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THE CONTINUITY OF SOME OPERATORS ON HERZ-TYPE HARDY SPACES ON THE HEISENBERG GROUP

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Abstract. In this paper, the authors give the boundedness of some multipliers satisfying Michlin condition on Herz-type Hardy spaces on the Heisenberg group.

1. Introduction and Main Results

The Heisenberg group $\mathbb{H}^n$ is the lie group with underlying manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ and multiplication $(z,t)\cdot(z',t') = (z+z', t+t'+2Im(z\cdot\bar{z}'))$, where $z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$. If we identify $\mathbb{C}^n \times \mathbb{R}$ with $\mathbb{R}^{2n+1}$ by $z_j = x_j + ix_{j+n},\ j = 1, \cdots, n$, then the group law can be rewritten as $(x_1, x_2, \cdots, x_{2n}, t)\cdot(y_1, y_2, \cdots, y_{2n}, t') = (x_1+y_1, \cdots, x_n+y_n, t+t'-2\sum_{j=1}^{n}(x_j y_{j+n} - y_j x_{j+n}))$, The reverse element of (z,t) is $(-z,-t)$ and we write the identity of $\mathbb{H}^n$ as $0 = (0,0)$.

Set $X_j = \frac{\partial}{\partial x_j} + 2x_{j+n}\frac{\partial}{\partial t},\ X_{j+n} = \frac{\partial}{\partial x_j} - 2x_{j+n}\frac{\partial}{\partial t}, T = \frac{\partial}{\partial t}, j = 1, 2, \cdots, n,$ then X_j, X_{j+n}, T, is a basis for the left invariant vector fields on $\mathbb{H}^n$.

The corresponding complex vector fields are $Z_j = \frac{1}{2}(X_j - iX_{j+n}) = \frac{\partial}{\partial z_j} + i\bar{z}_j\frac{\partial}{\partial t},\ \bar{Z}_j = \frac{1}{2}(X_j + iX_{j+n}) = \frac{\partial}{\partial \bar{z}_j} - iz_j\frac{\partial}{\partial t},\ \ j = 1, \cdots, n.$

The dilation on the Heisenberg group is defined as follows: If $r > 0, u = (z,t) \in \mathbb{H}^n$, we let $ru = (rz, r^2 t)$, the homogeneous norm of $u : |u| \equiv (|z|^4 + t^2)^{1/4}$. $B(u,r) = \{v \in \mathbb{H}^n : |uv^{-1}| < r\}$ is the open ball with the center u and radius r. The Haar measure dV on $\mathbb{H}^n$ coincides with the Lebesgue measure on $\mathbb{C}^n \times \mathbb{R}$ which is denoted by $dzd\bar{z}dt$. We note that $|B((z,t),r)| = cr^Q$ $(Q :=$

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$2n+2,$ the homogeneous dimension of the Heisenberg group$)$, where $|B|$ denotes the measure of B. Let $J=(j^1,j^2,j^0)\in\mathbb{Z}_+^n\times\mathbb{Z}_+^n\times\mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of all nonnegative integers, we set $h(J)=|j^1|+|j^2|+2j^0$, where, if $j^1=(j_1^1,\cdots,j_n^1)$, then $|j^1|=\sum_{k=1}^{n}j_k^1$. If $P(z,t)=\sum_J a_J(z,t)^J$ is a polynomial where $(z,t)^J=z^{j^1}\bar{z}^{j^2}t^{j^0}$, then we call $\max\{h(J):a_J\neq 0\}$ the homogeneous degree of $P(z,t)$. The set of all polynomials whose homogeneous degree $\leq s$ is denoted by $\mathcal{P}_s$. Schwartz space on $\mathbb{H}^n$ write as $\mathcal{S}(\mathbb{H}^n)$.

Fix $\lambda>0$, let $\mathcal{H}_\lambda$ be the Bargmann's space:

$$\mathcal{H}_\lambda=\{F \text{ holomorphic on } \mathbb{C}^n:\|F\|^2=(\frac{2\lambda}{\pi})^n\int_{\mathbb{C}^n}|F(\zeta)|^2e^{-2\lambda|\zeta|^2}d\zeta<+\infty\}.$$

Then, $\mathcal{H}_\lambda$ is a Hilbert space and the monomials

$$F_{\alpha,\lambda}(\zeta)=\sqrt{\frac{(2\lambda)^{|\alpha|}}{\alpha!}}\zeta^\alpha,\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)\in\mathbb{Z}_+^n$$

form an orthonormal basis for $\mathcal{H}_\lambda$, where $\alpha!=\alpha_1!\alpha_2!\cdots\alpha_n!$, $|\alpha|=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $\zeta^\alpha=\zeta_1^{\alpha_1}\zeta_2^{\alpha_2}\cdots\zeta_n^{\alpha_n}$. Suppose $W_{k,\lambda}$ and $W_{k,\lambda}^+$ are the closed operators on $\mathcal{H}_\lambda$ such that

$$W_{k,\lambda}F_{\alpha,\lambda}=(2(\alpha_k+1)\lambda)^{1/2}F_{\alpha+e_k,\lambda},$$
$$W_{k,\lambda}^+F_{\alpha,\lambda}=(2\alpha_k\lambda)^{1/2}F_{\alpha-e_k,\lambda},\qquad \text{for}\quad \lambda>0,$$

and

$$W_{k,\lambda}=W_{k,-\lambda}^+,$$
$$W_{k,\lambda}^+=W_{k,-\lambda},\qquad \text{for}\quad \lambda<0,$$

where $e_k=(0,\cdots,1,\cdots,0)\in\mathbb{Z}_+^n$ with the 1 in the $k-$th position. Then

$$\Pi_\lambda(z,t)=\exp^{i\lambda t}\exp^{(-z\cdot W_\lambda+\bar{z}\cdot W_\lambda^+)}$$

is an irreducible unitary representation of $\mathbb{H}^n$ on $\mathcal{H}_\lambda$, where $z\cdot W_\lambda=\sum_{k=1}^{n}z_k\cdot W_{k,\lambda}$. The group Fourier transform of $f\in L^1(\mathbb{H}^n)\cap L^2(\mathbb{H}^n)$ is an operator-valued function defined by

$$\hat{f}(\lambda)=\int_{\mathbb{H}^n}f(z,t)\Pi_\lambda(z,t)dV.$$

Obviously, $\|\hat{f}(\lambda)\|\leq\|f\|_{L^1}$. Here, $\|\cdot\|$ denotes the operator norm. Similar as in $\mathbb{R}^n$, for $f\in L^1\cap L^2(\mathbb{H}^n)$, we also have

Plancherel Theorem.

$$\|\hat{f}\|_{\mathcal{L}^2}^2 := \frac{2^{n-1}}{\pi^{n+1}} \int_{-\infty}^{\infty} \|\hat{f}(\lambda)\|_{H-S}^2 |\lambda|^n d\lambda = \|f\|_{L^2}^2,$$

where $\|\cdot\|_{H-S}$ denotes the Hilbert-Schmidt operator norm.

Inversion Theorem.

$$\int_{-\infty}^{\infty} tr(\Pi_\lambda^*(z,t)\hat{f}(\lambda))|\lambda|^n d\lambda = \frac{(2\pi)^{n+1}}{4^n} f(u).$$

For $(\lambda, m, \alpha) \in \mathbb{R}^* \times \mathbb{Z}^n \times \mathbb{Z}_+^n$, where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, we use the notations

$$m_i^+ = \max\{m_i, 0\}, \qquad m_i^- = -\min\{m_i, 0\},$$
$$m^+ = (m_1^+, m_2^+, \cdots, m_n^+), \qquad m^- = (m_1^-, m_2^-, \cdots, m_n^-).$$

The partial isometry operator $W_\alpha^m(\lambda)$ on $\mathcal{H}_{|\lambda|}$ by

$$W_\alpha^m(\lambda)F_{\beta,\lambda} = (-1)^{|m^+|}\delta_{\alpha+m^+,\beta}F_{\alpha+m^-,\lambda}, \quad \text{for } \lambda > 0;$$
$$W_\alpha^m(\lambda) = [W_\alpha^m(-\lambda)]^*, \quad \text{for } \lambda < 0.$$

Thus $\{W_\alpha^m(\lambda) : m \in \mathbb{Z}^n, \alpha \in \mathbb{Z}_+^n\}$ is an orthonormal basis for the Hilbert-Schmidt operators on $\mathcal{H}_{|\lambda|}$. Given a function $f \in L^2(\mathbb{H}^n)$ such that

$$f(z,t) = \sum_{m,\alpha} f_m(r_1, \cdots, r_n, t)e^{i(m_1\theta_1+\cdots+m_n\theta_n)}, \quad \text{where} \quad z_j = r_j e^{i\theta_j},$$

then,

$$\hat{f}(\lambda) = \sum_{m,\alpha} R_f(\lambda, m, \alpha)W_\alpha^m(\lambda),$$

where

$$R_f(\lambda, m, \alpha) = \int_{\mathbb{H}^n} f_m(r_1, \cdots, r_n, t)e^{i\lambda t} l_{\alpha_1}^{|m_1|}(2|\lambda|r_1^2)\cdots l_{\alpha_n}^{|m_n|}(2|\lambda|r_n^2)dV,$$

and $l_\alpha^{|m|}$ is the Larguerre function of type $|m|$ and degree $|\alpha|$.

Let P be a polynomial in $z_j, \bar{z}_j, t$ on $\mathbb{H}^n$, and we define the difference-differential operator Δ_P acting on the Fourier transform of $f \in L^1 \cap L^2(\mathbb{H}^n)$ by

$$\Delta_P\big(\sum_{m,\alpha} R_f(\lambda, m, \alpha)W_\alpha^m(\lambda)\big) = \sum_{m,\alpha} R_{Pf}(\lambda, m, \alpha)W_\alpha^m(\lambda),$$

namely, $\Delta_P \hat{f}(\lambda) = \widehat{P(\cdot)f(\cdot)}(\lambda)$. In [1] and [2], the authors gave the explicit expressions for Δ_t, Δ_{z_j} and $\Delta_{\bar{z}_j}$. For convenience, we shall write $\Delta^J_{(z,t)} = \Delta^J$.

For more about the knowledge on the Heisenberg group, we refer the reader to monograph [4,5,6]

In [1], Liu proved the following results.

Theorem A. ([1]). Let $f \in H^p(\mathbb{H}^n), 0 < p \leq 1$. Then

$$\|\hat{f}(\lambda)W^0_\alpha(\lambda)\| \leq C\|f\|_{H^p}((2|\alpha|+n)|\lambda|)^{\frac{Q}{2}(\frac{1}{p}-1)}.$$

Theorem B. ([1]). Let $0 < p \leq 1$ and $\tau > Q(\frac{1}{p} - \frac{1}{2})$ be even. If an operator valued function $M(\lambda) = \sum_{\alpha \in \mathbb{Z}^n_+} B(\alpha, 0, \alpha)W^0_\alpha(\lambda)$ satisfies

$$\|W^0_\alpha(\lambda)\Delta^J M(\lambda)\|_{H-S} \leq C((2|\alpha|+n)|\lambda|)^{-\frac{h(J)}{2}}, \quad 0 \leq h(J) \leq \tau,$$

then the right-multiplier T_M defined by

$$\widehat{(T_M f)}(\lambda) = \hat{f}(\lambda)M(\lambda), \quad f \in H^p(\mathbb{H}^n) \cap \mathcal{S}(\mathbb{H}^n)$$

can be extended to a bounded operator on $H^p(\mathbb{H}^n)$.

The above theorems are the extension of the analogy results in [3].

In this paper, we mainly generalize the above results to Herz-type Hardy spaces. Before we state our main results, we first introduce some concepts of Herz-type Hardy spaces.

Let us begin with the definition of the Herz spaces. In the whole paper, we let $B_k = \{u \in \mathbb{H}^n : |u| \leq 2^k\}$ and $E_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$, χ_k be the characteristic function of the set E_k, C is a absolute constant independent of the main parameters involved, but whose value may different from each occasion.

Definition 1.1. Let $-\infty < \alpha < \infty, 0 < p < \infty, 1 < q < \infty$,
(i) the homogeneous Herz spaces $\dot{K}^{\alpha,p}_q(\mathbb{H}^n)$ is defined by

$$\dot{K}^{\alpha,p}_q(\mathbb{H}^n) = \{f \in L^q_{\mathrm{loc}}(\mathbb{H}^n \setminus \{0\}) : \|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{H}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q(\mathbb{H}^n)} \equiv \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{H}^n)}\right\}^{1/p}.$$

(ii) The nonhomogeneous Herz spaces $K^{\alpha,p}_q(\mathbb{H}^n)$ is defined by

$$K^{\alpha,p}_q(\mathbb{H}^n) = L^q(\mathbb{H}^n) \cap \dot{K}^{\alpha,p}_q(\mathbb{H}^n),$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{H}^n)} \equiv \|f\chi_{B_0}\|_{L^q(\mathbb{H}^n)} + \sum_{k=1}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{H}^n)}.$$

With the usual modifications made when $p = \infty$ or $q = \infty$.

Obviously , $\dot{K}_p^{0,p}(\mathbb{H}^n) = K_p^{0,p}(\mathbb{H}^n) = L^p(\mathbb{H}^n)$ for $0 < p \le \infty$ and $\dot{K}_q^{\alpha/q,q}(\mathbb{H}^n) = L^q_{|x|^\alpha}(\mathbb{H}^n)$ and $K_q^{n(1-1/q),1}(\mathbb{H}^n) = A^q(\mathbb{H}^n)$ for $0 < q < \infty$. Here $A^q(\mathbb{H}^n)$ is the Beurling algebra.

Before we introduce the Herz-type Hardy spaces, we fix some notations. Let $\phi \in \mathcal{S}(\mathbb{H}^n)$ with $\operatorname{supp}\phi \subset B_0, \int_{\mathbb{H}^n}\phi(x)dV(x) \neq 0$ and $\phi^t(x) = \frac{1}{t^Q}\phi(\frac{x}{t})$ for any $t > 0$. Let

$$M_\phi(f)(x) = \sup_{t>0}|f * \phi^t(x)|.$$

Definition 1.2 Let $0 < p < \infty, 1 < q < \infty$ and $\alpha \in \mathbb{R}$.

(i) The homogeneous Herz-type Hardy spaces $H\dot{K}_q^{\alpha,p}(\mathbb{H}^n)$ is defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{H}^n) = \{f \in S'(\mathbb{H}^n) : M_\phi(f) \in \dot{K}_q^{\alpha,p}(\mathbb{H}^n)\}.$$

Moreover, we defined $\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{H}^n)} = \|M_\phi(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{H}^n)}$.

(ii) The non-homogeneous Herz-type Hardy Space $HK_q^{\alpha,p}(\mathbb{H}^n)$ is defined by

$$HK_q^{\alpha,p}(\mathbb{H}^n) = \{f \in S'(\mathbb{H}^n) : M_\phi(f) \in K_q^{\alpha,p}(\mathbb{H}^n)\}.$$

Moreover, we define $\|f\|_{HK_q^{\alpha,p}(\mathbb{H}^n)} = \|M_\phi(f)\|_{K_q^{\alpha,p}(\mathbb{H}^n)}$.

when $p = \infty$ and $q = \infty$, we just make the usual modifications.

Our main results as follows:

Theorem 1.1. Let $0 < p < \infty, \frac{Q}{2} \le \beta < \infty, \tau > \beta$ is even, $M(\lambda) = \sum_{\alpha\in\mathbb{Z}^n_+} B(\alpha,0,\alpha)W^0_\alpha(\lambda)$ satisfies

$$\|W^0_\alpha(\lambda)\Delta^J M(\lambda)\|_{H-S} \le C((2|\alpha|+n)|\lambda|)^{-\frac{h(J)}{2}}, \quad 0 \le h(J) \le \tau.$$

Then, the right-multiplier T_M defined by

$$\widehat{(T_M f)}(\lambda) = \hat{f}(\lambda)M(\lambda), \quad f \in H\dot{K}_2^{\beta,p}(\mathbb{H}^n) \cap \mathcal{S}(\mathbb{H}^n)$$

can be extended to a bounded operator on $H\dot{K}_2^{\beta,p}(\mathbb{H}^n)$.

Theorem 1.2. Let $f \in H\dot{K}_q^{\beta,p}(\mathbb{H}^n), 0 < p < \infty, 1 < q < \infty$ and $\beta \ge Q(1-1/q)$, then $\|\hat{f}(\lambda)W^0_\alpha(\lambda)\| \le C\|f\|_{H\dot{K}_q^{\beta,p}(\mathbb{H}^n)}((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(\beta+Q(1-1/q))}$.

Our results are also true for the non-homogeneous Herz-type Hardy Space $HK_q^{\alpha,p}(\mathbb{H}^n)$ and we omit the details here.

2. Proof of the Main Theorems

We first state definitions of atom and molecule about Herz-type Hardy spaces on $\mathbb{H}^n$.

Definition 2.1. Let $1 < q < \infty, Q(1-1/q) \leq \alpha < \infty$ and non-negative integer $s \geq [\alpha + Q(1/q-1)]$.

(1) A function $a(x)$ on $\mathbb{H}^n$ is called a central (α, q, s)-atom, if it satisfies

(i) $\text{supp } a \subset B(0,r) = \{x \in \mathbb{H}^n : |x| \leq r\}$;
(ii) $\|a\|_{L^q(\mathbb{H}^n)} \leq |B(0,r)|^{-\alpha/Q}$;
(iii) $\int a(x)x^\beta dx = 0, \beta$ is a multi-index with $\beta = (J_1, J_2, I), x^\beta = (x_1, x_2, t)^\beta$ $= x_1^{J_1} x_2^{J_2} t^I$ for all $|\beta| = J_1 + J_2 + 2I \leq s$.

(2) A function $a(x)$ on $\mathbb{H}^n$ is called a central $(\alpha, q, s)-$atom of restrict type , if it satisfied (ii) (iii) and (i)' $\text{supp} a \subset B(0,r), \quad r \geq 1$.

Definition 2.2. Let $1 < q < \infty, Q(1-1/q) \leq \alpha < \infty$, non-negative integer $s \geq [\alpha + Q(1/q-1)], \epsilon > \max\{s/Q, \alpha/Q + 1/q - 1\}, a = 1 - 1/q - \alpha/Q + \epsilon$ and $b = 1 - 1/q + \epsilon$.

(1) A function $M \in L^q(\mathbb{H}^n)$ is called a central $(\alpha, q, s, \epsilon)-$molecule, if it satisfies

(i) $R_q(M) := \|M\|_{L^q(\mathbb{H}^n)}^{a/b} \||x|^{Qb} M(x)\|_{L^q(\mathbb{H}^n)}^{1-a/b} < \infty$;
(ii) $\int M(x)x^\beta dV(x) = 0, \quad$ for all $\quad |\beta| \leq s$.

(2) A function $M \in L^q(\mathbb{H}^n)$ is called a central $(\alpha, q, s, \epsilon)-$molecule of restrict type , if it satisfies (i) (ii) and (iii) $\|\mathrm{M}\|_{\mathrm{L^q(\mathbb{H}^n)}} \leq 1$.

Then we have the decomposition theorem of Herz-type Hardy spaces.

Proposition 2.1. *Let $0 < p < \infty, 1 < q < \infty$ and $Q(1-1/q) \leq \alpha < \infty$, then the following three conditions are equivalents:*

(1) $f \in H\dot{K}_q^{\alpha,p}$;

(2) $f(x) =^{S'} \sum_{k=-\infty}^{\infty} \lambda_k a_k(x)$, *where each a_k is central $(\alpha, q, s)-$atoms with the support B_k, and $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$. Moreover ,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbb{H}^n)} \sim \inf \Big(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \Big)^{1/p},$$

where the infimum is taken over all above decompositions of f;

(3) $f =^{S'} \sum_{j=-\infty}^{\infty} u_j M_j$, *where each* M_j *is a dyadic central* (α, q, s, ϵ)*-molecule*

with $R_q(M_j) \leq c < \infty, c$ *is independent of* M_j, *and* $\sum_{j=-\infty}^{\infty} |u_j|^p < \infty$.

you can find $(1) \Leftrightarrow (2)$ *in* [7], *by the procedure of* [3], *we can prove* $(1) \Leftrightarrow (3)$. *Here, we omit it.*

Their exists similar results on non-homogeneous Herz-type Hardy spaces.

Lemma 2.1. *Let* a *be a* (β, q, s)*-atom with the center 0.*

For $q \geq 2$, *we have*

(i) $\|\Delta^J \hat{a}\|_{\mathcal{L}^2} \leq C\|a\|_{L^q(\mathbb{H}^n)}^{1-\frac{h(J)+Q(\frac{1}{2}-\frac{1}{q})}{\beta}}$;

For $1 \leq q < \infty$, *we have*

(ii) $\|\Delta^J \hat{a}(\lambda)\| \leq C\|a\|_{L^q(\mathbb{H}^n)}^{1-\frac{1}{\beta}(h(J)+Q/q')}$;

(iii) $\|\Delta^J \hat{a}(\lambda) W_\alpha^0(\lambda)\| \leq C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s-h(J)+1)} \|a\|_{L^q(\mathbb{H}^n)}^{1-\frac{1}{\beta}(s+1+Q/q')}$;

where $1/q' + 1/q = 1$.

Proof. Suppose supp $a \subset B(0, r)$.
When $q \geq 2$,

$$\begin{aligned}\|\Delta^J \hat{a}\|_{\mathcal{L}^2} &= \|(\cdot)^J a(\cdot)\|_{L^2(\mathbb{H}^n)} \leq Cr^{h(J)} \|a\|_{L^2(\mathbb{H}^n)} \\ &\leq Cr^{h(J)} \|a\|_{L^q(\mathbb{H}^n)} |B(0,r)|^{(1/2-1/q)}.\end{aligned}$$

For $|B(0,r)| \leq \|a\|_{L^q(\mathbb{H}^n)}^{-\frac{Q}{\beta}}$, then

$$\begin{aligned}\|\Delta^J \hat{a}\|_{\mathcal{L}^2} &\leq C|B(0,r)|^{\frac{h(J)}{Q}+\frac{1}{2}-\frac{1}{q}} \|a\|_{L^q(\mathbb{H}^n)} \\ &\leq C\|a\|_{L^q(\mathbb{H}^n)}^{-[\frac{h(J)}{\beta}+\frac{Q(\frac{1}{2}-\frac{1}{q})}{\beta}]} \|a\|_{L^q(\mathbb{H}^n)} \\ &\leq C\|a\|_{L^q(\mathbb{H}^n)}^{1-\frac{h(J)+Q(\frac{1}{2}-\frac{1}{q})}{\beta}}.\end{aligned}$$

This proves (i).

For (ii), since $1 \leq q < \infty$,

$$\begin{aligned}\|\Delta^J \hat{a}(\lambda)\| &= \|\widehat{(\cdot)^J a(\cdot)}\| \leq C\|(\cdot)^J a(\cdot)\|_{L^1(\mathbb{H}^n)} \\ &\leq Cr^{h(J)} \|a\|_{L^1(\mathbb{H}^n)} \\ &\leq Cr^{h(J)} \|a\|_{L^q(\mathbb{H}^n)} |B(0,r)|^{1/q'}\end{aligned}$$

$$\leq C|B(0,r)|^{\frac{h(J)}{Q}+\frac{1}{q'}}\|a\|_{L^q(\mathbb{H}^n)}$$
$$\leq C\|a\|_{L^q(\mathbb{H}^n)}^{-\frac{Q}{\beta}(\frac{h(J)}{Q}+\frac{1}{q'})}\|a\|_{L^q(\mathbb{H}^n)}$$
$$\leq C\|a\|_{L^q(\mathbb{H}^n)}^{1-\frac{h(J)+\frac{Q}{q'}}{\beta})}.$$

We now prove (iii).

Set $u=(z,t),\ p(z,t)=\displaystyle\sum_{2k+l\leq s-h(J)}\frac{(i\lambda t)^k}{k!}\cdot\frac{(z\cdot W_\lambda-\bar{z}\cdot W_\lambda^+)^l}{l!}$.

By the vanishing property of atom,

$$\Delta^J\hat{a}(\lambda)=\widehat{(\cdot)^J a(\cdot)}(\lambda)=\int_{\mathbb{H}^n}(z,t)^J a(z,t)(\Pi_\lambda(z,t))-p(z,t))dV(u).$$

Set ${\mathcal{H}_{|\lambda|}}^N$ be the subspace of $\mathcal{H}_{|\lambda|}$ spanned by $\{W_\alpha^0(\lambda):|\alpha|\leq N\}$. Remark that $z\cdot W_\lambda-\bar{z}\cdot W_\lambda^+$ is bounded from $\mathcal{H}_{|\lambda|}^N$ to $\mathcal{H}_{|\lambda|}^{N+1}$ and whose bound $\leq((2|\alpha|+n)|\lambda|)^{1/2}|z|$. Then we get

$$\begin{aligned}
\|\Delta^J\hat{a}(\lambda)W_\alpha^0(\lambda)\| &\leq C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s-h(J)+1)}\int_{\mathbb{H}^n}|(z,t)|^{s+1}|a(z,t)|dV(u)\\
&\leq C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s-h(J)+1)}r^{s+1}\int_{\mathbb{H}^n}|a(z,t)|dV(u)\\
&\leq C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s-h(J)+1)}r^{s+1}\|a\|_{L^1(\mathbb{H}^n)}\\
&\leq C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s-h(J)+1)}r^{s+1}\|a\|_{L^q(\mathbb{H}^n)}|B(0,r)|^{1/q'}\\
&\leq C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s-h(J)+1)}\|a\|_{L^q(\mathbb{H}^n)}|B(0,r)|^{\frac{s+1}{Q}+\frac{1}{q'}}\\
&\leq C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s-h(J)+1)}\|a\|_{L^q(\mathbb{H}^n)}^{-\frac{Q}{\beta}(\frac{s+1}{Q}+\frac{1}{q'})}\\
&\leq C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s-h(J)+1)}\|a\|_{L^q(\mathbb{H}^n)}^{-\frac{1}{\beta}(s+1+\frac{Q}{q'})}.
\end{aligned}$$

Then we finish the proof of lemma 2.1.

Proof of Theorem 1.1. We only need to prove if a is a dyadic central $(\beta,q,\tau-1)-$atom, then T_Ma is a $(p,q,[\beta+Q(1/q-1)],\tau/Q-1/2)-$molecule and $R_2(T_Ma)\leq C$.

If a is a dyadic central $(\beta,q,\tau-1)-$atom supported on $B(0,2^j),j\in\mathbb{Z}$, then $a(x)$ satisfies:

(1) $\text{supp } a\subset B(0,2^j)$,

(2) $\|a\|_{L^q(\mathbb{H}^n)}\leq|B(0,2^j)|^{-\frac{\beta}{Q}}\leq C2^{-j\beta}$,

(3) $\displaystyle\int a(x)x^J dx=0$ for all $h(J)\leq s$.

We need to prove $T_M a$ is a $(\beta, q, [\beta+Q(1/q-1)], \tau/Q-1/2)-$molecule and satisfies the following conditions:

(1) $\|T_M a\|_{L^q(\mathbb{H}^n)} \le C2^{-j\beta}$,

(2) $\|a\|_{L^q(\mathbb{H}^n)}^{a/b} \|a(\cdot)|\cdot|^{Qb}\|_{L^q(\mathbb{H}^n)}^{1-a/b} \le C < \infty$,

(3) $\int a(x)x^J dV(x) = 0$, for all $h(J) \le [\beta + Q(1/q-1)]$.

According to plancherel Theorem,

$$\begin{aligned}
\|T_M a\|_{L^2(\mathbb{H}^n)} &= \|\hat{a}(\lambda)M(\lambda)\|_{\mathcal{L}^2} \\
&= \Big(\frac{2^{n-1}}{\pi^{n+1}}\int_{-\infty}^{\infty}\|\hat{a}M\|_{H-S}^2|\lambda|^n d\lambda\Big)^{1/2} \\
&\le C\Big(\frac{2^{n-1}}{\pi^{n+1}}\int_{-\infty}^{\infty}\|\hat{a}\|_{H-S}^2|\lambda|^n d\lambda\Big)^{1/2} \\
&= C\|\hat{a}\|_{\mathcal{L}^2} \\
&\le C\|a\|_{L^2(\mathbb{H}^n)}.
\end{aligned}$$

Because $|(z,t)|^\tau \le C(|t|+|z|^2)^{\tau/2}$ and $\frac{\tau}{2}$ is an integer, we have

$$\begin{aligned}
\||\cdot|^\tau T_M(a)(\cdot)\|_{L^2(\mathbb{H}^n)} &\le C\|(|z|^2+|t|)^{\tau/2}T_M a\|_{L^2(\mathbb{H}^n)} \\
&\le C\|\big(\widehat{(|z|^2+|t|)^{\tau/2}}T_M a\big)\|_{\mathcal{L}^2} \\
&\le C\sum_{h(J)=\tau}\|\Delta^J(\widehat{T_M a})\|_{\mathcal{L}^2} \\
&\le C\sum_{h(J)=\tau}\|\Delta^J(\hat{a}M)\|_{\mathcal{L}^2} \\
&\le C\sum_{h(J')+h(J'')=\tau}\|(\Delta^{J'}\hat{a})(\Delta^{J''}M)\|_{\mathcal{L}^2}.
\end{aligned}$$

Next, we consider

$$\|(\Delta^{J'}\hat{a})(\Delta^{J''}M)\|_{\mathcal{L}^2} \quad \text{for} \quad h(J')+h(J'')=\tau.$$

If $h(J')=\tau$, from (i) of Lemma 2.1, we get

$$\|(\Delta^J\hat{a})M\|_{\mathcal{L}^2} \le C\|\Delta^J\hat{a}\|_{\mathcal{L}^2} \le C\|a\|_{L^2(\mathbb{H}^n)}^{1-\frac{h(J)}{\beta}} \le C\|a\|_{L^2(\mathbb{H}^n)}^{1-\frac{\tau}{\beta}}.$$

Suppose $0 \le h(J') < \tau$, then

$$\begin{aligned}
&\|(\Delta^{J'}\hat{a})(\Delta^{J''}M(\lambda))\|^2_{\mathcal{L}^2}\\
&\le C\int_{-\infty}^{\infty}\|(\Delta^{J'}\hat{a}(\lambda))(\Delta^{J''}M(\lambda))\|^2_{H-S}|\lambda|^n d\lambda\\
&\le C\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{\infty}\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}\|\Delta^{J'}\hat{a}(\lambda)W^0_\alpha(\lambda)W^0_\alpha(\lambda)\Delta^{J''}M(\lambda)\|^2_{H-S}|\lambda|^n d\lambda\\
&\le C\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{\infty}\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}\|\Delta^{J'}\hat{a}(\lambda)W^0_\alpha(\lambda)\|^2\|W^0_\alpha(\lambda)\Delta^{J''}M(\lambda)\|^2_{H-S}|\lambda|^n d\lambda.
\end{aligned}$$

Fix k_0 such that $2^{k_0} \le \|a\|^{2/\beta}_{L^2} \le 2^{k_0+1}$, by (iii) of Lemma 2.1, we have

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{k_0}\int_{2^k<(2|\alpha|+n)||\lambda|\le 2^{k+1}}\|\Delta^{J'}\hat{a}(\lambda)W^0_\alpha(\lambda)\|^2\|W^0_\alpha(\lambda)\Delta^{J''}M(\lambda)\|^2_{H-S}|\lambda|^n d\lambda.\\
&\le C\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{k_0}\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}\|\Delta^{J'}\hat{a}(\lambda)W^0_\alpha(\lambda)\|^2((2|\alpha|+n)|\lambda|)^{-h(J'')}|\lambda|^n d\lambda\\
&\le C\|a\|^{2(1-\frac{1}{\beta}(s+1+Q/2))}_{L^2(\mathbb{H}^n)}\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{k_0}\\
&\quad\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}((2|\alpha|+n)|\lambda|)^{(\tau-h(J'))}((2|\alpha|+n)|\lambda|)^{-h(J'')}|\lambda|^n d\lambda\\
&\le C\|a\|^{2(1-\frac{1}{\beta}(s+1+Q/2))}_{L^2(\mathbb{H}^n)}\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{k_0}\\
&\quad\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}((2|\alpha|+n)|\lambda|)^{(\tau-h(J')-h(J''))}|\lambda|^n d\lambda\\
&\le C\|a\|^{2-\frac{1}{\beta}(2\tau+Q)}_{L^2(\mathbb{H}^n)}\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{k_0}\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}|\lambda|^n d\lambda\\
&\le C\|a\|^{2-\frac{1}{\beta}(2\tau+Q)}_{L^2(\mathbb{H}^n)}\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{k_0}\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}\frac{2^{(k+1)n}}{(2|\alpha|+n)^n}d\lambda\\
&\le C\|a\|^{2-\frac{1}{\beta}(2\tau+Q)}_{L^2(\mathbb{H}^n)}\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{k_0}\int_{\frac{2^k}{(2|\alpha|+n)}}^{\frac{2^{k+1}}{(2|\alpha|+n)}}\frac{2^{(k+1)n}}{(2|\alpha|+n)^n}d\lambda\\
&\le C\|a\|^{2-\frac{1}{\beta}(2\tau+Q)}_{L^2(\mathbb{H}^n)}\sum_{\alpha\in\mathbb{Z}^n_+}\sum_{k=-\infty}^{k_0}\frac{2^{(k+1)(n+1)}}{(2|\alpha|+n)^{(n+1)}}\\
&\le C\|a\|^{2-\frac{1}{\beta}(2\tau+Q)}_{L^2(\mathbb{H}^n)}2^{k_0(n+1)}\\
&\le C\|a\|^{2-\frac{1}{\beta}(2\tau+Q)}_{L^2(\mathbb{H}^n)}\|a\|^{Q/\beta}_{L^2(\mathbb{H}^n)}\\
&\le C\|a\|^{2-2\tau/\beta}_{L^2(\mathbb{H}^n)}.
\end{aligned}$$

By (i) of Lemma 2.1, we have

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}_+^n}\sum_{k=k_0+1}^{\infty}\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}\|\Delta^{J'}\hat{a}(\lambda)W_\alpha^0(\lambda)\|^2\|W_\alpha^0(\lambda)\Delta^{J''}M(\lambda)\|_{H-S}^2|\lambda|^n d\lambda.\\
&\le C\sum_{\alpha\in\mathbb{Z}_+^n}\sum_{k=k_0+1}^{\infty}\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}\|\Delta^{J'}\hat{a}(\lambda)W_\alpha^0(\lambda)\|^2((2|\alpha|+n)|\lambda|)^{-h(J'')}|\lambda|^n d\lambda.\\
&\le C\sum_{\alpha\in\mathbb{Z}_+^n}\sum_{k=k_0+1}^{\infty}\int_{2^k<(2|\alpha|+n)|\lambda|\le 2^{k+1}}\|\Delta^{J'}\hat{a}(\lambda)W_\alpha^0(\lambda)\|_{H-S}^2|\lambda|^n d\lambda 2^{-kh(J'')}\\
&\le C\sum_{k=k_0+1}^{\infty}2^{-kh(J'')}\int_{-\infty}^{\infty}\|\Delta^{J'}\hat{a}(\lambda)\|_{H-S}^2|\lambda|^n d\lambda\\
&\le C\|\Delta^{J'}\hat{a}(\lambda)\|_{\mathcal{L}^2}^2\sum_{k=k_0+1}^{\infty}2^{-kh(J'')}\\
&\le C2^{-k_0h(J'')}\|a\|_{L^2(\mathbb{H}^n)}^{2-2h(J')/\beta}\\
&\le C\|a\|_{L^2(\mathbb{H}^n)}^{-2h(J'')/\beta}\|a\|_{L^2(\mathbb{H}^n)}^{2-2h(J')/\beta}\\
&\le C\|a\|_{L^2(\mathbb{H}^n)}^{2-2\tau/\beta}.
\end{aligned}$$

From the above estimates, we get

$$R_2(T_Ma) := \|T_Ma\|_{L^2(\mathbb{H}^n)}^{1-\beta/\tau}\|T_Ma(\cdot)|\cdot|^\tau\|_{L^2(\mathbb{H}^n)}^{\beta/\tau} \le C < \infty.$$

Finally, we prove the cancelation property of T_Ma.

From (iii) of Lemma 2.1, if $h(J') + h(J'') \le [\beta - Q/2]$, then

$$\begin{aligned}
&\|\Delta^{J'}\hat{a}(\lambda)W_\alpha^0(\lambda)\Delta^{J''}M(\lambda)\|_{H-S}^2\\
&\le C\|\Delta^{J'}\hat{a}(\lambda)W_\alpha^0(\lambda)\|^2\|W_\alpha^0(\lambda)\Delta^{J''}M(\lambda)\|_{H-S}^2\\
&\le C((2|\alpha|+n)|\lambda|)^{(\tau-1-h(J')+1)}((2|\alpha|+n)|\lambda|)^{-h(J'')}\|a\|_{L^2(\mathbb{H}^n)}^{2(1-\frac{1}{\beta}(\tau-1+1+Q/2))}\\
&\le C((2|\alpha|+n)|\lambda|)^{(\tau-h(J')-h(J''))}\|a\|_{L^2(\mathbb{H}^n)}^{2-\frac{2}{\beta}(\tau+Q/2)}\\
&= C((2|\alpha|+n)|\lambda|)^{(\tau-h(J')-h(J''))}\|a\|_{L^2(\mathbb{H}^n)}^{2-\frac{1}{\beta}(2\tau+Q)}\\
&\le C|\lambda|^{\tau-h(J')-h(J'')}.
\end{aligned}$$

Hence $\Delta^J(\hat{a}(\lambda)M(\lambda)) \to 0$ as $\lambda \to 0$ in the sense of weak convergence.

This implies

$$\int_{\mathbb{H}^n} T_Ma(z,t)(z,t)^J dV(u) = 0 \quad \text{for} \quad 0 \le h(J) \le [\beta - Q/2].$$

Then we finish the proof of Theorem 1.1.

Proof. Proof of Theorem 1.2. By (ii) of Lemma 2.1,

$$\|\hat{a}(\lambda)W_\alpha^0(\lambda)\| \le \|\hat{a}(\lambda)\| \le C\|a\|_{L^q(\mathbb{H}^n)}^{1-\frac{Q}{\beta}(1-1/q)}.$$

If $\|a\|_{L^q(\mathbb{H}^n)} \le C((2|\alpha|+n)|\lambda|)^{\beta/2}$, then

$$\|\hat{a}(\lambda)W_\alpha^0(\lambda)\| \le C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(\beta+Q(1/q-1))}.$$

If $\|a\|_{L^q(\mathbb{H}^n)} > C((2|\alpha|+n)|\lambda|)^{\beta/2}$, by (iii) of Lemma 2.1,

$$\begin{aligned}\|\hat{a}(\lambda)W_\alpha^0(\lambda)\| &\le C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s+1)}\|a\|_{L^q(\mathbb{H}^n)}^{1-\frac{1}{\beta}(s+1+Q/q')}\\ &\le C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(s+1)}((2|\alpha|+n)|\lambda|)^{\frac{\beta}{2}(1-\frac{1}{\beta}(s+1+Q/q''))}\\ &\le C((2|\alpha|+n)|\lambda|)^{\frac{1}{2}(\beta+Q(1/q-1))}.\end{aligned}$$

Then we done.

3. Applications

Let $\mathcal{L} = -\dfrac{1}{2}\displaystyle\sum_{k=1}^{n}(Z_k\bar{Z}_k + \bar{Z}_kZ_k)$ be the sub-Laplacian on $\mathbb{H}^n$, then $\mathcal{L}$ admits a spectral resolution $\mathcal{L} = \displaystyle\int_0^\infty \lambda dE(\lambda)$, where $E(\lambda)$ is spectral measure. According to the Littlewood-Paley-Stein theory [8], if

$$f(\lambda) = \int_0^\infty e^{-\lambda t}\phi(s)ds$$

for some $\phi \in L^\infty(0,\infty)$, then the operator $f(\mathcal{L}) = \displaystyle\int_0^\infty f(\lambda)dE(\lambda)$ is bounded on $L^p(\mathbb{H}^n), \quad 1 < p < \infty$.

An easy computation shows that the operator

$$\hat{\mathcal{L}}(\lambda) = \sum_{\alpha\in\mathbb{Z}_+^n}((2|\alpha|+n)|\lambda|)W_\alpha^0(\lambda).$$

If f is a bounded Borel measurable function on $[0,\infty)$, one may define the operator $f(\mathcal{L})$ by $f(\mathcal{L}) = \displaystyle\int_0^\infty f(\lambda)dE(\lambda)$. Clearly $m(\mathcal{L})$ is bounded on $L^2(\mathbb{H}^n)$.

As a simple corollary of Theorem 1.1, the following corollary is convenient for application.

Corollary 3.1. *Let* p, τ, β *as Theorem 1. Suppose* f *is a function of* $C^{\tau}(\mathbb{R}^*)$ *such that* $|f^{(j)}(r)| \leq C_n r^{-j}$ *for* $0 \leq j \leq \tau$. *Set* $M(\lambda) = \sum_{\alpha \in \mathbb{Z}_+^n} f((2|\alpha| + n)|\lambda|) W_\alpha^0(\lambda)$, *then, the multiplier* T_M *defined by* $\widehat{(T_M f)}(\lambda) = \widehat{f}(\lambda) M(\lambda)$ *is bounded on* $H\dot{K}_2^{\beta,p}(\mathbb{H}^n)$.

Example 1. The potential integral operators $\pounds^{it}$ and $(I + \pounds)^{it}$, $t \in \mathbb{R}$, are the right-multipliers defined respectively by

$$\widehat{(\pounds^{it})}\,(\lambda) = \sum_{\alpha \in \mathbb{Z}_+^n} ((2|\alpha| + n)|\lambda|)^{it} W_\alpha^0(\lambda)$$

and

$$(\widehat{I + \pounds})^{it}\,(\lambda) = \sum_{\alpha \in \mathbb{Z}_+^n} (1 + (2|\alpha| + n)|\lambda|)^{it} W_\alpha^0(\lambda)$$

are bounded operators on $H\dot{K}_2^{\beta,p}(\mathbb{H}^n)$, where β, p as in Theorem 1.1.

Example 2. The Riesz transforms defined as $\mathcal{R}_j = Z_j \pounds^{-\frac{1}{2}}$ and $\mathcal{R}_{j+n} = \bar{Z}_j \pounds^{-\frac{1}{2}}$, $j = 1, \cdots, n$ are bounded operators on $H\dot{K}_2^{\beta,p}(\mathbb{H}^n)$, where β, p as in Theorem 1.1.

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