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HYBRID VISCOSITY ITERATIVE APPROXIMATION OF ZEROS OF M-ACCRETIVE OPERATORS IN BANACH SPACES

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Abstract. In this paper, let X be a reflexive Banach space which either is uniformly smooth or has a weakly continuous duality map. We prove, under the convergence of no parameter sequences to zero, the strong convergence of their iterative scheme to a zero of m-accretive operator A in X, which solves a variational inequality on the set $A^{-1}(0)$ of zeros of A. Such a result includes their main result as a special case. Furthermore, we also give a weak convergence theorem for hybrid viscosity iterative approximation method involving a maximal monotone operator in a Hilbert space.

1. INTRODUCTION

Let X be a real Banach space with the dual space X^*. The normalized duality mapping J from X into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of X^* is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between X and X^*. It is known that the norm of X is said to be Gateaux differentiable (and X is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

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exists for every x, y in $U = \{x \in X : \|x\| = 1\}$ the unit sphere of X. The norm of X is said to be uniformly Frechet differentiable (and X is said to be uniformly smooth) if the limit in (1.1) is attained uniformly for $(x, y) \in U \times U$. Every uniformly smooth Banach space X is reflexive and smooth.

Recall that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in X weakly converging to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak*ly to $J_\varphi(x)$).

Let C be a nonempty closed convex subset of a real Banach space X, and $T : C \to C$ be a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $\mathrm{Fix}(T)$ the set of fixed points of T, i.e., $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that T is a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

As in [3], we use the notation Π_C to denote the collection of all contractions on C, i.e.,

$$\Pi_C = \{f : C \to C \text{ a contraction}\}.$$

Note that each $f \in \Pi_C$ has a unique fixed point in C.

Recall also that an operator A (possibly multivalued) with domain $D(A)$ and range $R(A)$ in X is called accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i$ for $i = 1, 2$, there exists a $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1)\rangle \geq 0,$$

where $J : X \to 2^{X^*}$ is the normalized duality mapping. An accretive operator A is called m-accretive if $R(I + rA) = X$ for each $r > 0$. Throughout the paper, we assume that A is m-accretive and has a zero (that is, the inclusion $0 \in Az$ is solvable). The set of zeros of A is denoted by F. For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then J_r is a nonexpansive mapping from X to $C := \overline{D(A)}$ which is assumed convex, and $\mathrm{Fix}(J_r) = F$ for all $r > 0$. We also denote by A_r the Yosida approximation of A, i.e., $A_r = \frac{1}{r}(I - J_r)$.

Recently, Kim and Xu [15] and Xu [9] studied the sequence generated by the iterative scheme

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)J_{r_n}x_n, \quad \forall n \geq 0, \tag{1.2}$$

and proved strong convergence of the iterative scheme (1.2) in the framework of uniformly smooth Banach spaces and a reflexive Banach space which has a weakly continuous duality map, respectively.

Inspired by the iterative scheme (1.2), Qin and Su [4] introduced the following iterative scheme:

$$(1.3) \qquad \begin{cases} y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \end{cases}$$

where $u \in C$ is an arbitrary (but fixed) element of C and $\{\alpha_n\}$ in $(0,1)$, $\{\beta_n\}$ in $[0,1]$. If $\beta_n = 0$, then (1.3) reduces to (1.2). Subsequently, Ceng, Khan, Ansari and Yao [12] proposed and analyzed a variant of the iterative scheme (1.3). They proved, under the same assumptions on the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$ as in Theorem 1.1 in Qin and Su [4], that $\{x_n\}$ defined by such a variant converges strongly to a zero of an m-accretive operator A.

In 2009, Chen, Liu and Shen [13] suggested and analyzed an iterative scheme for viscosity approximation of a zero of an m-accretive operator in a reflexive Banach space which has a weakly continuous duality map.

Theorem CLS. (see [13, Theorem 2.1]). *Let X be a real reflexive Banach space and has a weakly continuous duality map J_φ with gauge φ and A be an m-accretive operator in X such that $C = \overline{D(A)}$ is convex with $F = A^{-1}0 \neq \emptyset$, and $f : C \to C$ be a fixed contraction mapping, $\{\alpha_n\}_{n=0}^\infty$ in $(0,1)$ and $\{\beta_n\}_{n=0}^\infty$ in $[0,1]$, suppose $\{\alpha_n\}_{n=0}^\infty, \{\beta_n\}_{n=0}^\infty$ and $\{r_n\}_{n=0}^\infty$ satisfy the following conditions:*

(*i*) $\sum_{n=0}^\infty \beta_n = \infty$ *and* $\beta_n \to 0$ $(n \to \infty)$*;*

(*ii*) $\beta_n \in [0,a)$ *for some* $a \in (0,1)$ *and* $r_n \geq \epsilon$ *for all* n*;*

(*iii*) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty,\ \sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$*.*

Let $\{x_n\}_{n=1}^\infty$ be the composite process defined by

$$(1.4) \qquad \begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n, \quad \forall n \geq 0. \end{cases}$$

Then $\{x_n\}_{n=0}^\infty$ converges strongly to a zero of A.

On the other hand, Yao, Chen and Yao [16] combined the viscosity approximation method [17] and the modified Mann iteration [15], and gave the following hybrid viscosity approximation method:

Let C be a nonempty closed convex subset of a Banach space X, $T : C \to C$ a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$ and $f \in \Pi_C$. For any arbitrary $x_0 \in C$, define $\{x_n\}$ in the following way:

$$(1.5) \qquad \begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n, \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$. They proved under certain different control conditions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ that $\{x_n\}$ converges strongly to a fixed point of T. Their result is the extension and improvement of the main result in Kim and Xu [15].

Very recently, under the convergence of no parameter sequences to zero, Ceng and Yao [14] proved the strong convergence of the sequence $\{x_n\}$ generated by (1.5) to a fixed point of T, which solves a variational inequality on $\mathrm{Fix}(T)$.

Theorem CY (See [14, Theorem 3.1]). Let C be a nonempty closed convex subset of a uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$ and $f \in \Pi_C$ with contractive constant $\alpha \in (0,1)$. Given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$, the following control conditions are satisfied:

(C1) $0 \leq \beta_n \leq 1-\alpha, \forall n \geq n_0$ for some integer $n_0 \geq 1$;

(C2) $\sum_{n=0}^{\infty} \beta_n = \infty$;

(C3) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(C4) $\lim_{n\to\infty}(\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}) = 0$.

For an arbitrary $x_0 \in C$, let $\{x_n\}$ be defined by (1.5). Then,

$$x_n \to Q(f) \iff \beta_n(f(x_n) - x_n) \to 0,$$

where $Q(f) \in \mathrm{Fix}(T)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in \mathrm{Fix}(T).$$

In this paper, let X be a real reflexive Banach space which, either is uniformly smooth or has a weakly continuous duality map J_φ. Assume that A is an m-accretive operator in X. Combining Theorem CLS with Theorem CY as above, we prove, under the convergence of no parameter sequences to zero, the strong convergence of the sequence $\{x_n\}$ generated by (1.4) to a zero of A, which solves a variational inequality on $A^{-1}0$. Such a result includes Theorem CLS as a special case. Furthermore, we also give a weak convergence theorem for hybrid viscosity iterative approximation method (1.4) involving a maximal monotone operator in a Hilbert space H. The results presented in this paper can be viewed as the supplement, improvement and extension of some known results in the literature, e.g., [4, 9-18].

2. Preliminaries

Let X be a real Banach space with the topological dual space X^* and $\langle x, x^*\rangle$ be the pairing between $x \in X$ and $x^* \in X^*$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. Let 2^{X^*} denote the family of all subsets of X^*. Recall that the normalized duality mapping $J : X \to 2^{X^*}$ is defined as

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X.$$

In order to establish new strong and weak convergence theorems for hybrid viscosity iterative approximation of zeros of m-accretive operators, we need the following lemmas. The first lemma is a very well-known (subdifferential) inequality; see, e.g., [1].

Lemma 2.1. ([1]). *Let X be a real Banach space and J the normalized duality map on X. Then for any given $x, y \in X$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.2. ([8, Lemma 2]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.3. ([5]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$s_{n+1} \leq (1 - \mu_n)s_n + \mu_n\nu_n, \quad \forall n \geq 1,$$

where $\{\mu_n\}, \{\nu_n\}$ are sequences of real numbers such that

(i) $\{\mu_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \mu_n = \infty$, or equivalently,

$$\prod_{n=1}^{\infty}(1 - \mu_n) := \lim_{n\to\infty} \prod_{k=1}^{n}(1 - \mu_k) = 0;$$

(ii) $\limsup_{n\to\infty} \nu_n \leq 0$, or

$(ii)'$ $\sum_{n=1}^{\infty} \mu_n\nu_n$ is convergent.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4. ([17, Theorem 4.1]). *Let X be a uniformly smooth Banach space, C be a nonempty closed convex subset of X, $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f \in \Pi_C$. Then $\{x_t\}$ defined by*

$$x_t = tf(x_t) + (1 - t)Tx_t$$

converges strongly to a point in $\mathrm{Fix}(T)$. If we define $Q : \Pi_C \to \mathrm{Fix}(T)$ by

$$Q(f) := \lim_{t\to 0} x_t, \tag{2.1}$$

then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad \forall p \in \mathrm{Fix}(T). \tag{2.2}$$

In particular, if $f = u \in C$ is a constant, then (2.1) is reduced to the sunny nonexpansive retraction of Reich from C onto $\mathrm{Fix}(T)$*,*

$$\langle Q(u)-u, J(Q(u)-p)\rangle \le 0, \quad \forall p \in \mathrm{Fix}(T). \tag{2.3}$$

Recall that an operator A with domain $D(A)$ and range $R(A)$ in X is said to be accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there is a $j(x_2-x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1)\rangle \ge 0.$$

An accretive operator A is m-accretive if $R(I + \lambda A) = X$ for all $\lambda > 0$.

Denote by F the set of zeros of A, i.e.,

$$F := A^{-1}0 = \{x \in D(A) : 0 \in Ax\}.$$

Throughout the rest of this paper it is always assumed that A is m-accretive and F is nonempty. For each $r > 0$, we denote by J_r the resolvent of A, i.e., $J_r = (I + rA)^{-1}$. Note that if A is m-accretive, then $J_r : X \to X$ is nonexpansive and $\mathrm{Fix}(J_r) = A^{-1}0$ for all $r > 0$. Indeed, observe that

$$\begin{aligned} x \in A^{-1}0 &\Leftrightarrow 0 \in Ax \\ &\Leftrightarrow x \in (I + rA)x \\ &\Leftrightarrow x = (I + rA)^{-1}x \\ &\Leftrightarrow x = J_r x \\ &\Leftrightarrow x \in \mathrm{Fix}(J_r). \end{aligned}$$

We also denote by A_r the Yosida approximation of A, i.e., $A_r = \frac{1}{r}(I - J_r)$. It is known that J_r is a nonexpansive mapping from X to $C := \overline{D(A)}$, which will be assumed convex.

Lemma 2.5. (The Resolvent Identity [7]). *For each $\lambda, \mu > 0$ and each $x \in X$,*

$$J_\lambda x = J_\mu\Big(\frac{\mu}{\lambda}x + (1 - \frac{\mu}{\lambda})J_\lambda x\Big).$$

Recall that a gauge is a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : X \to 2^{X^*}$ defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|),\ \|x^*\| = \varphi(\|x\|)\}, \quad \forall x \in X.$$

Following Browder [2], we say that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak*ly to $J_\varphi(x)$). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall t \geq 0.$$

Then

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad \forall x \in X,$$

where ∂ denotes the subdifferential in the sense of convex analysis. The first part of the following lemma is an immediate consequence of the subdifferential inequality, and the proof of the second part can be found in [19].

Lemma 2.6. *Assume that X has a weakly continuous duality map J_φ with gauge φ.*

(i) *For all $x, y \in X$, there holds the inequality*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

(ii) *Assume a sequence $\{x_n\}$ in X is weakly convergent to a point x. Then there holds the identity*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y-x\|), \quad \forall y \in X.$$

Lemma 2.7. ([18, Theorem 3.1 and its proof]). *Let X be a reflexive Banach space and have a weakly continuous duality map J_φ with gauge φ, C be a nonempty closed convex subset of X, $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$, *and $f \in \Pi_C$. Then $\{x_t\}$ defined by*

$$x_t = tf(x_t) + (1-t)Tx_t$$

converges strongly to a point in $\mathrm{Fix}(T)$. *If we define $Q : \Pi_C \to \mathrm{Fix}(T)$ by*

$$Q(f) := \lim_{t\to 0} x_t, \tag{2.4}$$

then $Q(f)$ solves the variational inequality

$$\langle (I-f)Q(f), J_\varphi(Q(f)-p)\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{2.5}$$

In particular, if $f = u \in C$ is a constant, then (2.4) is reduced to the sunny nonexpansive retraction of Reich-type from C onto $\mathrm{Fix}(T)$,

$$\langle Q(u)-u, J_\varphi(Q(u)-p)\rangle \leq 0, \quad \forall p \in \mathrm{Fix}(T). \tag{2.6}$$

Recall that X satisfies Opial's property [20] provided, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \neq x.$$

It is known [20] that each l^p $(1 \leq p < \infty)$ enjoys this property, while L^p does not unless $p = 2$. It is known [21] that any separable Banach space can be equivalently renormed so that it satisfies Opial's property. We denote by $\omega_w(x_n)$ the weak ω-limit set of $\{x_n\}$, i.e.,

$$\omega_w(x_n) = \{\bar{x} \in X : x_{n_i} \rightharpoonup \bar{x} \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Finally, recall that in a Hilbert space, there holds the following equality

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x, y \in H$ and $\lambda \in [0,1]$; see Takahashi [22].

We also use the following elementary lemmas.

Lemma 2.8. ([23]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=0}^{\infty} b_n < \infty$ and $a_{n+1} \leq a_n + b_n$ for all $n \geq 0$. Then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.9. ([3]). *Demiclosedness Principle. Assume that T is a nonexpansive self-mapping of a nonempty closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.*

3. Main Results

We now state and prove the main results of this paper.

Theorem 3.1. *Let X be a reflexive Banach space. Assume, in addition, X either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ. Let A be an m-accretive operator in X such that $C = \overline{D(A)}$ is convex with $F := A^{-1}0 \neq \emptyset$, and $f \in \Pi_C$ with contractive constant $\alpha \in (0,1)$. Given sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$, the following control conditions are satisfied:*

(C1) $0 \leq \beta_n \leq 1-\alpha, \forall n \geq n_0$ for some integer $n_0 \geq 1$;

(C2) $\sum_{n=0}^{\infty} \beta_n = \infty$;

(C3) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(C4) $\lim_{n\to\infty}(\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}) = 0$;

(C5) $r_n \geq \epsilon, \forall n \geq 0$ for some $\epsilon > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

For an arbitrary $x_0 \in C$, let $\{x_n\}$ be generated by

$$(3.1) \qquad \begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) J_{r_n} x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n, \quad \forall n \geq 0. \end{cases}$$

Then,

$$x_n \to Q(f) \iff \beta_n(f(x_n) - x_n) \to 0,$$

where one of the following two statements holds:

(i) if X is uniformly smooth, then $Q(f) \in F$ solves the variational inequality

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in F;$$

(ii) if X has a weakly continuous duality map J_φ with gauge φ, then $Q(f) \in F$ solves the variational inequality

$$\langle (I-f)Q(f), J_\varphi(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in F.$$

Proof. First, let us show that $\{x_n\}$ is bounded. Indeed, taking an element $p \in F = A^{-1}0$ arbitrarily, we obtain that $p = J_{r_n} p$ and

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n \|f(x_n) - p\| + (1-\beta_n)\|y_n = p\| \\ &\leq \beta_n \|f(x_n) - f(p)\| + \beta_n \|f(p) - p\| \\ &\quad + (1-\beta_n)[\alpha_n \|x_n - p\| + (1-\alpha_n)\|J_{r_n} x_n - J_{r_n} p\|] \\ &\leq [\alpha\beta_n + \alpha_n(1-\beta_n) + (1-\beta_n)(1-\alpha_n)]\|x_n - p\| + \beta_n\|f(p) - p\| \\ &= [1 - (1-\alpha)\beta_n]\|x_n - p\| + \beta_n \|f(p) - p\| \\ &\leq \max\{\|x_n - p\|, \frac{\|f(p)-p\|}{1-\alpha}\}. \end{aligned}$$

By induction, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{\|f(p)-p\|}{1-\alpha}\}.$$

Hence $\{x_n\}$ is bounded, and so are the sequences $\{J_{r_n} x_n\}, \{y_n\}$ and $\{f(x_n)\}$.

Suppose that $x_n \to Q(f) \in F$ $(n \to \infty)$. Then $Q(f) = J_{r_n} Q(f)$ for all $n \geq 0$. From (3.1) it follows that

$$\begin{aligned} \|y_n - Q(f)\| &= \|\alpha_n(x_n - Q(f)) + (1-\alpha_n)(J_{r_n} x_n - Q(f))\| \\ &\leq \alpha_n \|x_n - Q(f)\| + (1-\alpha_n)\|J_{r_n} x_n - Q(f)\| \\ &\leq \alpha_n \|x_n - Q(f)\| + (1-\alpha_n)\|x_n - Q(f)\| \\ &= \|x_n - Q(f)\| \to 0 \quad (n \to \infty), \end{aligned}$$

that is, $y_n \to Q(f)$. Again from (3.1) we obtain that

$$\begin{aligned}\|\beta_n(f(x_n)-x_n)\| &= \|x_{n+1}-x_n-(1-\beta_n)(y_n-x_n)\| \\ &\le \|x_{n+1}-x_n\|+(1-\beta_n)\|y_n-x_n\| \\ &\le \|x_{n+1}-Q(f)\|+\|x_n-Q(f)\| \\ &\quad +(1-\beta_n)(\|y_n-Q(f)\|+\|x_n-Q(f)\|) \\ &\le \|x_{n+1}-Q(f)\|+2\|x_n-Q(f)\|+\|y_n-Q(f)\|.\end{aligned}$$

Since $x_n \to Q(f)$ and $y_n \to Q(f)$, we obtain $\beta_n(f(x_n)-x_n) \to 0$.

Conversely, Suppose that $\beta_n(f(x_n)-x_n) \to 0$ $(n \to \infty)$. Put $\gamma_n = (1-\beta_n)\alpha_n, \forall n \ge 0$. Then it follows from (C1) and (C3) that

$$\alpha_n \ge \gamma_n = (1-\beta_n)\alpha_n \ge (1-(1-\alpha))\alpha_n = \alpha\alpha_n, \quad \forall n \ge n_0,$$

and hence

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1. \tag{3.2}$$

Define

$$x_{n+1} = \gamma_n x_n + (1-\gamma_n) z_n. \tag{3.3}$$

Observe that

$$\begin{aligned}& z_{n+1}-z_n \\ &= \frac{x_{n+2}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{x_{n+1}-\gamma_n x_n}{1-\gamma_n} \\ &= \frac{\beta_{n+1}f(x_{n+1})+(1-\beta_{n+1})y_{n+1}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n f(x_n)+(1-\beta_n)y_n-\gamma_n x_n}{1-\gamma_n} \\ &= (\frac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\beta_n f(x_n)}{1-\gamma_n}) - \frac{(1-\beta_n)[\alpha_n x_n + (1-\alpha_n)J_{r_n}x_n]-\gamma_n x_n}{1-\gamma_n} \\ &\quad + \frac{(1-\beta_{n+1})[\alpha_{n+1}x_{n+1}+(1-\alpha_{n+1})J_{r_{n+1}}x_{n+1}]-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}} \\ &= (\frac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\beta_n f(x_n)}{1-\gamma_n}) \\ &\quad + \frac{(1-\beta_{n+1})(1-\alpha_{n+1})J_{r_{n+1}}x_{n+1}}{1-\gamma_{n+1}} - \frac{(1-\beta_n)(1-\alpha_n)J_{r_n}x_n}{1-\gamma_n} \\ &= (\frac{\beta_{n+1}f(x_{n+1})}{1-\gamma_{n+1}} - \frac{\beta_n f(x_n)}{1-\gamma_n}) + (J_{r_{n+1}}x_{n+1} - J_{r_n}x_n) \\ &\quad - \frac{\beta_{n+1}}{1-\gamma_{n+1}} J_{r_{n+1}}x_{n+1} + \frac{\beta_n}{1-\gamma_n} J_{r_n} x_n\end{aligned}$$

$$
\begin{aligned}
&= \Big(\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\Big) f(x_{n+1}) + (f(x_{n+1}) - f(x_n))\frac{\beta_n}{1-\gamma_n} + (J_{r_{n+1}}x_{n+1} - J_{r_n}x_n) \\
&\quad -\Big(\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\Big) J_{r_{n+1}}x_{n+1} - (J_{r_{n+1}}x_{n+1} - J_{r_n}x_n)\frac{\beta_n}{1-\gamma_n} \\
&= \Big(\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\Big)(f(x_{n+1}) - J_{r_{n+1}}x_{n+1}) + (f(x_{n+1}) - f(x_n))\frac{\beta_n}{1-\gamma_n} \\
&\quad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n}(J_{r_{n+1}}x_{n+1} - J_{r_n}x_n).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\le \Big|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\Big| \|f(x_{n+1}) - J_{r_{n+1}}x_{n+1}\| + \|f(x_{n+1}) - f(x_n)\|\frac{\beta_n}{1-\gamma_n} \\
&\quad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n}\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| \\
&\le \Big|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\Big| (\|f(x_{n+1})\| + \|J_{r_{n+1}}x_{n+1}\|) + \|x_{n+1} - x_n\|\frac{\alpha\beta_n}{1-\gamma_n} \\
&\quad + \frac{1-\gamma_n-\beta_n}{1-\gamma_n}\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\|.
\end{aligned} \tag{3.4}
$$

Here, we consider two cases.

Case 1. $r_n \le r_{n+1}$. In this case, Lemma 2.5 (the resolvent identity) implies that

$$J_{r_{n+1}}x_{n+1} = J_{r_n}\Big(\frac{r_n}{r_{n+1}}x_{n+1} + \Big(1 - \frac{r_n}{r_{n+1}}\Big)J_{r_{n+1}}x_{n+1}\Big).$$

Using the nonexpansivity of J_{r_n} we get

$$
\begin{aligned}
&\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| \\
&\le \Big\|\frac{r_n}{r_{n+1}}x_{n+1} + \Big(1 - \frac{r_n}{r_{n+1}}\Big)J_{r_{n+1}}x_{n+1} - x_n\Big\| \\
&= \Big\|\frac{r_n}{r_{n+1}}(x_{n+1} - x_n) + \Big(1 - \frac{r_n}{r_{n+1}}\Big)(J_{r_{n+1}}x_{n+1} - x_n)\Big\| \\
&\le \frac{r_n}{r_{n+1}}\|x_{n+1} - x_n\| + \Big(1 - \frac{r_n}{r_{n+1}}\Big)\|J_{r_{n+1}}x_{n+1} - x_n\| \\
&\le \frac{r_n}{r_{n+1}}\|x_{n+1} - x_n\| + \Big(1 - \frac{r_n}{r_{n+1}}\Big)(\|J_{r_{n+1}}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|) \\
&= \|x_{n+1} - x_n\| + \frac{r_{n+1} - r_n}{r_{n+1}}\|J_{r_{n+1}}x_{n+1} - x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + \frac{r_{n+1} - r_n}{\epsilon}\|J_{r_{n+1}}x_{n+1} - x_{n+1}\|.
\end{aligned} \tag{3.5}
$$

Substituting (3.5) in (3.4), we have

$$\begin{aligned}
&\|z_{n+1}-z_n\| \\
&\le \left|\frac{\beta_{n+1}}{1-\gamma_{n+1}}-\frac{\beta_n}{1-\gamma_n}\right|(\|f(x_{n+1})\| \\
&\quad +\|J_{r_{n+1}}x_{n+1}\|)+\|x_{n+1}-x_n\|\frac{\alpha\beta_n}{1-\gamma_n} \\
&\quad +\frac{1-\gamma_n-\beta_n}{1-\gamma_n}[\|x_{n+1}-x_n\|+\frac{r_{n+1}-r_n}{\epsilon}\|J_{r_{n+1}}x_{n+1}-x_{n+1}\|] \\
&\le \|x_{n+1}-x_n\|+\left|\frac{\beta_{n+1}}{1-\gamma_{n+1}}-\frac{\beta_n}{1-\gamma_n}\right|(\|f(x_{n+1})\|+\|J_{r_{n+1}}x_{n+1}\|) \\
&\quad +\frac{r_{n+1}-r_n}{\epsilon}\|J_{r_{n+1}}x_{n+1}-x_{n+1}\| \\
&\le \|x_{n+1}-x_n\|+M_1\left(\left|\frac{\beta_{n+1}}{1-\gamma_{n+1}}-\frac{\beta_n}{1-\gamma_n}\right|+|r_{n+1}-r_n|\right),
\end{aligned} \tag{3.6}$$

where M_1 is a constant such that

$$M_1 \ge \max\{\|f(x_n)\|+\|J_{r_n}x_n\|, \frac{\|J_{r_n}x_n-x_n\|}{\epsilon}\}, \quad \forall n\ge 1.$$

Case 2. $r_{n+1}\le r_n$. Similarly we can derive (3.6).

Therefore, from (3.6) and conditions (C4), (C5), we conclude that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence by Lemma 2.2 we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0. \tag{3.7}$$

It follows from (3.2) and (3.3) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\gamma_n)\|z_n-x_n\|=0. \tag{3.8}$$

From (3.1), we have

$$x_{n+1}-x_n=\beta_n(f(x_n)-x_n)+(1-\beta_n)(y_n-x_n),$$

which hence implies that

$$\begin{aligned}
\alpha\|y_n-x_n\| &= (1-(1-\alpha))\|y_n-x_n\| \\
&\le (1-\beta_n)\|y_n-x_n\| \\
&= \|x_{n+1}-x_n-\beta_n(f(x_n)-x_n)\| \\
&\le \|x_{n+1}-x_n\|+\|\beta_n(f(x_n)-x_n)\|.
\end{aligned}$$

Since $x_{n+1} - x_n \to 0$ and $\beta_n(f(x_n) - x_n) \to 0$, we get

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.9}$$

Observe that

$$y_n - x_n = (1 - \alpha_n)(J_{r_n}x_n - x_n). \tag{3.10}$$

It follows from (C3), (3.9) and (3.10) that

$$\lim_{n\to\infty} \|x_n - J_{r_n}x_n\| = 0. \tag{3.11a}$$

Since Lemma 2.5 (the resolvent identity) implies that

$$J_{r_n}x_n = J_\epsilon(\frac{\epsilon}{r_n}x_n + (1 - \frac{\epsilon}{r_n})J_{r_n}x_n),$$

it follows from (3.11a) and the nonexpansivity of J_ϵ that

$$\begin{aligned} \|J_\epsilon x_n - x_n\| &\leq \|J_{r_n}x_n - J_\epsilon x_n\| + \|x_n - J_{r_n}x_n\| \\ &\leq \|\frac{\epsilon}{r_n}x_n + (1 - \frac{\epsilon}{r_n})J_{r_n}x_n - x_n\| + \|x_n - J_{r_n}x_n\| \\ &= (1 - \frac{\epsilon}{r_n})\|J_{r_n}x_n - x_n\| + \|x_n - J_{r_n}x_n\| \\ &\leq 2\|x_n - J_{r_n}x_n\| \to 0 \quad (n \to \infty). \end{aligned} \tag{3.11b}$$

Firstly, suppose that X is uniformly smooth. Let us show that

$$\limsup_{n\to\infty}\langle f(z) - z, J(x_n - z)\rangle \leq 0, \tag{3.12}$$

where $z = Q(f)$,

$$Q(f) := \lim_{t\to 0} x_t,$$

and x_t is the unique fixed point of the contraction mapping T_t given by

$$T_t x = tf(x) + (1 - t)J_\epsilon x, \quad t \in (0, 1).$$

By Lemma 2.4, $Q(f) \in \mathrm{Fix}(J_\epsilon) = F$ solves the variational inequality

$$\langle (I - f)Q(f), J(Q(f) - p)\rangle \leq 0, \quad \forall p \in F.$$

Note that

$$x_t - x_n = t(f(x_t) - x_n) + (1 - t)(J_\epsilon x_t - x_n).$$

We apply Lemma 2.1 to deriving

$$\begin{aligned}
&\|x_t-x_n\|^2\\
&\le (1-t)^2\|J_\epsilon x_t-x_n\|^2+2t\langle f(x_t)-x_n, J(x_t-x_n)\rangle\\
&\le (1-t)^2(\|J_\epsilon x_t-J_\epsilon x_n\|+\|J_\epsilon x_n-x_n\|)^2 \qquad (3.13)\\
&\quad +2t\langle f(x_t)-x_t, J(x_t-x_n)\rangle+2t\|x_t-x_n\|^2\\
&\le (1-t)^2\|x_t-x_n\|^2+a_n(t)+2t\langle f(x_t)-x_t, J(x_t-x_n)\rangle+2t\|x_t-x_n\|^2,
\end{aligned}$$

where

$$a_n(t)=\|J_\epsilon x_n-x_n\|(2\|x_t-x_n\|+\|J_\epsilon x_n-x_n\|)\to 0 \quad (\text{due to } (3.11b)).$$

The last inequality implies

$$\langle x_t-f(x_t), J(x_t-x_n)\rangle \le \frac{t}{2}\|x_t-x_n\|^2+\frac{1}{2t}a_n(t).$$

It follows that

$$\limsup_{n\to\infty}\langle x_t-f(x_t), J(x_t-x_n)\rangle \le M\frac{t}{2}, \tag{3.14}$$

where $M>0$ is a constant such that $M\le\|x_t-x_n\|^2$ for all $n\ge 1$ and $t\in(0,1)$. Taking the limsup as $t\to 0$ in (3.14) and noticing the fact that the two limits are interchangeable due to the fact that the duality map J is uniformly norm-to-norm continuous on any bounded subset of X, we obtain (3.12).

Now, let us show that $x_n\to z$ as $n\to\infty$. Indeed, observe that

$$\begin{aligned}
x_{n+1}-z &= \beta_n(f(x_n)-z)+(1-\beta_n)(y_n-z)\\
&= \beta_n(f(x_n)-z)+(1-\beta_n)(1-\alpha_n)(J_{r_n}x_n-z)+(1-\beta_n)\alpha_n(x_n-z).
\end{aligned}$$

Then, utilizing Lemma 2.1 we get

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \|(1-\beta_n)\alpha_n(x_n-z)+(1-\beta_n)(1-\alpha_n)(J_{r_n}x_n-z)\|^2\\
&\quad +2\beta_n\langle f(x_n)-z, J(x_{n+1}-z)\rangle\\
&\le [(1-\beta_n)\alpha_n\|x_n-z\|+(1-\beta_n)(1-\alpha_n)\|x_n-z\|]^2\\
&\quad +2\beta_n\langle f(x_n)-f(z), J(x_{n+1}-z)\rangle\\
&\quad +2\beta_n\langle f(z)-z, J(x_{n+1}-z)\rangle\\
&\le (1-\beta_n)^2\|x_n-z\|^2+2\alpha\beta_n\|x_n-z\|\|x_{n+1}-z\|\\
&\quad +2\beta_n\langle f(z)-z, J(x_{n+1}-z)\rangle\\
&\le (1-\beta_n)^2\|x_n-z\|^2+\alpha\beta_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2)\\
&\quad +2\beta_n\langle f(z)-z, J(x_{n+1}-z)\rangle.
\end{aligned}$$

It hence follows that for all $n \geq n_0$

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\leq \frac{1-(2-\alpha)\beta_n+\beta_n^2}{1-\alpha\beta_n}\|x_n-z\|^2+\frac{2\beta_n}{1-\alpha\beta_n}\langle f(z)-z, J(x_{n+1}-z)\rangle \\
&= (1-\frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n})\|x_n-z\|^2+\frac{2\beta_n}{1-\alpha\beta_n}\langle f(z)-z, J(x_{n+1}-z)\rangle \\
&\quad +\frac{\beta_n^2}{1-\alpha\beta_n}\|x_n-z\|^2 \\
&\leq (1-\frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n})\|x_n-z\|^2+\frac{2\beta_n}{1-\alpha\beta_n}\langle f(z)-z, J(x_{n+1}-z)\rangle \\
&\quad +\frac{(1-\alpha)\beta_n}{1-\alpha\beta_n}\|x_n-z\|^2 \\
&= (1-\frac{(1-\alpha)\beta_n}{1-\alpha\beta_n})\|x_n-z\|^2+\frac{2\beta_n}{1-\alpha\beta_n}\langle f(z)-z, J(x_{n+1}-z)\rangle,
\end{aligned}$$

due to (C1). For every $n \geq n_0$, put

$$\mu_n = \frac{(1-\alpha)\beta_n}{1-\alpha\beta_n}$$

and

$$\nu_n = \frac{2}{1-\alpha}\langle f(z)-z, J(x_{n+1}-z)\rangle.$$

It follows that

$$\|x_{n+1}-z\|^2 \leq (1-\mu_n)\|x_n-z\|^2+\mu_n\nu_n, \quad \forall n \geq n_0. \tag{3.15}$$

It is readily seen from (C2) and (3.12) that

$$\sum_{n=0}^{\infty}\mu_n = \infty \quad \text{and} \quad \limsup_{n\to\infty}\nu_n \leq 0.$$

Therefore, applying Lemma 2.3 to (3.15), we conclude that $x_n \to z$ as $n \to \infty$.

Secondly, suppose that X has a weakly continuous duality map J_φ with gauge φ. Let us show that

$$\limsup_{n\to\infty}\langle f(z)-z, J_\varphi(x_n-z)\rangle \leq 0, \tag{3.16}$$

where $z = Q(f)$,

$$Q(f) := \lim_{t\to 0} x_t,$$

and x_t is the unique fixed point of the contraction mapping T_t given by

$$T_t x = tf(x)+(1-t)J_\epsilon x, \quad t \in (0,1).$$

By Lemma 2.7, $Q(f) \in \mathrm{Fix}(J_\epsilon) = F$ solves the variational inequality

$$\langle (I-f)Q(f), J_\varphi(Q(f)-p)\rangle \leq 0, \quad \forall p \in F. \tag{3.17}$$

We take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z)-z, J_\varphi(x_n-z)\rangle = \lim_{k\to\infty}\langle f(z)-z, J_\varphi(x_{n_k}-z). \tag{3.18}$$

Since X is reflexive and $\{x_n\}$ is bounded, we may assume that $x_{n_k} \rightharpoonup \bar{x}$. Note that $\|x_n - J_{r_n}x_n\| \to 0$ as $n \to \infty$. Hence, $\{J_{r_{n_k}}x_{n_k}\}$ converges weakly to $\bar{x}$, and

$$\|A_{r_n}x_n\| = \frac{1}{r_n}\|x_n - J_{r_n}x_n\| \leq \frac{1}{\epsilon}\|x_n - J_{r_n}x_n\| \to 0.$$

Consequently, taking the limit as $k \to \infty$ in the relation

$$A_{r_{n_k}}x_{n_k} \in AJ_{r_{n_k}}x_{n_k},$$

we get $0 \in A\bar{x}$; i.e., $\bar{x} \in F$. Thus, from (3.17) and (3.18) it follows that

$$\limsup_{n\to\infty}\langle f(z)-z, J_\varphi(x_n-z)\rangle = \langle f(z)-z, J_\varphi(\bar{x}-z) \leq 0.$$

This implies that (3.16) holds.

Now, let us show that $x_n \to z$ as $n \to \infty$. Indeed, observe that

$$\begin{aligned}\Phi(\|y_n - z\|) &= \Phi(\|\alpha_n(x_n-z) + (1-\alpha_n)(J_{r_n}x_n - z)\|)\\ &\leq \Phi(\alpha_n\|x_n - z\| + (1-\alpha_n)\|J_{r_n}x_n - z\|)\\ &\leq \Phi(\|x_n - z\|).\end{aligned}$$

Therefore, we apply Lemma 2.6 to getting

$$\begin{aligned}&\Phi(\|x_{n+1}-z\|)\\ &= \Phi(\|\beta_n(f(x_n)-z) + (1-\beta_n)(y_n - p)\|)\\ &= \Phi(\|\beta_n(f(x_n)-f(z)+f(z)-z) + (1-\beta_n)(y_n-z)\|)\\ &\leq \Phi(\|(1-\beta_n)(y_n-z) + \beta_n(f(x_n)-f(z))\|) + \beta_n\langle f(z)-z, J_\varphi(x_{n+1}-z)\rangle\\ &\leq \Phi((1-\beta_n)\|y_n-z\| + \beta_n\|f(x_n)-f(z)\|) + \beta_n\langle f(z)-z, J_\varphi(x_{n+1}-z)\rangle\\ &\leq \Phi((1-\beta_n)\|y_n-z\| + \alpha\beta_n\|x_n-z\|) + \beta_n\langle f(z)-z, J_\varphi(x_{n+1}-z)\rangle\\ &\leq (1-(1-\alpha)\beta_n)\Phi(\|x_n-z\|) + \beta_n\langle f(z)-z, J_\varphi(x_{n+1}-z)\rangle.\end{aligned}$$

Applying Lemma 2.3, we get

$$\Phi(\|x_n - z\|) \to 0 \quad (n \to \infty),$$

which hence implies that $\|x_n - z\| \to 0$ $(n \to \infty)$, i.e., $x_n \to z$ $(n \to \infty)$. This completes the proof. ∎

Corollary 3.1. *Let X be a reflexive Banach space. Assume, in addition, X either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ. Let A be an m-accretive operator in X such that $C = \overline{D(A)}$ is convex with $F := A^{-1}0 \neq \emptyset$, and $f \in \Pi_C$. Given sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$, the following control conditions are satisfied:*

($C1$) $\lim_{n\to\infty} \beta_n = 0$;

($C2$) $\sum_{n=0}^{\infty} \beta_n = \infty$;

($C3$) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

($C4$) $r_n \geq \epsilon, \forall n \geq 0$ *for some* $\epsilon > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then for an arbitrary $x_0 \in C$, the sequence $\{x_n\}$ defined by (3.1) converges strongly to a zero $Q(f)$ of A, where one of the following two statements holds:

(i) *if X is uniformly smooth, then $Q(f) \in F$ solves the variational inequality*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in F;$$

(ii) *if X has a weakly continuous duality map J_φ with gauge φ, then $Q(f) \in F$ solves the variational inequality*

$$\langle (I-f)Q(f), J_\varphi(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in F.$$

Proof. Repeating the same argument as in the proof of Theorem 3.1, we know that $\{x_n\}$ is bounded, and so are the sequences $\{J_{r_n} x_n\}, \{y_n\}$ and $\{f(x_n)\}$. Since $\lim_{n\to\infty} \beta_n = 0$, it is easy to see that there hold the following:

(i) $\beta_n(f(x_n) - x_n) \to 0$ $(n \to \infty)$;

(ii) $0 \leq \beta_n \leq 1 - \alpha, \forall n \geq n_0$ for some integer $n_0 \geq 1$;

(iii) $\lim_{n\to\infty}\left(\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}\right) = 0$.

Therefore, all conditions of Theorem 3.1 are satisfied. So, utilizing Theorem 3.1 we obtain the desired result. ∎

Corollary 3.2. *Let X be a reflexive Banach space. Assume, in addition, X either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ. Let A be an m-accretive operator in X such that $C = \overline{D(A)}$ is convex with $F := A^{-1}0 \neq \emptyset$, and $f \in \Pi_C$ with contractive constant $\alpha \in (0,1)$. Given sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$, the following control conditions are satisfied:*

(*C1*) $0 \le \beta_n \le 1-\alpha, \forall n \ge n_0$ *for some integer* $n_0 \ge 1$*;*

(*C2*) $\lim_{n\to\infty}(\beta_n - \beta_{n+1}) = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$*;*

(*C3*) $\lim_{n\to\infty}(\alpha_n - \alpha_{n+1}) = 0$ *and* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$*;*

(*C4*) $r_n \ge \epsilon, \forall n \ge 0$ *for some* $\epsilon > 0$ *and* $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$.

For an arbitrary $x_0 \in C$*, let* $\{x_n\}$ *be defined by (3.1). Then,*

$$x_n \to Q(f) \iff \beta_n(f(x_n) - x_n) \to 0,$$

where one of the following two statements holds:

(i) *if* X *is uniformly smooth, then* $Q(f) \in F$ *solves the variational inequality*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C,\ p \in F;$$

(ii) *if* X *has a weakly continuous duality map* J_φ *with gauge* φ*, then* $Q(f) \in F$ *solves the variational inequality*

$$\langle (I-f)Q(f), J_\varphi(Q(f)-p)\rangle \le 0, \quad f \in \Pi_C,\ p \in F.$$

Proof. Observe that

$$\begin{aligned}
&\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n} \\
&= \frac{(\beta_{n+1}-\beta_n) - \beta_{n+1}\alpha_n + \beta_n\alpha_{n+1} + \beta_{n+1}\beta_n\alpha_n - \beta_n\beta_{n+1}\alpha_{n+1}}{(1-(1-\beta_{n+1})\alpha_{n+1})(1-(1-\beta_n)\alpha_n)} \\
&= \frac{(\beta_{n+1}-\beta_n) - \beta_{n+1}(\alpha_n-\alpha_{n+1}) - \alpha_{n+1}(\beta_{n+1}-\beta_n) + \beta_n\beta_{n+1}(\alpha_n-\alpha_{n+1})}{(1-(1-\beta_{n+1})\alpha_{n+1})(1-(1-\beta_n)\alpha_n)} \\
&= \frac{(\beta_{n+1}-\beta_n)(1-\alpha_{n+1}) - \beta_{n+1}(\alpha_n-\alpha_{n+1})(1-\beta_n)}{(1-(1-\beta_{n+1})\alpha_{n+1})(1-(1-\beta_n)\alpha_n)}.
\end{aligned}$$

Since $\lim_{n\to\infty}(\beta_n - \beta_{n+1}) = 0$ and $\lim_{n\to\infty}(\alpha_n - \alpha_{n+1}) = 0$, it follows that

$$\lim_{n\to\infty}\left(\frac{\beta_{n+1}}{1-(1-\beta_{n+1})\alpha_{n+1}} - \frac{\beta_n}{1-(1-\beta_n)\alpha_n}\right) = 0.$$

Consequently, all conditions of Theorem 3.1 are satisfied. So, utilizing Theorem 3.1 we obtain the desired result. ∎

Remark 3.1. Repeating the same argument as in the proof of Theorem 3.1, we know that $\{x_n\}$ is bounded, and so are the sequences $\{J_{r_n}x_n\}, \{y_n\}$ and $\{f(x_n)\}$. Under the assumptions of Theorem CLS in Section 1, it can be easily seen that all conditions of Corollary 3.2 are satisfied. Thus, we conclude that $\{x_n\}$ converges strongly to a zero of A. This shows that Corollary 3.2 includes Theorem CLS as a special case. Note that the following condition in Corollary 3.2

$$\lim_{n\to\infty}(\alpha_n - \alpha_{n+1}) = 0, \quad \lim_{n\to\infty}(\beta_n - \beta_{n+1}) = 0 \quad \text{and} \quad \lim_{n\to\infty}(r_n - r_{n+1}) = 0$$

is much weaker than the following one in Theorem CLS

$$\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty, \quad \sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty.$$

In addition, Corollary 3.2 removes the condition "$\beta_n \to 0$" in Theorem CLS as well. Therefore, the advantages of Theorem 3.1 in the present paper are that weaker and fewer restrictions are imposed on the parameter sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{r_n\}$.

Corollary 3.3. *Let X be a reflexive Banach space. Assume, in addition, X either is uniformly smooth or has a weakly continuous duality map J_φ with gauge φ. Let A be an m-accretive operator in X such that $C = \overline{D(A)}$ is convex with $F := A^{-1}0 \neq \emptyset$. Let the real sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy the control conditions (C1)-(C4) in Corollary 3.1. For arbitrary $x_0, u \in C$, let $\{x_n\}$ be defined by*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)J_{r_n}x_n, \\ x_{n+1} = \beta_n u + (1-\beta_n)y_n, \quad \forall n \geq 0. \end{cases}$$

Then,

$$x_n \to Q(u) \iff \beta_n(u - x_n) \to 0,$$

where one of the following two statements holds:

(i) *if X is uniformly smooth, then $Q(u) \in F$ solves the variational inequality*

$$\langle Q(u) - u, J(Q(u) - p)\rangle \leq 0, \quad u \in C,\ p \in F;$$

(ii) *if X has a weakly continuous duality map J_φ with gauge φ, then $Q(f) \in F$ solves the variational inequality*

$$\langle Q(u) - u, J_\varphi(Q(u) - p)\rangle \leq 0, \quad u \in C,\ p \in F.$$

Proof. Put $f(x) = u$ for all $x \in C$. Then by Corollary 3.1 we obtain the desired result. ∎

It is known [6] that when $X = H$ a Hilbert space, m-accretive operators coincide with maximal monotone operators. Next we give a weak convergence theorem for hybrid viscosity iterative approximation method (3.1) involving a maximal monotone operator A in a Hilbert space H.

Theorem 3.2. *Let H be a Hilbert space. Let A be a maximal monotone operator in H such that $C = \overline{D(A)}$ is convex with $F := A^{-1}0 \neq \emptyset$, and $f \in \Pi_C$ with contractive constant $\alpha \in (0,1)$. Given sequences $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$, the following control conditions are satisfied:*

($C1$) $\sum_{n=0}^{\infty} \beta_n < \infty$;

($C2$) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

($C3$) $r_n \geq \epsilon, \forall n \geq 0$ *for some* $\epsilon > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

Then for an arbitrary $x_0 \in C$, the sequence $\{x_n\}$ defined by (3.1) converges weakly to a zero of A

Proof. Take a zero p of A arbitrarily. Repeating the same argument as in the proof of Theorem 3.1, we know that $\{x_n\}$ is bounded, and so are the sequences $\{J_{r_n}x_n\}, \{y_n\}$ and $\{f(x_n)\}$.

Observe that

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
\leq\ &(1-\beta_n)\|y_n - p\|^2 + \beta_n\|f(x_n) - p\|^2 \\
\leq\ &\|y_n - p\|^2 + \beta_n\|f(x_n) - p\|^2 \\
=\ &\|\alpha_n(x_n - p) + (1-\alpha_n)(J_{r_n}x_n - p)\|^2 + \beta_n\|f(x_n) - p\|^2 \\
=\ &\alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|J_{r_n}x_n - p\|^2 \\
&-\alpha_n(1-\alpha_n)\|x_n - J_{r_n}x_n\|^2 + \beta_n\|f(x_n) - p\|^2 \\
\leq\ &\alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - J_{r_n}x_n\|^2 \\
&+\beta_n\|f(x_n) - p\|^2 \\
=\ &\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_n - J_{r_n}x_n\|^2 + \beta_n\|f(x_n) - p\|^2 \\
\leq\ &\|x_n - p\|^2 + \beta_n\|f(x_n) - p\|^2.
\end{aligned} \tag{3.19}
$$

Since $\sum_{n=0}^{\infty} \beta_n < \infty$ and $\{f(x_n)\}$ is bounded, we get $\sum_{n=0}^{\infty} \beta_n\|f(x_n) - p\|^2 < \infty$. Utilizing Lemma 2.8, we deduce that $\lim_{n\to\infty} \|x_n - p\|$ exists. Furthermore it follows from (3.19) that for all $n \geq 0$

$$\alpha_n(1-\alpha_n)\|x_n - J_{r_n}x_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n\|f(x_n) - p\|^2. \tag{3.20}$$

Since $\beta_n \to 0$ and $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, it follows from (3.20) that $\lim_{n\to\infty} \|x_n - J_{r_n}x_n\| = 0$. As proved in the proof of Theorem 3.1, we have $\|x_n - J_\epsilon x_n\| \leq 2\|x_n - J_{r_n}x_n\|$ for all $n \geq 1$. This implies immediately that

$$\lim_{n\to\infty} \|x_n - J_\epsilon x_n\| = 0. \tag{3.21}$$

Now, let us show that $\omega_w(x_n) \subset F$. Indeed, let $\bar{x} \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$. Since $(I - J_\epsilon)x_n \to 0$, by Lemma 2.9 we known that $\bar{x} \in \mathrm{Fix}(J_\epsilon) = F$.

Finally, let us show that $\omega_w(x_n)$ is a singleton. Indeed, let $\{x_{m_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{m_j} \rightharpoonup \hat{x}$. Then $\hat{x}$ is also a zero of A. If $\bar{x} \neq \hat{x}$, by Opial's property of H, we reach the following contradiction:

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{x}\| &= \lim_{i\to\infty} \|x_{n_i} - \bar{x}\| \\
&< \lim_{i\to\infty} \|x_{n_i} - \hat{x}\| = \lim_{j\to\infty} \|x_{m_j} - \hat{x}\| \\
&< \lim_{j\to\infty} \|x_{m_j} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\|.
\end{aligned}$$

This implies that $\omega_w(x_n)$ is a singleton. Consequently, $\{x_n\}$ converges weakly to a zero of A. ■

Remark 3.2. Compared with Theorem CLS in Section 1, Theorem 3.2 is a weak convergence result. It can be viewed as the supplement of Theorem CLS.

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