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COMPACTNESS OF THE DIFFERENCES OF WEIGHTED COMPOSITION OPERATORS FROM WEIGHTED BERGMAN SPACES TO WEIGHTED-TYPE SPACES ON THE UNIT BALL

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Abstract. Let φ_1 and φ_2 be holomorphic self-maps of the open unit ball $\mathbb{B}$ in $\mathbb{C}^N$, u_1 and u_2 be holomorphic functions on $\mathbb{B}$ and let weighted composition operators W_{φ_1,u_1}; $W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ be bounded. This paper characterizes the compactness of the difference of these operators from the weighted Bergman space A^p_α, $0 < p < \infty$, $\alpha > -1$, to the weighted-type space H^∞_v of holomorphic functions on $\mathbb{B}$ in terms of inducing symbols φ_1, φ_2, u_1 and u_2. For the case $p > 1$ we find an asymptotically equivalent expression to the essential norm of the operator.

1. INTRODUCTION

Let $\mathbb{B}^N = \mathbb{B}$ be the open unit ball in the complex vector space $\mathbb{C}^N$, $\mathbb{B}^1 = \mathbb{D}$ the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{B})$ the space of all holomorphic functions on $\mathbb{B}$ and $H^\infty(\mathbb{B}) = H^\infty$ the space of all bounded holomorphic functions on $\mathbb{B}$ with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{B}} |f(z)|$. Let $z = (z_1, \ldots, z_N)$ and $w = (w_1, \ldots, w_N)$ be points in $\mathbb{C}^N$, $\langle z, w\rangle = \sum_{k=1}^N z_k \bar{w}_k$ and $|z| = \sqrt{\langle z, z\rangle}$.

Let dv be the normalized volume measure on $\mathbb{B}$ and $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dv(z)$, $\alpha > -1$, be the weighted Lebesgue measure on $\mathbb{B}$, where $c_\alpha = \frac{\Gamma(N+\alpha+1)}{N!\Gamma(\alpha+1)}$ is a normalizing constant, that is, $v_\alpha(\mathbb{B}) = 1$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space $A^p_\alpha(\mathbb{B}) = A^p_\alpha$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}} |f(z)|^p dv_\alpha(z) < \infty.$$

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When $p \geq 1$, the weighted Bergman space with the norm $\|\cdot\|_{A^p_\alpha}$ becomes a Banach space. If $p \in (0,1)$, it is a Fréchet space with the translation invariant metric

$$d(f,g) = \|f-g\|^p_{A^p_\alpha}.$$

Let v be a positive continuous function on $\mathbb{B}$ (*weight*). The weighted-type space $H^\infty_v(\mathbb{B}) = H^\infty_v$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H^\infty_v} = \sup_{z\in\mathbb{B}} v(z)|f(z)| < \infty.$$

With the norm $\|\cdot\|_{H^\infty_v}$, H^∞_v is a Banach space. For various kinds of weights and related weighted-type spaces see, e.g., [1, 2, 15, 17, 35] as well as the references therein.

Let $\varphi : \mathbb{B} \to \mathbb{B}$ be a holomorphic self-map of $\mathbb{B}$ and $u \in H(\mathbb{B})$, then the weighted composition operator $W_{\varphi,u}$ on $H(\mathbb{B})$ is defined by

$$W_{\varphi,u}f(z) = u(z)f(\varphi(z)), \quad z \in \mathbb{B}.$$

When $u(z) \equiv 1$ on $\mathbb{B}$, the weighted composition operator $W_{\varphi,1} = C_\varphi$ is called the composition operator. Recently there has been a huge interest in studying weighted composition operators between spaces of analytic functions, see, e.g., the following papers which consider these and some related operators mostly when one of the spaces is a weighted or Bloch-type space: [3-14, 17-38, 40] and the references therein.

Let X and Y be topological vector spaces whose topologies are given by translation-invariant metrics d_X and d_Y, respectively, and $L : X \to Y$ be a linear operator. It is said that L is *metrically bounded* if there exists a positive constant K such that

$$d_Y(Lf, 0) \leq K d_X(f, 0)$$

for all $f \in X$. When X and Y are Banach spaces, the metrically boundedness coincides with the usual definition of bounded operators between Banach spaces.

If Y is a Banach space then the quantity $\|L\|_{A^p_\alpha \to Y}$ is defined as follows

$$\|L\|_{A^p_\alpha \to Y} := \sup_{\|f\|_{A^p_\alpha} \leq 1} \|Lf\|_Y.$$

It is easy to see that this quantity is finite if and only if the operator $L : A^p_\alpha \to Y$ is metrically bounded. For the case $p \geq 1$ this is the standard definition of the norm of the operator $L : A^p_\alpha \to Y$, between two Banach spaces. If we say that an operator is bounded it means that it is metrically bounded.

Recall that $L : X \to Y$ is *metrically compact* if it maps bounded sets into relatively compact sets. If X and Y are Banach spaces then metrically compactness becomes usual compactness. In this case if $L : X \to Y$ is a bounded linear operator, then the essential norm of the operator $L : X \to Y$, denoted by $\|L\|_{e,X\to Y}$, is defined as follows

$$\|L\|_{e,X\to Y} = \inf\{\|L + K\|_{X\to Y} \ : \ K \text{ is compact from } X \text{ to } Y\},$$

where $\|\cdot\|_{X\to Y}$ denote the operator norm. From this definition and since the set of all compact operators is a closed subset of the space of bounded operators it follows that operator L is compact if and only if $\|L\|_{e,X\to Y} = 0$. Some results on essential norms can be found, e.g., in [7, 8, 14, 17, 25, 26, 28, 31, 35] and [37].

Let φ_1, φ_2 be holomorphic self-maps of $\mathbb{B}$ and $u_1, u_2 \in H(\mathbb{B})$. Differences of weighted composition operators on $H(\mathbb{B})$ are defined as follows

$$(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})(f)(z) = u_1(z)f(\varphi_1(z)) - u_2(z)f(\varphi_2(z)), \quad z \in \mathbb{B}.$$

Some results on differences of weighted composition operators can be found, e.g., in [4, 7, 8, 10, 14, 16, 18] and [35] (see also the references therein).

Here we characterize the compactness of differences of weighted composition operators acting from the weighted Bergman space A^p_α to the weighted-type space H^∞_v on the unit ball. For the case $1 < p < \infty$ we also find an asymptotically equivalent expression to the essential norm of these operators.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $A/C \le B \le CA$.

2. Auxiliary Results

In order to deal with differences of weighted composition operators, we need the Bergman metric for the unit ball $\mathbb{B}$. Recall that for an $a \in \mathbb{B}$, the involutive automorphism of the unit ball $\mathbb{B}$ which interchanges 0 and a is given by

$$\sigma_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}, \quad \text{for all} \quad z \in \mathbb{B},$$

where $s_a = (1 - |a|^2)^{1/2}$, P_a is the orthogonal projection from $\mathbb{C}^N$ onto the one dimensional subspace $[a]$ generated by a, and Q_a is the orthogonal projection from $\mathbb{C}^N$ onto $\mathbb{C}^N \ominus [a]$. The pseudo-hyperbolic metric $\rho(z, w)$, $z, w \in \mathbb{B}$ is given by

$$\rho(z, w) = |\sigma_w(z)|.$$

It is well-known that $\rho(z, w)$ is a metric on $\mathbb{B}$. The Bergman metric for the unit ball $\mathbb{B}$ is defined by

$$\beta(z, w) = \frac{1}{2} \ln \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

In this section we shall prove several auxiliary results which will be used in the proofs of the main results in this paper.

The proof of the following lemma is standard, so it will be omitted (see, e.g., Proposition 3.11 in [7] or Lemma 3 in [19]).

Lemma 1. *Assume* $p > 0$, $\alpha > -1$, v *is a weight on* $\mathbb{B}$, φ_1, φ_2 *are holomorphic self-maps of* $\mathbb{B}$, u_1, u_2 *are holomorphic functions on* $\mathbb{B}$ *and the operator* $W_{\varphi_1,u_1} - W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ *is bounded. Then the operator* $W_{\varphi_1,u_1} - W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ *is metrically compact if and only if for every bounded sequence* $(f_n)_{n\in\mathbb{N}}$ *in* A^p_α *such that* $f_n \to 0$ *uniformly on compacts of* $\mathbb{B}$ *as* $n \to \infty$ *it follows that*

$$\lim_{n\to\infty} \|(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})f_n\|_{H^\infty_v} = 0.$$

The following lemma was proved in [23] (see also [33]).

Lemma 2. *Let* v *be a weight on* $\mathbb{B}$, φ *a holomorphic self-map of* $\mathbb{B}$ *and* $u \in H(\mathbb{B})$. *Then the operator* $W_{\varphi,u} : A^p_\alpha \to H^\infty_v$ *is bounded if and only if*

$$\sup_{z\in\mathbb{B}} \frac{v(z)|u(z)|}{(1-|\varphi(z)|^2)^{\frac{N+\alpha+1}{p}}} < \infty.$$

The following lemma is well-known, see, for example, [39, Theorem 2.1].

Lemma 3. *Suppose* $p \in (0,\infty)$ *and* $\alpha > -1$. *Then for all* $f \in A^p_\alpha$ *and* $z \in \mathbb{B}$, *the following inequality holds*

$$|f(z)| \le \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{N+\alpha+1}{p}}}. \tag{1}$$

The following two lemmas are important tools in the proofs of the main results.

Lemma 4. *There exists a constant* $C > 0$ *such that*

$$\left|(1-|z|^2)^{\frac{N+\alpha+1}{p}} f(z) - (1-|w|^2)^{\frac{N+\alpha+1}{p}} f(w)\right| \le C\|f\|_{A^p_\alpha}\, \rho(z,w) \tag{2}$$

for all $f \in A^p_\alpha$ *and for all* z, w *in* $\mathbb{B}$.

Proof. By Lemma 3 we have that if $f \in A^p_\alpha$ then $f \in H^\infty_{(1-|z|^2)^{(N+\alpha+1)/p}}$ and

$$\|f\|_{H^\infty_{(1-|z|^2)^{(N+\alpha+1)/p}}} \le \|f\|_{A^p_\alpha}. \tag{3}$$

Set

$$g(z) := (1-|z|^2)^{\frac{N+\alpha+1}{p}} f(z), \quad z \in \mathbb{B}.$$

By Exercise 3.16 in [39] we have

$$|g(z) - g(w)| \le C\|g\|_\infty \beta(z,w) = C\|f\|_{H^\infty_{(1-|z|^2)^{(N+\alpha+1)/p}}} \beta(z,w). \tag{4}$$

First assume $\rho(z,w) \le 1/2$. Then, since for $x \in (0,1)$

$$\frac{1}{2}\ln\frac{1+x}{1-x} = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{2j+1} < x\sum_{j=0}^{\infty} x^{2j} = \frac{x}{1-x^2},$$

we obtain in this case that

$$\beta(z,w) \le \frac{4}{3}\rho(z,w). \tag{5}$$

Combining (3), (4) and (5) inequality (2) holds when $\rho(z,w) \le 1/2$.

Now assume that $\rho(z,w) > 1/2$. Then by inequality (3) we have

(6)
$$\left|(1-|z|^2)^{\frac{N+\alpha+1}{p}} f(z) - (1-|w|^2)^{\frac{N+\alpha+1}{p}} f(w)\right| \le 2\|f\|_{H^\infty_{(1-|z|^2)^{(N+\alpha+1)/p}}}$$
$$\le 4\|f\|_{A^p_\alpha}\rho(z,w),$$

which is inequality (2) in this case, finishing the proof of the lemma. ∎

Lemma 5. *For each sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathbb{B}$ with $|w_n| \to 1$ as $n \to \infty$, there exists its subsequence $(\eta_k)_{k\in\mathbb{N}}$ and functions $(f_{n_k})_{k\in\mathbb{N}}$ in $H^\infty(\mathbb{B})$ such that*

$$\sum_{k=1}^{\infty} |f_{n_k}(z)| \le 1, \quad \textit{for all} \quad z \in \mathbb{B}, \tag{7}$$

and

$$f_{n_k}(\eta_k) > 1 - \frac{1}{2^k}, \quad k \in \mathbb{N}. \tag{8}$$

Proof. Set
$$f(z) = \frac{z_1+1}{2}$$
where $z = (z_1, \dots, z_N) \in \mathbb{B}$. Let $e_1 = (1, 0, \dots, 0)$, then

$$f(e_1) = 1. \tag{9}$$

We also have

$$|f(z)| < 1, \quad \text{for } z \in \overline{\mathbb{B}} \setminus \{e_1\}. \tag{10}$$

Now set

$$g(z) = \frac{z_1 - 1}{2},$$

where $z = (z_1, \dots, z_N) \in \mathbb{B}$, and $g_n(z) = g^{\frac{1}{n}}(z)$, then $\|g_n\|_\infty = 1$, $g_n(e_1) = 0$, and for each $z \in \mathbb{B}$

$$\lim_{n\to\infty} |g_n(z)| = 1. \tag{11}$$

Without loss of generality we may assume that $w_n \to e_1$ as $n \to \infty$, otherwise, if $w_n \to \zeta \in \partial\mathbb{B}$ as $n \to \infty$, then we will consider the functions $f_{n_k}(Uz)$, where U is the unitary transformation such that $U\zeta = e_1$. By the method of induction, we construct two sequences $(m_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ of positive integers, a sequence of complex numbers $(c_k)_{k\in\mathbb{N}}$ with $|c_k| \le 1$, and a subsequence $(\eta_k)_{k\in\mathbb{N}}$ of $(w_n)_{n\in\mathbb{N}}$ such that

$$\sup_{z\in\overline{\mathbb{B}}} \sum_{k=1}^{L} |c_k f^{m_k}(z) g_{n_k}(z)| < 1, \quad \text{for every } L \in \mathbb{N}, \tag{12}$$

and

$$c_L f^{m_L}(\eta_L) g_{n_L}(\eta_L) > 1 - \frac{1}{2^L}, \quad \text{for every } L \in \mathbb{N}. \tag{13}$$

First, take $m_1 = 1$. By (9) and the continuity of f, there exists $\eta_1 \in (w_n)_{n\in\mathbb{N}}$ such that

$$|f(\eta_1)| > \frac{1}{2}.$$

From this and (11), there exists $n_1 \in \mathbb{N}$ such that

$$|f^{m_1}(\eta_1) g_{n_1}(\eta_1)| > \frac{1}{2}. \tag{14}$$

Take a complex number c_1 such that

$$c_1 f^{m_1}(\eta_1) g_{n_1}(\eta_1) = |f^{m_1}(\eta_1) g_{n_1}(\eta_1)|, \tag{15}$$

which along with (14) gives inequality (13) for $L = 1$ (note that $|c_1| = 1$). On the other hand, from (9), (10), and the facts $\|g_{n_1}\|_\infty = 1$, $g_{n_1}(e_1) = 0$, inequality (12) hold for $L = 1$ with $f_{n_1}(z) := c_1 f^{m_1}(z) g_{n_1}(z)$.

Now suppose that $(m_k)_{k=1}^L$, $(n_k)_{k=1}^L$, $(c_k)_{k=1}^L$ and $(\eta_k)_{k=1}^L$ satisfy our conditions. Define

$$F_L(z) = \sum_{k=1}^{L} |c_k f^{m_k}(z) g_{n_k}(z)|, \quad z \in \mathbb{B}. \tag{16}$$

Take an open subset U_L of $\mathbb{B}$ such that $e_1 \in \overline{U_L}$, $\{\eta_1, \dots, \eta_L\} \cap U_L = \emptyset$ (for example $U_L = B(e_1, \varepsilon(1 - \max_{j=1,\dots,L}\{|\eta_j|\})) \cap \mathbb{B}$, for sufficiently small $\varepsilon > 0$) and

$$F_L(z) < \frac{1}{2^{L+2}}, \quad \text{for } z \in \overline{U_L}. \tag{17}$$

By (10), (16) and the fact $F_L(e_1) = 0$, it follows that there is an $m_{L+1} \in \mathbb{N}$ such that $m_L < m_{L+1}$,

$$|f^{m_{L+1}}(z)| < \frac{1}{2^{L+1}}, \quad \text{for } z \in \overline{\mathbb{B} \setminus U_L}$$

and

$$F_L(z) + |f^{m_{L+1}}(z)| < 1, \quad \text{for} \quad z \in \overline{\mathbb{B} \setminus U_L}. \tag{18}$$

By (9) and the assumption $w_n \to e_1$ as $n \to \infty$, we have that there is a point $\eta_{L+1} \in (w_n)_{n\in\mathbb{N}} \cap U_L$ such that

$$|f^{m_{L+1}}(\eta_{L+1})| > \frac{1 - \frac{1}{2^{L+1}}}{1 - \frac{1}{2^{L+2}}}.$$

From this and by (11) it follows that there is an $n_{L+1} > n_L$ such that

$$|f^{m_{L+1}}(\eta_{L+1}) g_{n_{L+1}}(\eta_{L+1})| > \frac{1 - \frac{1}{2^{L+1}}}{1 - \frac{1}{2^{L+2}}}. \tag{19}$$

Since $\|g_{n_{L+1}}\|_\infty = 1$ and from (18) we have

$$F_L(z) + |f^{m_{L+1}}(z) g_{n_{L+1}}(z)| < 1, \quad \text{for} \quad z \in \overline{\mathbb{B} \setminus U_L}.$$

From this, (17) and since $\|f^{m_{L+1}} g_{n_{L+1}}\|_\infty < 1$ we have

$$\sup_{z\in\overline{\mathbb{B}}} \left[F_L(z) + \left(1 - \frac{1}{2^{L+2}}\right) |f^{m_{L+1}}(z) g_{n_{L+1}}(z)| \right] < 1. \tag{20}$$

Now take $b_{L+1} \in \mathbb{C}$ such that

$$b_{L+1} f^{m_{L+1}}(\eta_{L+1}) g_{n_{L+1}}(\eta_{L+1}) = |f^{m_{L+1}}(\eta_{L+1}) g_{n_{L+1}}(\eta_{L+1})|,$$

and let $c_{L+1} = b_{L+1}(1 - \frac{1}{2^{L+2}})$, then $|c_{L+1}| = 1 - \frac{1}{2^{L+2}} < 1$. For such chosen c_{L+1} inequality (20) means that inequality (12) holds for $L+1$. On the other hand, from (19) we get

$$c_{L+1} f^{m_{L+1}}(\eta_{L+1}) g_{n_{L+1}}(\eta_{L+1}) = \left(1 - \frac{1}{2^{L+2}}\right) |f^{m_{L+1}}(\eta_{L+1}) g_{n_{L+1}}(\eta_{L+1})| > 1 - \frac{1}{2^{L+1}}.$$

Hence (13) holds for $L+1$, finishing the inductive proof of this claim. Now note that the sequence $(f_{n_k})_{k\in\mathbb{N}}$ can be defined as follows

$$f_{n_k}(z) = c_k f^{m_k}(z) g_{n_k}(z), \quad k \in \mathbb{N}.$$

■

3. Main Results

Here we formulate and prove the main results of this paper.

Theorem 1. *Assume* $p > 0$, $\alpha > -1$, v *is a weight on* $\mathbb{B}$, φ_1, φ_2 *are nonconstant holomorphic self-maps of* $\mathbb{B}$, u_1, u_2 *are holomorphic functions on* $\mathbb{B}$ *and* W_{φ_1,u_1}; $W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ *are bounded operators. Then the operator* $W_{\varphi_1,u_1} - W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ *is metrically compact if and only if the following conditions hold*

$$(a)\qquad \lim_{|\varphi_1(z)|\to 1} \frac{v(z)|u_1(z)|}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\,\rho(\varphi_1(z),\varphi_2(z)) = 0;$$

$$(b)\qquad \lim_{|\varphi_2(z)|\to 1} \frac{v(z)|u_2(z)|}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\,\rho(\varphi_1(z),\varphi_2(z)) = 0;$$

(c)

$$\lim_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}\to 1} v(z)\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right| = 0.$$

Proof. First assume that the operator $W_{\varphi_1,u_1} - W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ is metrically compact. If $\|\varphi_1\|_\infty < 1$, then *(a)* vacuously holds. Hence assume that $\|\varphi_1\|_\infty = 1$. Suppose to the contrary that *(a)* is not true. Then there is a sequence $(z_n)_{n\in\mathbb{N}}$ such that $|\varphi_1(z_n)| \to 1$ as $n\to\infty$, $\varphi_1(z_n) \neq 0$, $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\,\rho(\varphi_1(z_n),\varphi_2(z_n)) = \delta > 0. \tag{21}$$

Since $|\varphi_1(z_n)| \to 1$ as $n \to \infty$, by Lemma 5, there exists functions $f_n \in H^\infty(\mathbb{B})$, $n \in \mathbb{N}$, such that

$$\sum_{n=1}^{\infty} |f_n(z)| \le 1, \quad \text{for all} \quad z \in \mathbb{B}, \tag{22}$$

and

$$f_n(\varphi_1(z_n)) > 1 - \frac{1}{2^n}, \quad n \in \mathbb{N}. \tag{23}$$

Now, we define

$$k_n(z) = \frac{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}{(1-\langle z,\varphi_1(z_n)\rangle)^{\frac{2(N+\alpha+1)}{p}}}, \quad n \in \mathbb{N}.$$

It is well-known that $\|k_n\|_{A^p_\alpha} = 1$ for each $n \in \mathbb{N}$.

Set

$$g_n(z) = f_n(z)\frac{\langle \sigma_{\varphi_2(z_n)}(z), \sigma_{\varphi_2(z_n)}(\varphi_1(z_n))\rangle}{|\sigma_{\varphi_2(z_n)}(\varphi_1(z_n))|}k_n(z), \quad \text{when } \varphi_1(z_n) \neq \varphi_2(z_n);$$
$$g_n(z) \equiv 0, \quad \text{when } \varphi_1(z_n) = \varphi_2(z_n).$$

Clearly $\sup_{n\in\mathbb{N}} \|g_n\|_{A^p_\alpha} \leq 1$ and $g_n \to 0$ uniformly on compacts of $\mathbb{B}$ as $n \to \infty$. Since $W_{\varphi_1,u_1} - W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ is metrically compact, by Lemma 1 we get

$$\lim_{n\to\infty} \|(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})g_n\|_{H^\infty_v} = 0. \tag{24}$$

On the other hand, from the definition of the space H^∞_v, the definition of functions g_n, and by using (23) we have that

$$\begin{aligned}
&\|(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})g_n\|_{H^\infty_v}\\
&\geq v(z_n)|u_1(z_n)g_n(\varphi_1(z_n)) - u_2(z_n)g_n(\varphi_2(z_n))|\\
&= v(z_n)|u_1(z_n)f_n(\varphi_1(z_n))k_n(\varphi_1(z_n))\sigma_{\varphi_2(z_n)}(\varphi_1(z_n))|\\
&\geq \frac{v(z_n)|u_1(z_n)|\rho(\varphi_1(z_n),\varphi_2(z_n))}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\Big(1-\frac{1}{2^n}\Big).
\end{aligned} \tag{25}$$

Letting $n \to \infty$ in (25) and using (21) and (24), we obtain

$$\begin{aligned}
0 &= \lim_{n\to\infty} \|(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})g_n\|_{H^\infty_v}\\
&\geq \lim_{n\to\infty} \frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z_n),\varphi_2(z_n))\\
&= \delta > 0,
\end{aligned}$$

which is a contradiction. This shows that

$$\lim_{n\to\infty} \frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z_n),\varphi_2(z_n)) = 0,$$

for every sequence $(z_n)_{n\in\mathbb{N}}$ such that $|\varphi_1(z_n)| \to 1$ as $n \to \infty$, which implies *(a)*.

Condition *(b)* is proved similarly. Hence we omit its proof.

Now, we prove *(c)*. If $\min\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\} < 1$, then *(c)* vacuously holds. Hence assume $\min\{\|\varphi_1\|_\infty, \|\varphi_2\|_\infty\} = 1$. Suppose to the contrary that *(c)* does not hold. Then there is a sequence $(z_n)_{n\in\mathbb{N}}$ such that $\min\{|\varphi_1(z_n)|, |\varphi_2(z_n)|\} \to 1$ as $n \to \infty$ and

$$\lim_{n\to\infty} v(z_n)\left|\frac{u_1(z_n)}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z_n)}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right| = \beta > 0. \tag{26}$$

We may also assume that there is the following limit

$$l := \lim_{n\to\infty} \rho(\varphi_1(z_n), \varphi_2(z_n)) \geq 0. \tag{27}$$

Assume that $0 < l$. Then we have that for sufficiently large n, say $n \geq n_0$

$$\begin{aligned} 0 < \frac{\beta}{2} &\leq v(z_n)\left|\frac{u_1(z_n)}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z_n)}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right| \\ &\leq \frac{2}{l}\Bigg(\frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z_n),\varphi_2(z_n)) \\ &\quad + \frac{v(z_n)|u_2(z_n)|}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z_n),\varphi_2(z_n))\Bigg). \end{aligned} \tag{28}$$

Letting $n \to \infty$ in (28) and using *(a)* and *(b)* we arrive at a contradiction. Thus, we can assume that $l = 0$.

Let the sequences of functions $(f_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ be defined as above. Set

$$h_n(z) = f_n(z)k_n(z), \quad n \in \mathbb{N}.$$

Then $\sup_{n\in\mathbb{N}} \|h_n\|_{A^p_\alpha} \leq 1$ and $h_n \to 0$ uniformly on compacts of $\mathbb{B}$ as $n \to \infty$. Hence by Lemma 1

$$\lim_{n\to\infty} \|(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})h_n\|_{H_v^\infty} = 0. \tag{29}$$

Since $W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ is bounded, then by Lemma 2 we have that

$$M := \sup_{z\in\mathbb{B}} \frac{v(z)|u_2(z)|}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}} < \infty. \tag{30}$$

We have

$$\begin{aligned} &\|(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})h_n\|_{H_v^\infty} \\ \geq\ & v(z_n)|u_1(z_n)h_n(\varphi_1(z_n)) - u_2(z_n)h_n(\varphi_2(z_n))| \\ =\ & v(z_n)|u_1(z_n)f_n(\varphi_1(z_n))k_n(\varphi_1(z_n)) - u_2(z_n)f_n(\varphi_2(z_n))k_n(\varphi_2(z_n))| \\ \geq\ & v(z_n)\left|u_1(z_n)\frac{f_n(\varphi_1(z_n))}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}} - u_2(z_n)\frac{f_n(\varphi_1(z_n))}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right| \\ & - v(z_n)\left|u_2(z_n)\frac{f_n(\varphi_1(z_n))}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}} - u_2(z_n)f_n(\varphi_2(z_n))k_n(\varphi_2(z_n))\right| \\ \geq\ & v(z_n)\left|\frac{u_1(z_n)}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z_n)}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right|\left(1-\frac{1}{2^n}\right) \\ & - \frac{v(z_n)|u_2(z_n)|}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}} \\ & \times\left|(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}h_n(\varphi_1(z_n)) - (1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}h_n(\varphi_2(z_n))\right|. \end{aligned} \tag{31}$$

From (30), applying Lemma 4 to the functions h_n, $n \in \mathbb{N}$, with the points $z = \varphi_1(z_n)$ and $w = \varphi_2(z_n)$, and by using the fact $\sup_{n\in\mathbb{N}} \|h_n\|_{A^p_\alpha} \le 1$, we get

$$\begin{aligned}(32)\qquad &\frac{v(z_n)|u_2(z_n)|}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\\ &\Big|(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}} h_n(\varphi_1(z_n)) - (1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}} h_n(\varphi_2(z_n))\Big|\\ &\le CM\rho(\varphi_1(z_n), \varphi_2(z_n)).\end{aligned}$$

Using (32) in (31), then letting $n \to \infty$ is such obtained inequality, using (29) and $l = 0$, we obtain that $\beta = 0$, which is a contradiction. This proves *(c)*.

Now we assume that conditions *(a)-(c)* hold. Assume $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in A^p_α such that $f_n \to 0$ uniformly on compacts of $\mathbb{B}$. To prove that $W_{\varphi_1,u_1} - W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ is a compact operator, in view of Lemma 1, it is enough to show that $\|(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})f_n\|_{H^\infty_v} \to 0$ as $n \to \infty$. Suppose to the contrary that this is not true. Then for some $\varepsilon > 0$ there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that

$$\|(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})f_{n_k}\|_{H^\infty_v} \ge 2\varepsilon > 0$$

for every $k \in \mathbb{N}$. We may assume that $(f_{n_k})_{k\in\mathbb{N}}$ is $(f_n)_{n\in\mathbb{N}}$. Then there is a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{B}$ such that

$$(33)\qquad v(z_n)|u_1(z_n)f_n(\varphi_1(z_n)) - u_2(z_n)f_n(\varphi_2(z_n))| \ge \varepsilon > 0, \quad n \in \mathbb{N}.$$

We may also assume that the sequences $(\varphi_1(z_n))_{n\in\mathbb{N}}$ and $(\varphi_2(z_n))_{n\in\mathbb{N}}$ converge. If it were $\max\{|\varphi_1(z_n)|, |\varphi_2(z_n)|\} \to q < 1$, then from (33), since for the test function $f(z) \equiv 1 \in A^p_\alpha$ from the boundedness of the operators $W_{\varphi_i,u_i} : A^p_\alpha \to H^\infty_v$, $i = 1, 2$, we have that $u_1, u_2 \in H^\infty_v$ and since $f_n(\varphi_i(z_n)) \to 0$ as $n \to \infty$, $i = 1, 2$, we would obtain a contradiction. Hence we may assume $\max\{|\varphi_1(z_n)|, |\varphi_2(z_n)|\} \to 1$ as $n \to \infty$. We can suppose that $|\varphi_1(z_n)| \to 1$ and $\varphi_2(z_n) \to z_0$ as $n \to \infty$. Also, we can suppose that limit in (27) exists. Assume that $l > 0$. Then by *(a)* and *(b)*, we get

$$(34)\qquad \begin{aligned}&\lim_{|\varphi_1(z_n)|\to 1} \frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}} = 0 \ \text{ and}\\ &\lim_{|\varphi_2(z_n)|\to 1} \frac{v(z_n)|u_2(z_n)|}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}} = 0,\end{aligned}$$

where if $|z_0| < 1$ we regard that the second equality in (34) vacuously holds.

From (33) and Lemma 3, it follows that

$$
\begin{aligned}
0<\varepsilon \leq\; & \frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}|(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}} f_n(\varphi_1(z_n))| \\
& +\frac{v(z_n)|u_2(z_n)|}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}|(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}} f_n(\varphi_2(z_n))| \\
\leq\; & \left(\frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}+\frac{v(z_n)|u_2(z_n)|}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right)\|f_n\|_{A^p_\alpha}.
\end{aligned} \tag{35}
$$

Letting $n\to\infty$ in (35) and using (34) we obtain a contradiction. Thus, we conclude that $l=0$ which implies that $|\varphi_2(z_n)|\to 1$ as $n\to\infty$.

From (33), Lemmas 2, 3 and 4, and using *(a)* and *(b)* we have

$$
\begin{aligned}
0<\varepsilon \leq\; & v(z_n)|u_1(z_n)f_n(\varphi_1(z_n))-u_2(z_n)f_n(\varphi_2(z_n))| \\
\leq\; & \frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}|(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}} f_n(\varphi_1(z_n)) \\
& -(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}} f_n(\varphi_2(z_n))| \\
& +v(z_n)\left|\frac{u_1(z_n)}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z_n)}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right| \\
& \quad (1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}|f_n(\varphi_2(z_n))| \\
\leq\; & C\rho(\varphi_1(z_n),\varphi_2(z_n))\frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\|f_n\|_{A^p_\alpha} \\
& +v(z_n)\left|\frac{u_1(z_n)}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z_n)}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right|\|f_n\|_{A^p_\alpha}\to 0,
\end{aligned}
$$

as $n\to\infty$, which is a contradiction. The proof is complete. ■

From Theorem 1 with $u_1(z)=u_2(z)\equiv 1$, we obtain the following corollary.

Corollary 1. *Assume $p>0$, $\alpha>-1$, v is a weight on $\mathbb{B}$, φ_1, φ_2 are analytic self-maps of $\mathbb{B}$ and C_{φ_1}, $C_{\varphi_2}: A^p_\alpha\to H^\infty_v$ are bounded. Then the operator $C_{\varphi_1}-C_{\varphi_2}: A^p_\alpha\to H^\infty_v$ is metrically compact if and only if*

$$
(a)\qquad \lim_{|\varphi_1(z)|\to 1}\frac{v(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z),\varphi_2(z))=0;
$$

$$
(b)\qquad \lim_{|\varphi_2(z)|\to 1}\frac{v(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z),\varphi_2(z))=0;
$$

(c)
$$\lim_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}\to 1} v(z)\left|\frac{1}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{1}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right| = 0.$$

Now we give an estimate of the essential norm $\|W_{\varphi_1,u_1} - W_{\varphi_2,u_2}\|_{e,A^p_\alpha\to H^\infty_v}$ for the case $p > 1$.

Theorem 2. *Assume* $p > 1$, $\alpha > -1$, v *is a weight on* $\mathbb{B}$, φ_1, φ_2 *are holomorphic self-maps of* $\mathbb{B}$ *and* u_1, u_2 *are holomorphic functions on* $\mathbb{B}$. *If* W_{φ_1,u_1}; $W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ *are bounded operators, then the essential norm* $\|W_{\varphi_1,u_1} - W_{\varphi_2,u_2}\|_{e,A^p_\alpha\to H^\infty_v}$ *is equivalent to the maximum of the following expressions:*

(i)
$$\limsup_{|\varphi_1(z)|\to 1} \frac{v(z)|u_1(z)|}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\,\rho(\varphi_1(z),\varphi_2(z));$$

(ii)
$$\limsup_{|\varphi_2(z)|\to 1} \frac{v(z)|u_2(z)|}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\,\rho(\varphi_1(z),\varphi_2(z));$$

(iii)
$$\limsup_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}\to 1} v(z)\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|.$$

Proof. First we show that the maximum of the expressions in (i)-(iii) is a lower bound for the essential norm. If $\|\varphi_1\|_\infty < 1$ then the expression in (i) is obviously a lower bound. Hence assume $\|\varphi_1\|_\infty = 1$. Find a sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{B}$ such that $|\varphi_1(z_n)| \to 1$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{v(z_n)|u_1(z_n)|\rho(\varphi_1(z_n),\varphi_2(z_n))}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}} = \limsup_{|\varphi_1(z)|\to 1} \frac{v(z)|u_1(z)|\rho(\varphi_1(z),\varphi_2(z))}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}.$$

Since $|\varphi_1(z_n)| \to 1$ as $n \to \infty$, as in Theorem 1 we can find functions $f_n \in H^\infty(\mathbb{B})$, $n \in \mathbb{N}$, satisfying (22) and (23). Let k_n and g_n be the sequences as in Theorem 1. By Problem 2.25 in [39], $g_n \to 0$ weakly in A^p_α as $n \to \infty$ (here we use the condition $p > 1$). Then for each compact operator $K : A^p_\alpha \to H^\infty_v$ we have $\lim_{n\to\infty} \|Kg_n\|_{H^\infty_v} = 0$. From this and since $\sup_{n\in\mathbb{N}} \|g_n\|_{A^p_\alpha} \le 1$, for each $n \in \mathbb{N}$, we obtain

$$\|W_{\varphi_1,u_1} - W_{\varphi_2,u_2} - K\|_{A^p_\alpha\to H^\infty_v} \ge \|(W_{\varphi_1,u_1} - W_{\varphi_2,u_2})g_n\|_{H^\infty_v} - \|Kg_n\|_{H^\infty_v}.$$

Hence

$$\begin{aligned}\|W_{\varphi_1,u_1}-W_{\varphi_2,u_2}-K\|_{A^p_\alpha\to H^\infty_v} &\ge \limsup_{n\to\infty}\|(W_{\varphi_1,u_1}-W_{\varphi_2,u_2})g_n\|_{H^\infty_v}\\ &\ge \limsup_{n\to\infty} v(z_n)|u_1(z_n)g_n(\varphi_1(z_n))-u_2(z_n)g_n(\varphi_2(z_n))|\\ &= \limsup_{n\to\infty} v(z_n)|u_1(z_n)f_n(\varphi_1(z_n))k_n(\varphi_1(z_n))|\,|\sigma_{\varphi_2(z_n)}(\varphi_1(z_n))|\\ &= \limsup_{n\to\infty}\frac{v(z_n)|u_1(z_n)|}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z_n),\varphi_2(z_n)),\end{aligned}$$

from which it follows that expression *(i)* is a lower bound for the essential norm.

That the expression in *(ii)* is a lower bound is proved similarly, so we omit it.

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence with $\min\{|\varphi_1(z_n)|,|\varphi_2(z_n)|\}\to 1$ as $n\to\infty$ and

$$\begin{aligned}&\lim_{n\to\infty} v(z_n)\left|\frac{u_1(z_n)}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z_n)}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right|\\ &=\lim_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}\to 1} v(z)\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|.\end{aligned}\tag{36}$$

If such a sequence does not exist then the estimate vacuously holds. We may also assume that the limit $\lim_{n\to\infty}\rho(\varphi_1(z_n),\varphi_2(z_n))$ exists and that is equal, say to l_1. If $l_1>0$ when $\min\{|\varphi_1(z_n)|,|\varphi_2(z_n)|\}\to 1$ as $n\to\infty$, then *(iii)* follows from *(i)* and *(ii)*. Thus we can assume that $l_1=0$ when $\min\{|\varphi_1(z_n)|,|\varphi_2(z_n)|\}\to 1$ as $n\to\infty$. Let $(f_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ be as in Theorem 1 and $h_n(z)=f_n(z)k_n(z)$, $n\in\mathbb{N}$. Then by Problem 2.25 in [39] we have $h_n\to 0$ weakly in A^p_α as $n\to\infty$.

Thus by using Lemma 4 and the fact $\sup_{n\in\mathbb{N}}\|h_n\|_{A^p_\alpha}\le 1$ it follows that

$$\begin{aligned}\|W_{\varphi_1,u_1}-W_{\varphi_2,u_2}-K\|_{A^p_\alpha\to H^\infty_v} &\ge \limsup_{n\to\infty}\|(W_{\varphi_1,u_1}-W_{\varphi_2,u_2})h_n\|_{H^\infty_v}\\ &\ge \limsup_{n\to\infty} v(z_n)|u_1(z_n)h_n(\varphi_1(z_n))-u_2(z_n)h_n(\varphi_2(z_n))|\\ &= \limsup_{n\to\infty} v(z_n)|u_1(z_n)f_n(\varphi_1(z_n))k_n(\varphi_1(z_n))-u_2(z_n)f_n(\varphi_2(z_n))k_n(\varphi_2(z_n))|\\ &\ge \limsup_{n\to\infty} v(z_n)\left|u_1(z_n)\frac{f_n(\varphi_1(z_n))}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}-u_2(z_n)\frac{f_n(\varphi_1(z_n))}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right|\\ &\quad-\limsup_{n\to\infty} v(z_n)\left|u_2(z_n)\frac{f_n(\varphi_1(z_n))}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}-u_2(z_n)f_n(\varphi_2(z_n))k_n(\varphi_2(z_n))\right|\\ &\ge \limsup_{n\to\infty} v(z_n)\left|\frac{u_1(z_n)}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z_n)}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right|\left(1-\frac{1}{2^n}\right)\end{aligned}$$

$$
\begin{aligned}
&- C\limsup_{n\to\infty}\frac{v(z)|u_2(z)|}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z_n),\varphi_2(z_n))\\
&=\limsup_{n\to\infty}v(z_n)\left|\frac{u_1(z_n)}{(1-|\varphi_1(z_n)|^2)^{\frac{N+\alpha+1}{p}}}-\frac{u_2(z_n)}{(1-|\varphi_2(z_n)|^2)^{\frac{N+\alpha+1}{p}}}\right|.
\end{aligned}
$$

Hence expression *(iii)* is also a lower bound for the essential norm, as claimed.

Now we prove the upper estimates. Consider the operators on $H(\mathbb{B})$ defined by

$$
P_k(f)(z)=f\left(\frac{k}{k+1}z\right),\quad k\in\mathbb{N}.
$$

It is easy to see that they are continuous on the compact open topology and that $P_k(f)\to f$ on compacts of $\mathbb{B}$ as $k\to\infty$. Since the integral means

$$
M_p(f,r)=\left(\int_{\mathbb{S}}|f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p}
$$

are nondecreasing in r, then by the polar coordinates it follows that $\|P_k(f)\|_{A^p_\alpha}\le\|f\|_{A^p_\alpha}$, $k\in\mathbb{N}$, which implies $\sup_{k\in\mathbb{N}}\|P_k\|_{A^p_\alpha\to A^p_\alpha}\le 1$. It is also easy to see that the operators $(P_k)_{k\in\mathbb{N}}$ are also compact on A^p_α.

Let $r\in(0,1)$ be fixed and $f\in A^p_\alpha$ such that $\|f\|_{A^p_\alpha}\le 1$. Set

$$
g_k:=(I-P_k)f,\quad k\in\mathbb{N}.
$$

Then clearly $g_k\in A^p_\alpha$, $k\in\mathbb{N}$ and $\sup_{k\in\mathbb{N}}\|g_k\|_{A^p_\alpha}\le 2$.

We have

$$
\begin{aligned}
&\|W_{\varphi_1,u_1}-W_{\varphi_2,u_2}\|_{e,A^p_\alpha\to H^\infty_v}\\
&\le\sup_{\|f\|_{A^p_\alpha}\le 1}\|(W_{\varphi_1,u_1}-W_{\varphi_2,u_2})g_k\|_{H^\infty_v}\\
&\le\sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{|\varphi_1(z)|>r}v(z)|u_1(z)g_k(\varphi_1(z))-u_2(z)g_k(\varphi_2(z))|\\
&\quad+\sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{|\varphi_2(z)|>r}v(z)|u_1(z)g_k(\varphi_1(z))-u_2(z)g_k(\varphi_2(z))|\\
&\quad+\sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{\max\{|\varphi_1(z)|,|\varphi_2(z)|\}\le r}v(z)|u_1(z)g_k(\varphi_1(z))-u_2(z)g_k(\varphi_2(z))|\\
&\quad=I_{k,1}(r)+I_{k,2}(r)+I_{k,3}(r).
\end{aligned}
$$

First we estimate $I_{k,1}(r)$. Lemmas 4 and 2 and the fact $\sup_{k\in\mathbb{N}}\|g_k\|_{A^p_\alpha}\le 2$, yield

$$
\begin{aligned}
&v(z)|u_1(z)g_k(\varphi_1(z)) - u_2(z)g_k(\varphi_2(z))| \\
&\le \frac{v(z)|u_1(z)|}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\Big|(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}g_k(\varphi_1(z)) \\
&\quad -(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}g_k(\varphi_2(z))\Big| \\
&\quad +v(z)\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z)}{(1-|\varphi_2(z)^2)^{\frac{N+\alpha+1}{p}}}\right| \\
&\quad (1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}|g_k(\varphi_2(z))| \\
&\le 2C\frac{v(z)|u_1(z)|}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z),\varphi_2(z)) \\
&\quad +2v(z)\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z)}{(1-|\varphi_2(z)^2)^{\frac{N+\alpha+1}{p}}}\right|.
\end{aligned}
\tag{37}
$$

An analogous estimate is obtained for $I_{k,2}(r)$.

It is clear that for every $h \in H(\mathbb{B})$, $\lim_{k\to\infty}(I-P_k)h = 0$ and that the space $H(\mathbb{B})$ endowed with compact open topology is a Fréchet space. Hence, by the Banach-Steinhaus theorem, $(I-P_k)h$ converges to zero uniformly on compacts of $(H(\mathbb{B}), co)$. Since the unit ball of A^p_α is a compact subset of $(H(\mathbb{B}), co)$ we have

$$
\lim_{k\to\infty}\sup_{\|h\|_{A^p_\alpha}\le 1}\sup_{|\zeta|\le r}|(I-P_k)(h)(\zeta)| = 0. \tag{38}
$$

The boundedness of $W_{\varphi_1,u_1}; W_{\varphi_2,u_2} : A^p_\alpha \to H^\infty_v$ implies $u_1, u_2 \in H^\infty_v$. From this, since v is a weight, and (37) we get that for each $r \in (0,1)$ and for $|\varphi_2(z)| \le r$

$$
\limsup_{k\to\infty} I_{k,1}(r) \le 2C \sup_{|\varphi_1(z)|>r}\frac{v(z)|u_1(z)|}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z),\varphi_2(z)).
$$

If $|\varphi_2(z)| > r$, then we have

$$
\begin{aligned}
&\limsup_{k\to\infty} I_{k,1}(r) \\
&\le 2C \sup_{|\varphi_1(z)|>r}\frac{v(z)|u_1(z)|}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}}\rho(\varphi_1(z),\varphi_2(z)) \\
&\quad + 2\sup_{\min\{|\varphi_1(z)|,|\varphi_2(z)|\}>r} v(z)\left|\frac{u_1(z)}{(1-|\varphi_1(z)|^2)^{\frac{N+\alpha+1}{p}}} - \frac{u_2(z)}{(1-|\varphi_2(z)|^2)^{\frac{N+\alpha+1}{p}}}\right|.
\end{aligned}
$$

Letting $r \to 1$ in the last two inequalities an estimate for $\limsup_{r\to 1}\limsup_{k\to\infty} I_{k,1}(r)$ in terms of *(i)* and *(iii)* is obtained.

The quantity $\limsup_{r\to 1}\limsup_{k\to\infty} I_{k,2}(r)$ is estimated similarly.

Since $u_1, u_2 \in H_v^\infty$, v is a weight and (38), it follows that $\lim_{k\to\infty} I_{k,3}(r) = 0$. The upper estimate follows from these facts, finishing the proof of the theorem. ■

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References

1. K. D. Bierstedt and W. H. Summers, Biduals of weighted Banach spaces of analytic functions, *J. Austral. Math. Soc.* (*Series A*), **54** (1993), 70-79.
2. K. D. Bierstedt, J. Bonet and J. Taskinen, Associated weights and spaces of holomorphic functions, *Studia Math.*, **127** (1998), 70-79.
3. J. Bonet, P. Domanski and M. Lindström, Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions, *Canad. Math. Bull.*, **42(2)**, (1999), 139-148.
4. J. Bonet, M. Lindström and E. Wolf, Differences of composition operators between weighted Banach spaces of holomorphic functions, *J. Austral. Math. Soc.*, **84** (2008), 9-20.
5. D. Clahane and S. Stević, Norm equivalence and composition operators between Bloch/Lipschitz spaces of the unit ball, *J. Inequal. Appl.*, Vol. 2006, Article ID 61018, (2006), 11 pages.
6. M. D. Contreras and A. G. Hernández-Diaz, Weighted composition operators in weighted Banach spaces of analytic functions, *J. Austral. Math. Soc.* (*Serie A*), **69** (2000), 41-60.
7. C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, 1995.
8. T. Hosokawa and K. Izuchi, Essential norms of differences of composition operators on H^∞, *J. Math. Japan*, **57** (2005), 669-690.
9. T. Hosokawa, K. Izuchi and D. Zheng, Isolated points and essential components of composition operators on H^∞, *Proc. Amer. Math. Soc.*, **130** (2002), 1765-1773.
10. T. Hosokawa and S. Ohno, Differences of composition operators on the Bloch spaces, *J. Operator Theory*, **57** (2007), 229-242.
11. S. Li and S. Stević, Weighted composition operators from α-Bloch space to H^∞ on the polydisk, *Numer. Funct. Anal. Optimization*, **28(7)** (2007), 911-925.
12. S. Li and S. Stević, Weighted composition operators between H^∞ and α-Bloch spaces in the unit ball, *Taiwanese J. Math.*, **12** (2008), 1625-1639.

13. S. Li and S. Stević, Weighted composition operators from Zygmund spaces into Bloch spaces, *Appl. Math. Comput.*, **206(2)** (2008), 825-831.

14. M. Lindström and E. Wolf, Essential norm of the difference of weighted composition operators, *Monatsh. Math.*, **153** (2008), 133-143.

15. W. Lusky, On weighted spaces of harmonic and holomorphic functions, *J. London Math. Soc.*, **51** (1995), 309-320.

16. B. D. MacCluer, S. Ohno and R. Zhao, Topological structure of the space of composition operators on H^∞, *Integral Equations Operator Theory*, **40** (2001), 481-494.

17. A. Montes-Rodriguez, Weighted composition operators on weighted Banach spaces of analytic functions. *J. London Math. Soc.*, **61(3)** (2000), 872-884.

18. P. J. Nieminen, Compact differences of composition operators on Bloch and Lipschitz spaces, *Comput. Methods Funct. Theory*, **7(2)** (2007), 325-344.

19. S. Stević, Composition operators between H^∞ and the α-Bloch spaces on the polydisc, *Z. Anal. Anwend.*, **25** (2006), 457-466.

20. S. Stević, Weighted composition operators between mixed norm spaces and H_α^∞ spaces in the unit ball, *J. Inequal. Appl.*, Vol. 2007, Article ID 28629, 2007, 9 pages.

21. S. Stević, Generalized composition operators from logarithmic Bloch spaces to mixed-norm spaces, *Util. Math.*, **77** (2008), 167-172.

22. S. Stević, Norm of weighted composition operators from Bloch space to H_μ^∞ on the unit ball, *Ars. Combin.*, **88** (2008), 125-127.

23. S. Stević, Norms of some operators from Bergman spaces to weighted and Bloch-type space, *Util. Math.*, **76** (2008), 59-64.

24. S. Stević, On a new operator from H^∞ to the Bloch-type space on the unit ball, *Util. Math.*, **77** (2008), 257-263.

25. S. Stević, Essential norm of an operator from the weighted Hilbert-Bergman space to the Bloch-type space, *Ars Combin.*, **91** (2009), 123-127.

26. S. Stević, Essential norms of weighted composition operators from the Bergman space to weighted-type spaces on the unit ball, *Ars. Combin.*, **91** (2009), 391-400.

27. S. Stević, Integral-type operators from the mixed-norm space to the Bloch-type space on the unit ball, *Siberian Math. J.*, **50(6)** (2009), 1098-1105.

28. S. Stević, Norm and essential norm of composition followed by differentiation from α-Bloch spaces to H_μ^∞, *Appl. Math. Comput.*, **207** (2009), 225-229.

29. S. Stević, Norm of weighted composition operators from α-Bloch spaces to weighted-type spaces, *Appl. Math. Comput.*, **215** (2009), 818-820.

30. S. Stević, Norms of some operators on the Bergman and the Hardy space in the unit polydisk and the unit ball, *Appl. Math. Comput.*, **215** (6) (2009), 2199-2205.

31. S. Stević, On a new integral-type operator from the Bloch space to Bloch-type spaces on the unit ball, *J. Math. Anal. Appl.*, **354** (2009), 426-434.

32. S. Stević, Products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces, *Siberian Math. J.*, **50(4)** (2009), 726-736.

33. S. Stević, Weighted composition operators from weighted Bergman spaces to weighted-type spaces on the unit ball, *Appl. Math. Comput.*, **212** (2009), 499-504.

34. S. Stević, Weighted differentiation composition operators from H^∞ and Bloch spaces to nth weigthed-type spaces on the unit disk, *Appl. Math. Comput.*, **216** (2010), 3634-3641.

35. S. Stević and E. Wolf, Differences of composition operators between weighted-type spaces of holomorphic functions on the unit ball of $\mathbb{C}^n$, *Appl. Math. Comput.*, **215(5)** (2009), 1752-1760.

36. S. I. Ueki, Weighted composition operator on the Fock space, *Proc. Amer. Math. Soc.*, **135** (2007), 1405-1410.

37. S. I. Ueki and L. Luo, Essential norms of weighted composition operators between weighted Bergman spaces of the ball, *Acta Sci. Math.* (*Szeged*), **74** (2008), 829-843.

38. W. Yang, Weighted composition operators from Bloch-type spaces to weighted-type spaces, *Ars Combin.*, **93** (2009), 265-274.

39. K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer, New York, 2005.

40. X. Zhu, Weighted composition operators from logarithmic Bloch spaces to a class of weighted-type spaces in the unit ball, *Ars. Combin.*, **91** (2009), 87-95.

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