to\infty}\|(B(n))^{-1}\sum_{k=1}^{n}a_{nk}X_k\|^{1/\log\log n}=e^{1/\alpha}\ a.s.$$

Corollary 3.4. *Let* $0<\alpha<2$*,* $f>0$ *be a nondecreasing function and* $\{a_i, -\infty<i<\infty\}$ *be a sequence of real constants with*

$$0\neq\sum_{i=-\infty}^{\infty}|a_i|^\theta<\infty,$$

where $\theta=1$ *if* $1<\alpha<2$ *and* $0<\theta<\alpha$ *if* $0<\alpha\leq 1$*. Suppose that* $\{(B(n))^{-1}\sum_{k=1}^{n}X_k, n\geq 1\}$ *is bounded in probability. Then*

$$(3.14)\qquad \limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty}a_iX_{k-i}\|=0 \text{ or } \infty\ a.s.$$

according to

$$\int_1^\infty \frac{dx}{xf(x)}<\infty \text{ or } =\infty$$

respectively. In particular

$$(3.15)\qquad \limsup_{n\to\infty}\|(B(n))^{-1}\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty}a_iX_{k-i}\|^{1/\log\log n}=e^{1/\alpha}\ a.s.$$

Proof. We only prove the convergence parts and the divegence part is similar to Theorem 3.1. Suppose $\int_1^\infty\frac{dx}{xf(x)}<\infty$. Set

$$Y_{mn}=(B(nf(n)))^{-1}\sum_{k=1}^{n}\sum_{i=-m}^{m}a_iX_{k-i},$$

$$\widetilde{a}_m=0,\ \widetilde{a}_i=\sum_{j=i+1}^{m}a_j,\ i=0,\cdots,m-1,$$

$$\widetilde{\widetilde{a}}_{-m} = 0,\ \widetilde{\widetilde{a}}_i = \sum_{j=-m}^{i-1} a_j,\ i = -m+1, -m+2, \cdots, 0,$$
$$\widetilde{X}_k = \sum_{i=1}^{m} \widetilde{a}_i X_{k-i},\ \widetilde{\widetilde{X}}_k = \sum_{i=-m}^{0} \widetilde{\widetilde{a}}_i X_{k-i}.$$

Then

$$\begin{aligned} Y_{mn} &= \left(\sum_{i=-m}^{m} a_i\right) (B(nf(n)))^{-1}) \sum_{k=1}^{n} X_k \\ &\quad + (B(nf(n)))^{-1}(\widetilde{X}_0 - \widetilde{X}_n + \widetilde{\widetilde{X}}_{n+1} - \widetilde{\widetilde{X}}_1) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &(B(nf(n)))^{-1} \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_i X_{k-i} \\ &= Y_{mn} + (B(nf(n)))^{-1} \sum_{k=1}^{n} \sum_{|i|>m} a_i X_{k-i}. \end{aligned} \tag{3.17}$$

For every i, by Property 1 and (2.2) for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\{\|X_{n-i}\| > \varepsilon B(nf(n))\} \le c \sum_{n=1}^{\infty} \frac{1}{nf(n)} \le c \int_1^{\infty} \frac{dx}{xf(x)} < \infty,$$

hence by Borel-Cantelli Lemma

$$\lim_{n\to\infty} (B(nf(n)))^{-1} \|X_{n-i}\| = 0\ a.s.$$

Thus we have

$$\lim_{n\to\infty} (B(nf(n)))^{-1} \|\widetilde{X}_n\| = 0\ a.s.$$

and

$$\lim_{n\to\infty} (B(nf(n)))^{-1} \|\widetilde{\widetilde{X}}_{n+1}\| = 0\ a.s.$$

And it is obvious that

$$\lim_{n\to\infty} (B(nf(n)))^{-1} \|\widetilde{X}_0\| = \lim_{n\to\infty} (B(nf(n)))^{-1} \|\widetilde{\widetilde{X}}_1\| = 0\ a.s.$$

Hence

$$\lim_{n\to\infty} (B(nf(n)))^{-1} \|\widetilde{X}_0 - \widetilde{X}_n + \widetilde{\widetilde{X}}_{n+1} - \widetilde{\widetilde{X}}_1\| = 0\ a.s. \tag{3.18}$$

By (3.16)-(3.18) and Theorem 3.1

$$
\begin{aligned}
&\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty}a_iX_{k-i}\| \\
&=\limsup_{n\to\infty}\|Y_{mn}+\sum_{|i|>m}a_i(B(nf(n)))^{-1}\sum_{k=1}^{n}X_{k-i}\| \\
&\le\limsup_{n\to\infty}|\sum_{i=-m}^{m}a_i|\|(B(nf(n)))^{-1}\sum_{k=1}^{n}X_k\| \\
&\quad+\limsup_{n\to\infty}\sum_{|i|>m}|a_i|\|(B(nf(n)))^{-1}\sum_{k=1}^{n}X_{k-i}\| \\
&\le\sum_{|i|>m}|a_i|\sup_{n\ge1}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}X_{k-i}\|\ a.s.
\end{aligned}
\tag{3.19}
$$

Using Theorem 3.1, $\sup_{n\geq1}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}X_k\|<\infty$ almost surely, and note that $\{X_i,-\infty<i<\infty\}$ is stationary, by Lemma 2.2

$$
\begin{aligned}
&E(\sum_{i=-\infty}^{\infty}|a_i|\sup_{n\ge1}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}X_{k-i}\|)^{\theta} \\
&\le\sum_{i=-\infty}^{\infty}|a_i|^{\theta}E(\sup_{n\ge1}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}X_{k-i}\|)^{\theta} \\
&=\sum_{i=-\infty}^{\infty}|a_i|^{\theta}E(\sup_{n\ge1}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}X_{k}\|)^{\theta} \\
&<\infty.
\end{aligned}
$$

Hence

$$
\sum_{i=-\infty}^{\infty}|a_i|\sup_{n\ge1}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}X_{k-i}\|<\infty\ a.s.
$$

Let $m\to\infty$ in (3.19), we have

$$
\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=1}^{n}\sum_{i=-\infty}^{\infty}a_iX_{k-i}\|=0\ a.s.
$$

Remark 3.2. Theorem 3.1 and Corollaries 3.1-3.4 extend and generalize the results of [3-8, 10, 16, 18, 19] respectively.

4. Main Results About the Weighted Sums

In this section, the analogous studies of (1.1) and (1.4) are investigated for the weighted sums which do not include the partial sums. Meanwhile the index α maybe greater than or equal to 2.

Theorem 4.1 *Let* $\alpha > 0$, $0 < \delta < \min\{1, 2/\alpha\}$, $f > 0$ *be a nondecreasing function and* $\{a_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of real constants such that*

$$\begin{cases} k_n \le Mn, \ \forall \ n \ge 1, \\ \sup_{n\ge 1} \max_{1\le k\le k_n} |a_{nk}| < \infty \text{ and } \displaystyle\sum_{k=1}^{k_n} a_{nk}^2 = O(n^\delta), \end{cases}$$

where the constant $M > 0$ *does not dependent on* n *and there exist two strictly increasing sequences* $\{n(k), k \ge 1\}$ *and* $\{m(k), k \ge 1\}$ *such that (3.11) holds. Suppose* $\{(B(n))^{-1}\sum_{k=1}^{k_n} a_{nk}X_k, n \ge 1\}$ *is bounded in probability. Then*

$$\limsup_{n\to\infty}(B(nf(n))^{-1}\|\sum_{k=1}^{k_n} a_{nk}X_k\| = 0 \text{ or } \infty \ a.s. \tag{4.1}$$

according to

$$\int_1^\infty \frac{dx}{xf(x)} < \infty \text{ or } = \infty$$

respectively. In particular

$$\limsup_{n\to\infty}\|(B(n))^{-1}\sum_{k=1}^{k_n} a_{nk}X_k\|^{1/\log\log n} = e^{1/\alpha} \ a.s. \tag{4.2}$$

Proof. We only prove the convergence part of (4.1), the proof in the rest part is similar to Theorem 3.1 and Corollary 3.1. Suppose $\int_1^\infty \frac{dx}{xf(x)} < \infty$. Then by Lemma 2.1 in Chen [6], we assume that $\limsup_{x\to\infty} f(2x)/f(x) < \infty$. Hence by the same argument as Li et al. [13], we assume that $k_n = n$. So it is enough to prove that

$$\lim_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=1}^{n} a_{nk}X_k\| = 0 \ a.s. \tag{4.3}$$

Note that $\{(B(n))^{-1}\sum_{k=1}^{n} a_{nk}X_k, n \ge 1\}$ is bounded in probability implies that $(B(nf(n)))^{-1}\sum_{k=1}^{n} a_{nk}X_k \to 0$ in probability, hence by standard argument of symmetrization, without loss of generality to assume that $\{X, X_n, n \ge 1\}$ are symmetric.

Choose an integer $j \ge 2$ with $j(\min\{1, 2/\alpha\} - \delta') > 1$ for some $\delta', \delta < \delta' < \min\{1, 2/\alpha\}$. Set $M_0 = \sup_{n\ge 1}\max_{1\le k\le n}|a_{nk}|$. Then (4.3) holds if we can show that for every $\varepsilon > 0$

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=1}^{n} a_{nk}X_k\| \le 2j\cdot 2M_0\varepsilon \ a.s. \tag{4.4}$$

By (2.2)

$$\sum_{n=1}^{\infty} P\{\|X_n\| > \varepsilon B(nf(n))\} \leq c \sum_{n=1}^{\infty} \frac{1}{nf(n)} \leq c \int_1^{\infty} \frac{dx}{xf(x)} < \infty.$$

Then by the Borel-Cantelli Lemma, $\sum_{k=1}^{n} a_{nk} X_k I(\|X_k\| > \varepsilon B(nf(n)))$ is bounded almost surely. Therefore

$$\lim_{n\to\infty} (B(nf(n)))^{-1} \| \sum_{k=1}^{n} a_{nk} X_k I(\|X_k\| > \varepsilon B(nf(n)))\| = 0 \ a.s.$$

Thus to prove (4.4), it is enough to prove that

$$\limsup_{n\to\infty} (B(nf(n)))^{-1} \left\| \sum_{k=1}^{n} a_{nk} X_k I(\|X_k\| \leq \varepsilon B(nf(n))) \right\| \leq 2j \cdot 2M_0 \varepsilon \ a.s. \tag{4.5}$$

By the Borel-Cantelli Lemma, it is enough to prove that

$$\sum_{n=1}^{\infty} P\left\{ \left\| \sum_{k=1}^{n} a_{nk} X_k I(\|X_k\| \leq \varepsilon B(nf(n))) \right\| > 2j \cdot 2M_0 \varepsilon B(nf(n)) \right\} < \infty. \tag{4.6}$$

In view of Lemma 2.3, (4.6) follows from

$$\sum_{n=1}^{\infty} P\left\{ \max_{1\leq k\leq n} \|a_{nk} X_k I(\|X_k\| \leq \varepsilon B(nf(n)))\| > 2M_0 \varepsilon B(nf(n)) \right\} < \infty \tag{4.7}$$

and

$$\sum_{n=1}^{\infty} \left(P\left\{ \left\| \sum_{k=1}^{n} a_{nk} X_k I(\|X_k\| \leq \varepsilon B(nf(n))) \right\| > 2M_0 \varepsilon B(nf(n)) \right\} \right)^j < \infty. \tag{4.8}$$

Since $\max_{1\leq k\leq n} \|a_{nk} X_k I(\|X_k\| \leq \varepsilon B(nf(n)))\| \leq M_0 \varepsilon B(nf(n)))$, for every $n \geq 1$, we know that

$$P\{\max_{1\leq k\leq n} \|a_{nk} X_k I(\|X_k\| \leq \varepsilon B(nf(n)))\| > 2M_0 \varepsilon B(nf(n))\} = 0,$$

so (4.7) holds true. By Lemma 7.2 in Ledoux and Talagrand [12]

$$\lim_{n\to\infty} (B(nf(n)))^{-1} E\|a_{nk} X_k I(\|X_k\| \leq \varepsilon B(nf(n)))\| = 0.$$

Hence by Markov's inequality, Theorem 2.1 in de Acosta [1], when n large enough,

$$
\begin{aligned}
&P\{\|\sum_{k=1}^{n} a_{nk}X_k I(\|X_k\| \le \varepsilon B(nf(n)))\| > 2M_0\varepsilon B(nf(n))\} \\
&\le P\{|\|\sum_{k=1}^{n} a_{nk}X_k I(\|X_k\| \le \varepsilon B(nf(n)))\| \\
&\quad -E\|\sum_{k=1}^{n} a_{nk}X_k I(\|X_k\| \le \varepsilon B(nf(n)))\|| > M_0\varepsilon B(nf(n))\} \\
&\le c(B(nf(n)))^{-2}E\|X\|^2 I(\|X\| \le \varepsilon B(nf(n))) \sum_{k=1}^{n} |a_{nk}|^2 \\
&\le cn^{\delta}(B(nf(n)))^{-2}E\|X\|^2 I(\|X\| \le \varepsilon B(nf(n))).
\end{aligned}
$$

When $0 < \alpha < 2$, by Property 4

$$n^{\delta}(B(nf(n)))^{-2}E\|X\|^2 I(\|X\| \le \varepsilon B(nf(n))) \le cn^{\delta-1} \le cn^{\delta'-1}.$$

When $\alpha = 2$, let $p = 2(1+\delta-\delta')^{-1}$, hence $p > 2$. By Jensen's inequality and Property 4

$$
\begin{aligned}
&n^{\delta}(B(nf(n)))^{-2}E\|X\|^2 I(\|X\| \le \varepsilon B(nf(n))) \\
&\le n^{\delta}((B(nf(n)))^{-p}E\|X\|^p I(\|X\| \le \varepsilon B(nf(n))))^{2/p} \\
&\le cn^{\delta'-1}.
\end{aligned}
$$

When $\alpha > 2$, by Property 3, $E\|X\|^2 < \infty$, hence by Property 5

$$n^{\delta}(B(nf(n)))^{-2}E\|X\|^2 I(\|X\| \le \varepsilon B(nf(n))) \le cn^{\delta'-2/\alpha}.$$

Hence we always have

$$P\left\{\left\|\sum_{k=1}^{n} a_{nk}X_k I(\|X_k\| \le \varepsilon B(nf(n)))\right\| > 2M_0\varepsilon B(nf(n))\right\} \le cn^{\delta'-\min\{1,\ 2/\alpha\}}.$$

Since $j(\min\{1, 2/\alpha\} - \delta') > 1$, (4.8) follows at once. Hence (4.5) holds true.

By Theorem 4.1, we have the following corollaries.

Corollary 4.1. *Let $\alpha > 0$, $0 < \gamma < 1$, $2\gamma - 1 < 2/\alpha$ and $f > 0$ be a nondecreasing function. Suppose that $\{(B(n))^{-1}\sum_{k=0}^{n} C_{n-k}^{\gamma-1}X_k, n \ge 1\}$ is bounded in probability, where for any $\beta > -1$, $C_0^{\beta} = 1$ and $C_j^{\beta} = (\beta+1)\cdots(\beta+j)/j!$ for every $j \ge 1$. Then*

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\|\sum_{k=0}^{n}C_{n-k}^{\gamma-1}X_k\| = 0 \text{ or } \infty \ a.s. \tag{4.9}$$

according to

$$\int_1^\infty \frac{dx}{xf(x)} < \infty \text{ or } = \infty$$

respectively. In particular

$$\limsup_{n\to\infty}\|(B(n))^{-1}\sum_{k=0}^{n}C_{n-k}^{\gamma-1}X_k\|^{1/\log\log n} = e^{1/\alpha} \ a.s. \tag{4.10}$$

Proof. Let $a_{nk} = C_{n-k}^{\gamma-1}$ for $0 \le k \le n$ and $n \ge 1$. Note that for $\gamma > -1$

$$\lim_{n\to\infty} n^{-\gamma}C_n^\gamma = \Gamma^{-1}(\gamma+1),$$

where $\Gamma(\cdot)$ is a gamma function. Hence it is easy to show that $\{a_{nk}, 0 \le k \le n, n \ge 1\}$ satisfies the conditions of Theorem 4.1. So by Theorem 4.1, we have Corollary 4.1 at once.

Corollary 4.2. *Let $0 < \alpha < 4$, $0 < q < 1$ and $f > 0$ be a nondecreasing function. Suppose that $\{(B(n))^{-1}\sqrt{n}\sum_{k=0}^{n}C_n^kq^k(1-q)^{n-k}X_k, n \ge 1\}$ is bounded in probability. Then*

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\sqrt{n}\|\sum_{k=0}^{n}C_n^kq^k(1-q)^{n-k}X_k\| = 0 \text{ or } \infty \ a.s. \tag{4.5}$$

according as

$$\int_1^\infty \frac{dx}{xf(x)} < \infty \text{ or } = \infty.$$

In particular

$$\limsup_{n\to\infty}\left\|(B(n))^{-1}\sqrt{n}\sum_{k=0}^{n}C_n^kq^k(1-q)^{n-k}X_k\right\|^{1/\log\log n} = e^{1/\alpha} \ a.s. \tag{4.6}$$

Proof. Let $a_{nk} = \sqrt{n}C_n^kq^k(1-q)^{n-k}$ for $0 \le k \le n$ and $n \ge 1$. By Lemma 1 in Maejima [14] and the Stirling's formula, it is easy to show that $\{a_{nk}, 0 \le k \le n, n \ge 1\}$ satisfies the conditions of Theorem 4.1. So by Theorem 4.1, we complete the proof.

Corollary 4.3. *Let $0 < \alpha < 4$ and $f > 0$ be a nondecreasing function. Suppose that $\{(B(n))^{-1}\sqrt{n}e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}X_k, n \ge 1\}$ is bounded in probability. Then*

$$\limsup_{n\to\infty}(B(nf(n)))^{-1}\sqrt{n}e^{-n}\left\|\sum_{k=0}^{\infty}\frac{n^k}{k!}X_k\right\| = 0 \text{ or } \infty \ a.s. \tag{4.7}$$

according as

$$\int_1^\infty \frac{dx}{xf(x)} < \infty \text{ or } = \infty.$$

In particular

$$\limsup_{n\to\infty} \left\| B^{-1}(n)\sqrt{n}e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}X_k \right\|^{1/\log\log n} = e^{1/\alpha} \ a.s. \tag{4.8}$$

Proof. Let $t \in (0, \min\{1, \alpha\})$. By Theorem 16 of Chapter 7 in Petrov [17], there exists a constant $M > 1$ such that $\sum_{k\geq Mn+1}(\sqrt{n}e^{-n}\frac{n^k}{k!})^t < cn^{-1}$. Hence for some $t' \in (0, t)$, by Morkov inequality, c_r-inequality and Properties 3 and 5, for every $\varepsilon > 0$

$$P\left\{ \left\| \sqrt{n}e^{-n}\sum_{k\geq Mn+1}\frac{n^k}{k!}X_k \right\| \geq \varepsilon B(nf(n)) \right\} \leq cn^{-(1+t'/\alpha)}$$

By the Borel-Cantelli Lemma

$$\lim_{n\to\infty}(B(nf(n)))^{-1}\sqrt{n}e^{-n}\left\| \sum_{k\geq Mn+1}\frac{n^k}{k!}X_k \right\| = 0 \ a.s.$$

Let $a_{nk} = \sqrt{n}e^{-n}\frac{n^k}{k!}$ for $0 \leq k \leq Mn$ and $n \geq 1$. It is easy to show that $\{a_{nk}, 0 \leq k \leq Mn, n \geq 1\}$ satisfies the conditions of Theorem 4.1. So we complete the proof. ∎

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