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INTEGRAL REPRESENTATIONS AND GROWTH PROPERTIES FOR A CLASS OF SUPERFUNCTIONS IN A CONE

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Abstract. An integral representation for a class of superfunctions, associated with the Schrödinger operator, is investigated. Meanwhile, growth properties of them are also proved outside of some exceptional sets.

1. Introduction and Main Results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n (n \geq 2)$ the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \ldots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary, the closure and the complement of a set $\mathbf{S}$ in $\mathbf{R}^n$ are denoted by $\partial\mathbf{S}$, $\overline{\mathbf{S}}$ and $\mathbf{S}^c$, respectively.

For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at P and radius r in $\mathbf{R}^n$. $S_r = \partial B(O, r)$.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, x_n)$ by

$$x_1 = r\left(\prod_{j=1}^{n-1} \sin\theta_j\right) \ (n \geq 2),\ x_n = r\cos\theta_1,$$

and if $n \geq 3$, then

$$x_{n-m+1} = r\left(\prod_{j=1}^{m-1} \sin\theta_j\right)\cos\theta_m \ (2 \leq m \leq n-1),$$

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where $0 \leq r < +\infty$, $-\frac{1}{2}\pi \leq \theta_{n-1} < \frac{3}{2}\pi$, and if $n \geq 3$, then $0 \leq \theta_j \leq \pi$ $(1 \leq j \leq n-2)$.

Let D be an arbitrary domain in $\mathbf{R}^n$ and $\mathcal{A}_a$ denote the class of nonnegative radial potentials $a(P)$, i.e. $0 \leq a(P) = a(r)$, $P = (r, \Theta) \in D$, such that $a \in L^b_{loc}(D)$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

If $a \in \mathcal{A}_a$, then the stationary Schrödinger operator

$$Sch_a = -\Delta + a(P)I = 0,$$

where Δ is the Laplace operator and I is the identical operator, can be extended in the usual way from the space $C_0^\infty(D)$ to an essentially self-adjoint operator on $L^2(D)$ (see [14, Ch. 13]). We will denote it Sch_a as well. This last one has a Green's a-function $G^a_D(P, Q)$. Here $G^a_D(P, Q)$ is positive on D and its inner normal derivative $\partial G^a_D(P, Q)/\partial n_Q \geq 0$, where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into D. We denote this derivative by $PI^a_D(P, Q)$, which is called the Poisson a-kernel with respect to D.

We call a function $u \not\equiv -\infty$ that is upper semi-continuous in D a subfunction of the Schrödinger operator Sch_a if its values belong to the interval $[-\infty, \infty)$ and at each point $P \in D$ with $0 < r < r(P)$ the generalized mean-value inequality

$$u(P) \leq \int_{S(P,r)} u(Q) \frac{\partial G^a_{B(P,r)}(P, Q)}{\partial n_Q} d\sigma(Q)$$

is satisfied, where $G^a_{B(P,r)}(P, Q)$ is the Green a-function of Sch_a in $B(P, r)$ and $d\sigma(Q)$ is a surface measure on the sphere $S(P, r) = \partial B(P, r)$.

The class of subfunctions in D is denoted by $SbH(a, D)$. If $-u \in SbH(a, D)$, then we call u a superfunction and denote the class of superfunctions by $SpH(a, D)$. If a function u is both subfunction and superfunction, it is, clearly, continuous and is called an a-harmonic function associated with the operator Sch_a. The class of a-harmonic functions is denoted by $H(a, D) = SbH(a, D) \cap SpH(a, D)$. In terminology we follow A. I. Kheyfits (see [10, 11]), E. F. Beckenbach (see [3]) and L. Nirenberg (see [13]). The class $SbH(a, D)$ has been considered by various authors (see, for example, [4, 5, 15]). But a systematic study of subfunctions from the point of view of function theory began recently by B. Ya. Levin and A. I. Kheyfits (see [11]).

The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}^{n-1}_+$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Omega \subset \mathbf{S}^{n-1}$, the set $\{(r, \Theta) \in \mathbf{R}^n; r \in \Xi, (1, \Theta) \in \Omega\}$ in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Omega$. In particular, the half space $\mathbf{R}_+ \times \mathbf{S}^{n-1}_+ = \{(X, x_n) \in \mathbf{R}^n; x_n > 0\}$ will be denoted by $\mathbf{T}_n$.

By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1} (n \geq 2)$. We call it a cone. Then T_n is a special cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$, which is $\partial C_n(\Omega) - \{O\}$. Furthermore, we denote by dS_r the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_r.

We shall say that a set $E \subset C_n(\Omega)$ has a covering $\{r_j, R_j\}$ if there exists a sequence of balls $\{B_j\}$ with centers in $C_n(\Omega)$ such that $E \subset \cup_{j=0}^{\infty} B_j$, where r_j is the radius of B_j and R_j is the distance from the origin to the center of B_j.

From now on, we always assume $D = C_n(\Omega)$. For the sake of brevity, we shall write $G_\Omega^a(P, Q)$ instead of $G_{C_n(\Omega)}^a(P, Q)$, $PI_\Omega^a(P, Q)$ instead of $PI_{C_n(\Omega)}^a(P, Q)$, $SpH(a)$ (resp. $SbH(a)$) instead of $SpH(a, C_n(\Omega))$ (resp. $SbH(a, C_n(\Omega))$) and $H(a)$ instead of $H(a, C_n(\Omega))$.

For positive functions h_1 and h_2, we say that $h_1 \lesssim h_2$ if $h_1 \leq M h_2$ for some constant $M > 0$. If $h_1 \lesssim h_2$ and $h_2 \lesssim h_1$, we say that $h_1 \approx h_2$.

Let Ω be a domain on $\mathbf{S}^{n-1}$ with smooth boundary. Consider the Dirichlet problem

$$(\Lambda_n + \lambda)\varphi = 0 \quad \text{on} \quad \Omega,$$

$$\varphi = 0 \quad \text{on} \quad \partial\Omega,$$

where Λ_n is the spherical part of the Laplace opera Δ_n

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Lambda_n}{r^2}.$$

We denote the least positive eigenvlaue of this boundary value problem by λ and the normalized positive eigenfunction corresponding to λ by $\varphi(\Theta)$, $\int_\Omega \varphi^2(\Theta) dS_1 = 1$. In order to ensure the existence of λ and a smooth $\varphi(\Theta)$. We put a rather strong assumption on Ω: if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain $(0 < \alpha < 1)$ on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [8, p. 88-89] for the definition of $C^{2,\alpha}$-domain).

For any $(1, \Theta) \in \Omega$, we have (see [12, p. 7-8])

$$\varphi(\Theta) \approx dist((1, \Theta), \partial C_n(\Omega)),$$

which yields that

$$\delta(P) \approx r\varphi(\Theta), \tag{1.1}$$

where $P = (r, \Theta) \in C_n(\Omega)$ and $\delta(P) = dist(P, \partial C_n(\Omega))$.

Solutions of an ordinary differential equation

$$-Q''(r) - \frac{n-1}{r} Q'(r) + \left(\frac{\lambda}{r^2} + a(r) \right) Q(r) = 0, \quad 0 < r < \infty, \tag{1.2}$$

play on essential role in this paper. It is known (see, for example, [19]) that if the potential $a \in \mathcal{A}_a$, then the equation (1.2) has a fundamental system of positive solutions $\{V, W\}$ such that V is nondecreasing with

$$0 \le V(0+) \le V(r) \ \ as \ \ r \to +\infty,$$

and W is monotonically decreasing with

$$+\infty = W(0+) > W(r) \searrow 0 \ \text{ as } \ r \to +\infty.$$

Let $u(r, \Theta)$ be a function on $C_n(\Omega)$. For any given $r \in \mathbf{R}_+$, The integral

$$\int_\Omega u(r, \Theta)\varphi(\Theta) dS_1,$$

is denoted by $N_u(r)$, when it exists. The finite or infinite limit

$$\lim_{r\to\infty} V^{-1}(r) N_u(r)$$

is denoted by $\mathcal{U}_u$, when it exists.

We will also consider the class $\mathcal{B}_a$, consisting of the potentials $a \in \mathcal{A}_a$ such that there exists the finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0, \infty)$, and moreover, $r^{-1}|r^2 a(r) - k| \in L(1, \infty)$. If $a \in \mathcal{B}_a$, then the (sub)superfunctions are continuous (see [17]).

In the rest of paper, we assume that $a \in \mathcal{B}_a$ and we shall suppress this assumption for simplicity.

Denote

$$\iota_k^{\pm} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k+\lambda)}}{2},$$

then the solutions to the equation (1.2) have the asymptotic (see [9])

$$V(r) \approx r^{\iota_k^+}, \ W(r) \approx r^{\iota_k^-}, \text{ as } r \to \infty. \tag{1.3}$$

Remark 1. If $a=0$ and $\Omega = \mathbf{S}_+^{n-1}$, then $\iota_0^+ = 1$, $\iota_0^- = 1-n$ and $\varphi(\Theta) = (2ns_n^{-1})^{1/2} cos\theta_1$, where s_n is the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of $\mathbf{S}^{n-1}$.

We denote the Green a-potential with a positive measure v on $C_n(\Omega)$ by

$$G_\Omega^a \nu(P) = \int_{C_n(\Omega)} G_\Omega^a(P, Q) d\nu(Q).$$

The Poisson a-integral $PI_\Omega^a \mu(P)$ (resp. $PI_\Omega^a[g](P)) \not\equiv +\infty$ $(P \in C_n(\Omega))$ of μ (resp. g) relative to $C_n(\Omega)$ is defined as follows

$$PI_\Omega^a \mu(P) = \frac{1}{c_n} \int_{S_n(\Omega)} PI_\Omega^a(P, Q) d\mu(Q),$$

$$\text{(resp. } PI^a_\Omega[g](P) = \frac{1}{c_n}\int_{S_n(\Omega)} PI^a_\Omega(P,Q)g(Q)d\sigma_Q,)$$

where

$$PI^a_\Omega(P,Q) = \frac{\partial G^a_\Omega(P,Q)}{\partial n_Q}, \qquad c_n = \begin{cases} 2\pi & n=2, \\ (n-2)s_n & n \geq 3, \end{cases}$$

μ is a positive measure on $\partial C_n(\Omega)$ (resp. g is a continuous function on $\partial C_n(\Omega)$ and $d\sigma_Q$ is the surface area element on $S_n(\Omega)$) and $\frac{\partial}{\partial n_Q}$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$.

We define the positive measure μ' on $\mathbf{R}^n$ by

$$d\mu'(Q) = \begin{cases} t^{-1}W(t)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q) & Q=(t,\Phi)\in S_n(\Omega;(1,+\infty)), \\ 0 & Q\in \mathbf{R}^n - S_n(\Omega;(1,+\infty)). \end{cases}$$

Let ν be any positive measure $C_n(\Omega)$ such that $G^a_\Omega\nu(P) \not\equiv +\infty$ $(P\in C_n(\Omega))$. The positive measure ν' on $\mathbf{R}^n$ is defined by

$$d\nu'(Q) = \begin{cases} W(t)\varphi(\Phi)d\nu(Q) & Q=(t,\Phi)\in C_n(\Omega;(1,+\infty)), \\ 0 & Q\in \mathbf{R}^n - C_n(\Omega;(1,+\infty)). \end{cases}$$

So the positive measure ξ on $\mathbf{R}^n$ is defined by

$$d\xi(Q) = \begin{cases} t^{-1}W(t)d\xi'(Q) & Q=(t,\Phi)\in \overline{C_n(\Omega;(1,+\infty))}, \\ 0 & Q\in \mathbf{R}^n - \overline{C_n(\Omega;(1,+\infty))}, \end{cases}$$

where

$$d\xi'(Q) = \begin{cases} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q) & Q=(t,\Phi)\in S_n(\Omega;(1,+\infty)), \\ t\varphi(\Phi)d\nu(Q) & Q=(t,\Phi)\in C_n(\Omega;(1,+\infty)). \end{cases}$$

Remark 2. Let $a=0$ and $\Omega = \mathbf{S}^{n-1}_+$. Then

$$G^0_{\mathbf{S}^{n-1}_+}(x,y) = \begin{cases} \log|x-y^*| - \log|x-y| & n=2, \\ |x-y|^{2-n} - |x-y^*|^{2-n} & n\geq 3, \end{cases}$$

where $y^* = (Y, -y_n)$, that is, y^* is the mirror image of $y=(Y,y_n)$ with respect to ∂T_n. Hence, for the two points $x=(X,x_n)\in T_n$ and $y=(Y,y_n)\in\partial T_n$, we have

$$PI^0_{\mathbf{S}^{n-1}_+}(x,y) = \frac{\partial}{\partial n_y}G^0_{\mathbf{S}^{n-1}_+}(x,y) = \begin{cases} 2|x-y|^{-2}x_n & n=2, \\ 2(n-2)|x-y|^{-n}x_n & n\geq 3. \end{cases}$$

Remark 3. If $d\mu(Q) = |g(Q)|d\sigma_Q$ $(Q = (t, \Phi) \in S_n(\Omega))$, where $g(Q)$ is a continuous function on $\partial C_n(\Omega)$, then we have

$$d\mu''(Q) = \begin{cases} |g(Q)|t^{-1}W(t)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\sigma_Q & Q = (t, \Phi) \in S_n(\Omega; (1, +\infty)), \\ 0 & Q \in \mathbf{R}^n - S_n(\Omega; (1, +\infty)). \end{cases}$$

Remark 4. Let $a = 0$ and $\Omega = \mathbf{S}_+^{n-1}$. Then a positive measure δ on $\mathbf{R}^n$ is defined by

$$d\delta(y) = \begin{cases} |y|^{-n}d\delta'(y) & y = (Y, y_n) \in \overline{T_n}, \\ 0 & y \in \mathbf{R}^n - \overline{T_n}, \end{cases}$$

where

$$d\delta'(y) = \begin{cases} d\mu(y) & y = (Y, 0) \in \partial T_n, \\ y_n d\nu(y) & y = (Y, y_n) \in T_n. \end{cases}$$

Let $\epsilon > 0$, $\beta \geq 0$ and λ' be any positive measure on $\mathbf{R}^n$ having finite total mass. For each $P = (r, \Theta) \in \mathbf{R}^n - \{O\}$, the maximal function $M(P; \lambda', \beta)$ is defined by

$$M(P; \lambda', \beta) = \sup_{0<\rho<\frac{r}{2}} \frac{\lambda'(B(P, \rho))}{\rho^\beta}.$$

The set $\{P = (r, \Theta) \in \mathbf{R}^n - \{O\}; M(P; \lambda', \beta)r^\beta > \epsilon\}$ is denoted by $E(\epsilon; \lambda', \beta)$.

Remark 5. If $\lambda'(\{P\}) > 0$ $(P \neq O)$, then $M(P; \lambda', \beta) = +\infty$ for any positive number β. So we can find $\{P \in \mathbf{R}^n - \{O\}; \lambda'(\{P\}) > 0\} \subset E(\epsilon; \lambda', \beta)$.

As in T_n, Siegel-Talvila [16, Corollary 2.1] have proved

Theorem A. Let g be a measurable function on ∂T_n satisfying

$$\int_{\partial T_n} \frac{|g(y)|}{1 + |y|^n} dy < \infty. \tag{1.4}$$

Then the harmonic function $PI^0_{\mathbf{S}_+^{n-1}}[g](x) = \frac{1}{c_n}\int_{\partial T_n} PI^0_{\mathbf{S}_+^{n-1}}(x, y)g(y)dy$ satisfies $PI^0_{\mathbf{S}_+^{n-1}}[g] = o(|x| \sec^{n-1}\theta_1)$ as $|x| \to \infty$ in T_n, where $PI^0_{\mathbf{S}_+^{n-1}}(x, y)$ is the general Poisson kernel for the n-dimensional half space, see Remark 2.

Now we state our first result.

Theorem 1. Let $0 \leq \alpha \leq n$, ϵ be a sufficiently small positive number and μ be a positive measure on $\partial C_n(\Omega)$ such that

$$PI^a_\Omega\mu(P) \not\equiv +\infty \ (P = (r, \Theta) \in C_n(\Omega)).$$

Then there exists a covering $\{r_j, R_j\}$ of $E(\epsilon;\mu', n-\alpha)$ $(\subset C_n(\Omega))$ satisfying

$$\sum_{j=0}^{\infty}(\frac{r_j}{R_j})^{2-\alpha}V(\frac{R_j}{r_j})W(\frac{R_j}{r_j})<\infty, \tag{1.5}$$

such that

$$\lim_{r\to\infty, P\in C_n(\Omega)-E(\epsilon;\mu',n-\alpha)} V^{-1}(r)\varphi^{\alpha-1}(\Theta)PI^a_\Omega\mu(P)=0.$$

Corollary 1. Let μ be a positive measure on $S_n(\Omega)$ satisfying

$$\int_{S_n(\Omega)}\frac{1}{1+tW^{-1}(t)}d\mu(Q)<\infty. \tag{1.6}$$

Then the generalized harmonic function $PI^a_\Omega\mu(P)$ satisfies

$$\lim_{r\to\infty, P\in C_n(\Omega)} V^{-1}(r)\varphi^{n-1}(\Theta)PI^a_\Omega\mu(P)=0.$$

Our next aim is to be concerned with the solutions of the Dirichlet problem for the Schrödinger operator Sch_a on $C_n(\Omega)$ and the growth property of them.

Theorem 2. Let α, ϵ be defined as in Theorem 1 and g be a continuous function on $\partial C_n(\Omega)$ satisfying

$$\int_1^{\infty} t^{-1}V^{-1}(t)\left(\int_{\partial\Omega}|g(t,\Phi)|d_{\sigma_\Phi}\right)dt<+\infty, \tag{1.7}$$

where d_{σ_Φ} is the surface area element of $\partial\Omega$ at $\Phi\in\partial\Omega$. Then the function $PI^a_\Omega[g](P)$ $(P=(r,\Theta))$ satisfies

$$PI^a_\Omega[g]\in C^2(C_n(\Omega))\cap C^0(\overline{C_n(\Omega)}),$$

$$Sch_a PI^a_\Omega[g]=0 \text{ in } C_n(\Omega),$$

$$PI^a_\Omega[g]=g \text{ on } \partial C_n(\Omega)$$

and there exists a covering $\{r_j, R_j\}$ of $E(\epsilon;\mu'', n-\alpha)$ $(\subset C_n(\Omega)$, see Remark 3) satisfying (1.5) such that

$$\lim_{r\to\infty, P\in C_n(\Omega)-E(\epsilon;\mu'',n-\alpha)} V^{-1}(r)\varphi^{\alpha-1}(\Theta)PI^a_\Omega[g](P)=0. \tag{1.8}$$

Remark 6. In the case $a=0$ and $\Omega=\mathbf{S}^{n-1}_+$, (1.7) is equivalent to (1.4) from (1.3). In the case $\alpha=n$, (1.5) is a finite sum, then the set $E(\epsilon;\mu'',0)$ is a bounded set and (1.8) holds in $C_n(\Omega)$, which generalize Theorem A to the conical case.

Then we give a way to estimate the Green a-potential with measures on $C_n(\Omega)$. For a similar result, we refer the readers to the paper by B. Ya. Levin and A. I. Kheyfits [11, Corollary 6.1], who gave the growth properties of $G^a_\Omega\nu(P)$ at infinity in $C_n(\Omega)$ under the conditions

$$\int_{C_n(\Omega;(1,+\infty))} W(t)\varphi(\Phi)d\nu(Q) < +\infty \tag{1.9}$$

and

$$\int_{C_n(\Omega;(0,1))} V(t)\varphi(\Phi)d\nu(Q) < +\infty. \tag{1.10}$$

Theorem 3. Let $0 \leq \alpha < n$, ϵ be defined as in Theorem 1 and ν be a positive measure on $C_n(\Omega)$ such that

$$G^a_\Omega\nu(P) \not\equiv +\infty \ (P = (r,\Theta) \in C_n(\Omega)). \tag{1.11}$$

Then there exists a covering $\{r_j, R_j\}$ of $E(\epsilon;\nu', n-\alpha)$ $(\subset C_n(\Omega))$ satisfying (1.5) such that

$$\lim_{r\to\infty, P\in C_n(\Omega)-E(\epsilon;\nu',n-\alpha)} V^{-1}(r)\varphi^{\alpha-1}(\Theta)G^a_\Omega\nu(P) = 0.$$

Remark 7. By comparison the condition (1.11) is fairly briefer and easily applied. Moreover, $E(\epsilon;\nu', n-1)$ is a set of a-finite view in the sense of [11] (see [11, Definition 6.1] for the definition of a-finite view).

It is known that a positive superharmonic function $u(x)$ on T_n can be uniquely decomposed as

$$u(x) = d_1 x_n + c_n PI^0_{\mathbf{S}^{n-1}_+}\mu(x) + G^0_{\mathbf{S}^{n-1}_+}\nu(x), \tag{1.12}$$

where $d_1 \geq 0$, $d\mu$ is a positive measure on ∂T_n satisfying

$$\int_{\partial T_n} \frac{1}{1+|y|^n} d\mu(y) < \infty$$

and $d\nu$ is the Riesz associated measure of $u(x)$.

Motivated by the above result, we give an integral representation of a positive superfunction in a cone. It must be pointed out that the integral representations of generalized harmonic functions in a half space were developed by A. I. Kheyfits (see [10]).

Theorem 4. Let $0 < u(P) \in SpH(a)$, then there exist a unique positive measure μ on $\partial C_n(\Omega)$ satisfying (1.6) and a unique positive measure ν on $C_n(\Omega)$ satisfying (1.9)-(1.10) such that

$$u(P) = \mathcal{U}_u V(r)\varphi(\Theta) + c_n PI^a_\Omega\mu(P) + G^a_\Omega\nu(P). \tag{1.13}$$

Remark 8. V. S. Azarin treated the case $a = 0$ (see [2, Theorem 1]).

The following Theorem 5 follows readily from Theorems 1 and 3, which generalizes the growth properties of harmonic and superharmonic functions to the superfunctions on $C_n(\Omega)$.

Theorem 5. Let $0 \leq \alpha < n$, ϵ be defined as in Theorem 1 and $u(P)$ $(\not\equiv +\infty)$ $(P = (r, \Theta) \in C_n(\Omega))$ be defined by (1.13). Then there exists a covering $\{r_j, R_j\}$ of $E(\epsilon; \xi, n-\alpha)$ $(\subset C_n(\Omega))$ satisfying (1.5) such that

$$\lim_{r\to\infty, P\in C_n(\Omega)-E(\epsilon;\xi,n-\alpha)} V^{-1}(r)\varphi^{\alpha-1}(\Theta)\{u(P) - \mathcal{U}_u V(r)\varphi(\Theta)\} = 0.$$

We remark that $E(\epsilon; \xi, n-1)$ is a set of a-finite view.

As in T_n and $a = 0$ (cf. [7]), we have by Remarks 1, 4 and (1.3)

Corollary 2. Let ϵ be defined as in Theorem 1 and $u(x)$ $(\not\equiv +\infty)$ $(x = (X, x_n) \in T_n)$ be defined by (1.12). Then,

(i) there exists a covering $\{r_j, R_j\}$ of $E(\epsilon; \delta, n-1)$ ($\subset T_n$, see Remark 4) satisfying

$$\sum_{j=0}^{\infty}(\frac{r_j}{R_j})^{n-1} < \infty$$

such that

$$\lim_{|x|\to\infty, x\in T_n-E(\epsilon;\delta,n-1)} |x|^{-1}\{u(x) - d_1 x_n\} = 0.$$

(ii) there exists a covering $\{r_j, R_j\}$ of $E(\epsilon; \delta, n)$ $(\subset T_n)$ satisfying

$$\sum_{j=0}^{\infty}(\frac{r_j}{R_j})^{n} < \infty$$

such that

$$\lim_{|x|\to\infty, x\in T_n-E(\epsilon;\delta,n)} x_n^{-1}\{u(x) - d_1 x_n\} = 0.$$

2. Some Lemmas

In our discussions, the following estimates for the kernel functions $PI^a_\Omega(P, Q)$, $G^a_\Omega(P, Q)$ and $\partial G^a_{\Omega,R}(P, Q)/\partial R$ are fundamental, which follow from [11] and [2, Lemma 4 and Remark].

Lemma 1.

$$PI^a_\Omega(P, Q) \approx t^{-1}V(t)W(r)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}, \tag{2.1}$$

$$\text{(resp. } PI^a_\Omega(P,Q) \approx V(r)t^{-1}W(t)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}, \text{)} \tag{2.2}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in S_n(\Omega)$ satisfying $0<\frac{t}{r}\leq\frac{4}{5}$ (resp. $0<\frac{r}{t}\leq\frac{4}{5}$);

$$PI^0_\Omega(P,Q) \lesssim \frac{\varphi(\Theta)}{t^{n-1}}\frac{\partial\varphi(\Phi)}{\partial n_\Phi} + \frac{r\varphi(\Theta)}{|P-Q|^n}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}, \tag{2.3}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$.

Lemma 2.

$$G^a_\Omega(P,Q) \approx V(t)W(r)\varphi(\Theta)\varphi(\Phi), \tag{2.4}$$

$$\text{(resp. } G^a_\Omega(P,Q) \approx V(r)W(t)\varphi(\Theta)\varphi(\Phi), \text{)} \tag{2.5}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in C_n(\Omega)$ satisfying $0<\frac{t}{r}\leq\frac{4}{5}$ (resp. $0<\frac{r}{t}\leq\frac{4}{5}$);

Further, for any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(t,\Phi)\in C_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$, we have

$$G^0_\Omega(P,Q) \lesssim \frac{\varphi(\Theta)\varphi(\Phi)}{t^{n-2}} + \Pi_\Omega(P,Q), \tag{2.6}$$

where

$$\Pi_\Omega(P,Q) = \min\{\frac{1}{|P-Q|^{n-2}}, \frac{rt\varphi(\Theta)\varphi(\Phi)}{|P-Q|^n}\}.$$

Lemma 3. Let $G^a_{\Omega,R}(P,Q)$ be the Green a-function of the Schrӧdinger operator for $C_n(\Omega,(0,R))$, then

$$-\frac{\partial G^a_{\Omega,R}(P,Q)}{\partial R} \approx V(r)\{-W'(R)\}\varphi(\Theta)\varphi(\Phi) \tag{2.7}$$

for any $P=(r,\Theta)\in C_n(\Omega)$ and any $Q=(R,\Phi)\in S_n(\Omega;R)$.

Lemma 4. Let μ be a positive measure on $S_n(\Omega)$ such that there is a sequence of points $P_i=(r_i,\Theta_i)\in C_n(\Omega), r_i\to+\infty\ (i\to+\infty)$ satisfying $PI^a_\Omega\mu(P_i)<+\infty$ $(i=1,2,\ldots)$. Then for a positive number l,

$$\int_{S_n(\Omega;(l,+\infty))}\frac{W(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q)<+\infty \tag{2.8}$$

and

$$\lim_{R\to+\infty}\frac{W(R)}{V(R)}\int_{S_n(\Omega;(0,R))}\frac{V(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q)=0. \tag{2.9}$$

Proof. Take a positive number l satisfying $P_1 = (r_1, \Theta_1) \in C_n(\Omega)$, $r_1 \leq \frac{4}{5}l$. Then from (2.2), we have

$$V(r_1)\varphi(\Theta_1)\int_{S_n(\Omega;(l,+\infty))}\frac{W(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q) \lesssim \int_{S_n(\Omega)} PI_\Omega^a(P,Q)d\mu(Q) < +\infty,$$

which gives (2.8). For any positive number ϵ, from (2.8), we can take a number R_ϵ such that

$$\int_{S_n(\Omega;(R_\epsilon,+\infty))}\frac{W(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q) < \frac{\epsilon}{2}.$$

If we take a point $P_i = (r_i, \Theta_i) \in C_n(\Omega)$, $r_i \geq \frac{5}{4}R_\epsilon$, then we have from (2.1)

$$W(r_i)\varphi(\Theta_i)\int_{S_n(\Omega;(0,R_\epsilon])}\frac{V(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q) \lesssim \int_{S_n(\Omega)} PI_\Omega^a(P,Q)d\mu(Q) < +\infty.$$

If R $(R > R_\epsilon)$ is sufficiently large, then

$$\begin{aligned}
&\frac{W(R)}{V(R)}\int_{S_n(\Omega;(0,R))}\frac{V(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q)\\
\lesssim\ &\frac{W(R)}{V(R)}\int_{S_n(\Omega;(0,R_\epsilon])}\frac{V(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q) + \int_{S_n(\Omega;(R_\epsilon,R))}\frac{W(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q)\\
\lesssim\ &\frac{W(R)}{V(R)}\int_{S_n(\Omega;(0,R_\epsilon])}\frac{V(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q) + \int_{S_n(\Omega;(R_\epsilon,+\infty))}\frac{W(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}d\mu(Q)\\
\lesssim\ &\epsilon,
\end{aligned}$$

which gives (2.9).

Lemma 5. Let ν be a positive measure on $C_n(\Omega)$ such that there is a sequence of points $P_i = (r_i, \Theta_i) \in C_n(\Omega), r_i \to +\infty$ $(i \to +\infty)$ satisfying $G_\Omega^a\nu(P_i) < +\infty$ $(i = 1, 2, \ldots; Q \in C_n(\Omega))$. Then for a positive number l,

$$\int_{C_n(\Omega;(l,+\infty))} W(t)\varphi(\Phi)d\nu(Q) < +\infty$$

and

$$\lim_{R\to+\infty}\frac{W(R)}{V(R)}\int_{C_n(\Omega;(0,R))} V(t)\varphi(\Phi)d\nu(Q) = 0.$$

Proof. In order to prove Lemma 5, We have only to use (2.4) and (2.5) instead of (2.1) and (2.2) respectively in the proof of Lemma 4.

Lemma 6. Let $\epsilon > 0$, $\beta \geq 0$ and λ' be any positive measure on $\mathbf{R}^n$ having finite total mass. Then $E(\epsilon; \lambda', \beta)$ has a covering $\{r_j, R_j\}$ $(j = 1, 2, \ldots)$ satisfying

$$\sum_{j=1}^{\infty}\left(\frac{r_j}{R_j}\right)^{2-n+\beta}V\left(\frac{R_j}{r_j}\right)W\left(\frac{R_j}{r_j}\right) < \infty.$$

Proof. Set

$E_j(\epsilon;\lambda',\beta)=\{P=(r,\Theta)\in E(\epsilon;\lambda',\beta):2^j\le r<2^{j+1}\}\ (j=2,3,4,\ldots).$

If $P=(r,\Theta)\in E_j(\epsilon;\lambda',\beta)$, then there exists a positive number $\rho(P)$ such that

$$(\frac{\rho(P)}{r})^{2-n+\beta}V(\frac{r}{\rho(P)})W(\frac{r}{\rho(P)})\approx(\frac{\rho(P)}{r})^{\beta}\le\frac{\lambda'(B(P,\rho(P)))}{\epsilon}.$$

Since $E_j(\epsilon;\lambda',\beta)$ can be covered by the union of a family of balls $\{B(P_{j,i},\rho_{j,i}):P_{j,i}\in E_k(\epsilon;\lambda',\beta)\}$ $(\rho_{j,i}=\rho(P_{j,i}))$. By the Vitali Lemma (see [18]), there exists $\Lambda_j\subset E_j(\epsilon;\lambda',\beta)$, which is at most countable, such that $\{B(P_{j,i},\rho_{j,i}):P_{j,i}\in\Lambda_j\}$ are disjoint and $E_j(\epsilon;\lambda',\beta)\subset\cup_{P_{j,i}\in\Lambda_j}B(P_{j,i},5\rho_{j,i})$.

So

$$\cup_{j=2}^{\infty}E_j(\epsilon;\lambda',\beta)\subset\cup_{j=2}^{\infty}\cup_{P_{j,i}\in\Lambda_j}B(P_{j,i},5\rho_{j,i}).$$

On the other hand, note that $\cup_{P_{j,i}\in\Lambda_j}B(P_{j,i},\rho_{j,i})\subset\{P=(r,\Theta):2^{j-1}\le r<2^{j+2}\}$, so that

$$\begin{aligned}\sum_{P_{j,i}\in\Lambda_j}(\frac{5\rho_{j,i}}{|P_{j,i}|})^{2-n+\beta}V(\frac{|P_{j,i}|}{5\rho_{j,i}})W(\frac{|P_{j,i}|}{5\rho_{j,i}})&\approx\sum_{P_{j,i}\in\Lambda_j}(\frac{5\rho_{j,i}}{|P_{j,i}|})^{\beta}\\&\le 5^{\beta}\sum_{P_{j,i}\in\Lambda_j}\frac{\lambda'(B(P_{j,i},\rho_{j,i}))}{\epsilon}\\&\le\frac{5^{\beta}}{\epsilon}\lambda'(C_n(\Omega;[2^{j-1},2^{j+2}))).\end{aligned}$$

Hence we obtain

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{P_{j,i}\in\Lambda_j}(\frac{\rho_{j,i}}{|P_{j,i}|})^{2-n+\beta}V(\frac{|P_{j,i}|}{\rho_{j,i}})W(\frac{|P_{j,i}|}{\rho_{j,i}})&\approx\sum_{j=1}^{\infty}\sum_{P_{j,i}\in\Lambda_j}(\frac{\rho_{j,i}}{|P_{j,i}|})^{\beta}\\&\le\sum_{j=1}^{\infty}\frac{\lambda'(C_n(\Omega;[2^{j-1},2^{j+2})))}{\epsilon}\\&\le\frac{3\lambda'(\mathbf{R}^n)}{\epsilon}.\end{aligned}$$

Since $E(\epsilon;\lambda',\beta)\cap\{P=(r,\Theta)\in\mathbf{R}^n;r\ge 4\}=\cup_{j=2}^{\infty}E_j(\epsilon;\lambda',\beta)$. Then $E(\epsilon;\lambda',\beta)$ is finally covered by a sequence of balls $\{B(P_{j,i},\rho_{j,i}),B(P_1,6)\}$ $(j=2,3,\ldots;i=1,2,\ldots)$ satisfying

$$\sum_{j,i}(\frac{\rho_{j,i}}{|P_{j,i}|})^{2-n+\beta}V(\frac{|P_{j,i}|}{\rho_{j,i}})W(\frac{|P_{j,i}|}{\rho_{j,i}})\approx\sum_{j,i}(\frac{\rho_{j,i}}{|P_{j,i}|})^{\beta}\le\frac{3\lambda'(\mathbf{R}^n)}{\epsilon}+6^{\beta}<+\infty,$$

where $B(P_1,6)$ $(P_1=(1,0,\ldots,0)\in\mathbf{R}^n)$ is the ball which covers $\{P=(r,\Theta)\in\mathbf{R}^n;r<4\}$.

3. Proof of the Theorem 1

Take any point $P = (r, \Theta) \in C_n(\Omega; (R, +\infty)) - E(\epsilon; \mu', n-\alpha)$, where $R(\leq \frac{4}{5}r)$ is a sufficiently large number and ϵ is a sufficiently small positive number.

Write

$$PI^a_\Omega \mu(P) = B_1(P) + B_2(P) + B_3(P),$$

where

$$B_1(P) = \frac{1}{c_n} \int_{S_n(\Omega;(0,\frac{4}{5}r])} PI^a_\Omega(P, Q) d\mu(Q),$$

$$B_2(P) = \frac{1}{c_n} \int_{S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} PI^a_\Omega(P, Q) d\mu(Q)$$

and

$$B_3(P) = \frac{1}{c_n} \int_{S_n(\Omega;[\frac{5}{4}r,\infty))} PI^a_\Omega(P, Q) d\mu(Q).$$

The relation $G^a_\Omega(P, Q) \leq G^0_\Omega(P, Q)$ implies this inequality (see [1])

$$PI^a_\Omega(P, Q) \leq PI^0_\Omega(P, Q). \tag{3.1}$$

By (2.1), (2.2) and Lemma 4, we have the following growth estimates:

$$\begin{aligned} B_1(P) &\lesssim V(r)\varphi(\Theta) \frac{W(\frac{4}{5}r)}{V(\frac{4}{5}r)} \int_{S_n(\Omega;(0,\frac{4}{5}r])} \frac{V(t)}{t} \frac{\partial \varphi(\Phi)}{\partial n_\Phi} d\mu(Q) \\ &\lesssim \epsilon V(r)\varphi(\Theta). \end{aligned} \tag{3.2}$$

$$\begin{aligned} B_3(P) &\lesssim V(r)\varphi(\Theta) \int_{S_n(\Omega;[\frac{5}{4}r,\infty))} \frac{W(t)}{t} \frac{\partial \varphi(\Phi)}{\partial n_\Phi} d\mu(Q) \\ &\lesssim \epsilon V(r)\varphi(\Theta). \end{aligned} \tag{3.3}$$

By (3.1) and (2.3), we write

$$B_2(P) \lesssim B_{21}(P) + B_{22}(P),$$

where

$$B_{21}(P) = \int_{S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} V(t)\varphi(\Theta) d\mu'(Q)$$

and

$$B_{22}(P) = \int_{S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} \frac{tr\varphi(\Theta)}{|P-Q|^n W(t)} d\mu'(Q).$$

We first have

$$B_{21}(P) \lesssim \epsilon V(r)\varphi(\Theta) \tag{3.4}$$

from Lemma 4.

Next, we shall estimate $B_{22}(P)$. Take a sufficiently small positive number d_2 such that $S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))\subset B(P,\frac{1}{2}r)$ for any $P=(r,\Theta)\in\Lambda(d_2)$, where

$$\Lambda(d_2)=\{P=(r,\Theta)\in C_n(\Omega);\ \inf_{z\in\partial\Omega}|(1,\Theta)-(1,z)|<d_2,\ 0<r<\infty\},$$

and divide $C_n(\Omega)$ into two sets $\Lambda(d_2)$ and $C_n(\Omega)-\Lambda(d_2)$.

If $P=(r,\Theta)\in C_n(\Omega)-\Lambda(d_2)$, then there exists a positive d_2' such that $|P-Q|\geq d_2'r$ for any $Q\in S_n(\Omega)$, and hence

$$B_{22}(P)\lesssim \epsilon V(r)\varphi(\Theta) \tag{3.5}$$

from Lemma 4.

We shall consider the case $P\in\Lambda(d_2)$. Now put

$$H_i(P)=\{Q\in S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r));\ 2^{i-1}\delta(P)\leq|P-Q|<2^i\delta(P)\}.$$

Since $S_n(\Omega)\cap\{Q\in\mathbf{R}^n:|P-Q|<\delta(P)\}=\varnothing$, we have

$$B_{22}(P)=\sum_{i=1}^{i(P)}\int_{H_i(P)}\frac{tr\varphi(\Theta)}{|P-Q|^nW(t)}d\mu'(Q),$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P)\leq\frac{r}{2}<2^{i(P)}\delta(P)$.

By (1.1) we have $r\varphi(\Theta)\lesssim\delta(P)$ $(P=(r,\Theta)\in C_n(\Omega))$, and hence

$$\int_{H_i(P)}\frac{tr\varphi(\Theta)}{|P-Q|^nW(t)}d\mu'(Q)\ \lesssim\ \frac{r^{2-\alpha}}{W(r)}\varphi^{1-\alpha}(\Theta)\frac{\mu'(H_i(P))}{\{2^i\delta(P)\}^{n-\alpha}}$$

for $i=0,1,2,\ldots,i(P)$.

Since $P=(r,\Theta)\notin E(\epsilon;\mu',n-\alpha)$, we have

$$\begin{aligned}\frac{\mu'(H_i(P))}{\{2^i\delta(P)\}^{n-\alpha}}&\lesssim\frac{\mu'(B(P,2^i\delta(P)))}{\{2^i\delta(P)\}^{n-\alpha}}\\&\lesssim M(P;\mu',n-\alpha)\leq\epsilon r^{\alpha-n}\ (i=0,1,2,\ldots,i(P)-1)\end{aligned}$$

and

$$\frac{\mu'(H_{i(P)}(P))}{\{2^i\delta(P)\}^{n-\alpha}}\lesssim\frac{\mu'(B(P,\frac{r}{2}))}{(\frac{r}{2})^{n-\alpha}}\leq\epsilon r^{\alpha-n}.$$

So

$$B_{22}(P)\lesssim\epsilon V(r)\varphi^{1-\alpha}(\Theta). \tag{3.6}$$

Combining (3.2)-(3.6), we finally obtain that if L is sufficiently large and ϵ is a sufficiently small, then $PI_\Omega^a\mu(P)=o(V(r)\varphi^{1-\alpha}(\Theta))$ as $r\to\infty$, where $P=(r,\Theta)\in C_n(\Omega;(R,+\infty))-E(\epsilon;\mu',n-\alpha)$. Finally, there exists an additional finite ball B_0 covering $C_n(\Omega;(0,R])$, which together with Lemma 6, gives the conclusion of Theorem 1.

4. Proof of the Theorem 2

For any fixed $P = (r, \Theta) \in C_n(\Omega)$, take a number R satisfying $R > \max(1, \frac{5}{4}r)$. By (1.7) and (2.2), we have

$$\frac{1}{c_n}\int_{S_n(\Omega;(R,+\infty))} PI^a_\Omega(P,Q)|g(Q)|d\sigma_Q$$
$$\lesssim V(r)\varphi(\Theta)\int_R^\infty t^{-1}V^{-1}(t)\left(\int_{\partial\Omega}|g(t,\Phi)|d_{\sigma_\Phi}\right)dt < \infty.$$

Thus $PI^a_\Omega[g](P)$ is finite for any $P \in C_n(\Omega)$. Since $PI^a_\Omega(P,Q)$ is an a-harmonic function of $P \in C_n(\Omega)$ for any $Q \in S_n(\Omega)$, $PI^a_\Omega[g](P) \in H(a)$.

Now we study the boundary behavior of $PI^a_\Omega[g](P)$. Let $Q' = (t', \Phi') \in \partial C_n(\Omega)$ be any fixed point and L be any positive number such that $L > \max\{t' + 1, \frac{4}{5}R\}$.

Set $\chi_{S(L)}$ is the characteristic function of $S(L) = \{Q = (t, \Phi) \in \partial C_n(\Omega), t \leq L\}$ and write

$$PI^a_\Omega[g](P) = PI^a_{\Omega,1}[g](P) + PI^a_{\Omega,2}[g](P),$$

where

$$PI^a_{\Omega,1}[g](P) = \frac{1}{c_n}\int_{S_n(\Omega;(0,\frac{5}{4}L])} PI^a_\Omega(P,Q)g(Q)d\sigma_Q$$

and

$$PI^a_{\Omega,2}[g](P) = \frac{1}{c_n}\int_{S_n(\Omega;(\frac{5}{4}L,\infty))} P^a_\Omega(P,Q)g(Q)d\sigma_Q.$$

Notice that $PI^a_{\Omega,1}[g](P)$ is the Poisson a-integral of $g(Q)\chi_{S(\frac{5}{4}L)}$, we have

$$\lim_{P\to Q', P\in C_n(\Omega)} PI^a_{\Omega,1}[g](P) = g(Q').$$

Since $\lim_{\Theta\to\Phi'}\varphi(\Theta) = 0$, $PI^a_{\Omega,2}[g](P) = O(V(r)\varphi(\Theta))$ and therefore tends to zero. So the function $PI^a_\Omega[g](P)$ can be continuously extended to $\overline{C_n(\Omega)}$ such that

$$\lim_{P\to Q', P\in C_n(\Omega)} PI^a_\Omega[g](P) = g(Q')$$

for any $Q' = (t', \Phi') \in \partial C_n(\Omega)$ from the arbitrariness of L. Further, (1.8) is the conclusion of Theorem 1. Thus we complete the proof of Theorem 2.

5. Proof of the Theorem 3

For any point $P = (r, \Theta) \in C_n(\Omega; (R, +\infty)) - E(\epsilon; \nu', n-\alpha)$, where $R(\leq \frac{4}{5}r)$ is a sufficiently large number and ϵ is a sufficiently small positive number.

Write

$$G^a_\Omega\nu(P) = U_1(P) + U_2(P) + U_3(P),$$

where

$$U_1(P) = \int_{C_n(\Omega;(0,\frac{4}{5}r])} G^a_\Omega(P,Q)d\nu(Q),$$

$$U_2(P) = \int_{C_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} G^a_\Omega(P,Q)d\nu(Q)$$

and

$$U_3(P) = \int_{C_n(\Omega;[\frac{5}{4}r,\infty))} G^a_\Omega(P,Q)d\nu(Q).$$

If we use (2.4), (2.5) and Lemma 5 in place of (2.1), (2.2) and Lemma 2, we obtain the following growth estimates in the completely paralleled way to the proof of Theorem 1.

$$U_1(P) \lesssim \epsilon V(r)\varphi(\Theta). \tag{5.1}$$

$$U_3(P) \lesssim \epsilon V(r)\varphi(\Theta). \tag{5.2}$$

By (2.6) and (3.1), we have

$$U_2(P) \leq U_{21}(P) + U_{22}(P),$$

where

$$U_{21}(P) = \varphi(\Theta)\int_{C_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} V(t)d\nu'(Q) \text{ and}$$

$$U_{22}(P) = \int_{C_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} \Pi_\Omega(P,Q)d\nu(Q).$$

Then by Lemma 5, we immediately get

$$U_{21}(P) \lesssim \epsilon V(r)\varphi(\Theta). \tag{5.3}$$

To estimate $U_{22}(P)$, take a sufficiently small positive number c_2 independent of P such that

$$\Lambda(P) = \{(t,\Phi) \in C_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r));\ |(1,\Phi)-(1,\Theta)| < c_2\} \subset B(P,\frac{r}{2}) \tag{5.4}$$

and divide $C_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$ into two sets $\Lambda(P)$ and $\Lambda'(P)$, where $\Lambda'(P) = C_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r)) - \Lambda(P)$.

Write

$$U_{22}(P) = U^{(1)}_{22}(P) + U^{(2)}_{22}(P),$$

where

$$U^{(1)}_{22}(P) = \int_{\Lambda(P)} \Pi_\Omega(P,Q)d\nu(Q) \text{ and } U^{(2)}_{22}(P) = \int_{\Lambda'(P)} \Pi_\Omega(P,Q)d\nu(Q).$$

There exists a positive c_2' such that $|P-Q| \geq c_2' r$ for any $Q \in \Lambda'(P)$, and hence

$$
\begin{aligned}
U_{22}^{(2)}(P) &\lesssim \int_{C_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} \frac{rt\varphi(\Theta)\varphi(\Phi)}{|P-Q|^n} d\nu(Q) \\
&\lesssim V(r)\varphi(\Theta)\int_{C_n(\Omega;(\frac{4}{5}r,\infty))} d\nu'(Q) \\
&\lesssim \epsilon V(r)\varphi(\Theta)
\end{aligned} \tag{5.5}
$$

from Lemma 5.

Now we estimate $U_{22}^{(1)}(P)$. Set

$$I_i(P) = \{Q \in \Lambda(P);\ 2^{i-1}\delta(P) \leq |P-Q| < 2^i\delta(P)\},$$

where $i = 0, \pm 1, \pm 2, \ldots$.

Since $P = (r,\Theta) \notin E(\epsilon;\nu', n-\alpha)$ and hence $\nu'(\{P\}) = 0$ from Remark 5, we can divide $U_{22}^{(1)}(P)$ into $U_{22}^{(1)}(P) = U_{22}^{(11)}(P) + U_{22}^{(12)}(P)$, where

$$U_{22}^{(11)}(P) = \sum_{i=-\infty}^{-1} \int_{I_i(P)} \Pi_\Omega(P,Q)d\nu(Q) \text{ and } U_{22}^{(12)}(P) = \sum_{i=0}^{\infty} \int_{I_i(P)} \Pi_\Omega(P,Q)d\nu(Q).$$

Since $\delta(Q) + |P-Q| \geq \delta(P)$, we have $tf_\Omega(\Phi) \gtrsim \delta(Q) \gtrsim 2^{-1}\delta(P)$ for any $Q = (t,\Phi) \in I_i(p)$ $(i = -1, -2, \ldots)$. Then by (1.1)

$$
\begin{aligned}
\int_{I_i(P)} \Pi_\Omega(P,Q)d\nu(Q) &\lesssim \int_{I_i(P)} \frac{1}{|P-Q|^{n-2}W(t)\varphi(\Phi)} d\nu'(Q) \\
&\lesssim \frac{r^{2-\alpha}}{W(r)}\varphi^{1-\alpha}(\Theta)\frac{\nu'(B(P,2^i\delta(P)))}{\{2^i\delta(P)\}^{n-\alpha}} \\
&\lesssim V(r)\varphi^{1-\alpha}(\Theta)r^{n-\alpha}M(P;\nu',n-\alpha)\ (i = -1,-2,\ldots).
\end{aligned}
$$

Since $P = (r,\Theta) \notin E(\epsilon;\nu', n-\alpha)$, we obtain

$$U_{22}^{(11)}(P) \lesssim \epsilon V(r)\varphi^{1-\alpha}(\Theta). \tag{5.6}$$

By (5.4), we can take a positive integer $i(P)$ satisfying $2^{i(P)-1}\delta(P) \leq \frac{r}{2} < 2^{i(P)}\delta(P)$ and $I_i(P) = \varnothing$ $(i = i(P)+1, i(P)+2, \ldots)$.

Since $rf_\Omega(\Theta) \lesssim \delta(P)$ $(P = (r,\Theta) \in C_n(\Omega))$, we have

$$
\begin{aligned}
\int_{I_i(P)} \Pi_\Omega(P,Q)d\nu'(Q) &\lesssim r\varphi(\Theta)\int_{I_i(P)} \frac{t}{|P-Q|^n W(t)} d\nu'(Q) \\
&\lesssim V(r)\varphi^{1-\alpha}(\Theta)r^{n-\alpha}\frac{\nu'(I_i(P))}{\{2^i\delta(P)\}^{n-\alpha}}\ (i = 0,1,2,\ldots,i(P)).
\end{aligned}
$$

Since $P=(r,\Theta)\notin E(\epsilon;\nu',n-\alpha)$, we have

$$\begin{aligned}\tfrac{\nu'(I_i(P))}{\{2^i\delta(P)\}^{n-\alpha}} &\lesssim \frac{\nu'(B(P,2^i\delta(P)))}{\{2^i\delta(P)\}^{n-\alpha}}\\ &\lesssim M(P;\nu',n-\alpha)<\epsilon r^{\alpha-n}\ (i=0,1,2,\dots,i(P)-1)\end{aligned}$$

and

$$\frac{\nu'(I_{i(P)}(P))}{\{2^i\delta(P)\}^{n-\alpha}}\lesssim\frac{\nu'(\Lambda(P))}{(\frac{r}{2})^{n-\alpha}}<\epsilon r^{\alpha-n}.$$

Hence we obtain

$$U_{22}^{(12)}(P)\lesssim\epsilon V(r)\varphi^{1-\alpha}(\Theta). \tag{5.7}$$

Combining (5.1)-(5.3) and (5.5)-(5.7), we finally obtain that if R is sufficiently large and ϵ is a sufficiently small, then $G^a_\Omega\nu(P)=o(V(r)\varphi^{1-\alpha}(\Theta))$ as $r\to\infty$, where $P=(r,\Theta)\in C_n(\Omega;(R,+\infty))-E(\epsilon;\nu',n-\alpha)$. Finally, there exists an additional finite ball B_0 covering $C_n(\Omega;(0,R])$, which together with Lemma 6, gives the conclusion of Theorem 3.

6. Proof of the Theorem 4

For the a-harmonic function $\mathcal{U}_uV(r)\varphi(\Theta)$ on $C_n(\Omega)$, define

$$w(P)=u(P)-\mathcal{U}_uV(r)\varphi(\Theta).$$

Then $0\le w(P)\in SpH(a)$ and

$$\mathcal{U}_w=0. \tag{6.1}$$

Apply the Riesz decomposition theorem (see [11]) to $w(P)$ on $C_n(\Omega;(0,R))$, we obtain

$$\begin{aligned}w(P)\ =&\int_{S_n(\Omega;(0,R))}\frac{\partial G^a_{\Omega,R}(P,Q)}{\partial n_Q}d\mu_0\\ &-\int_{S_n(\Omega;R)}\frac{\partial G^a_{\Omega,R}(P,Q)}{\partial R}dS_R+\int_{C_n(\Omega;(0,R))}G^a_{\Omega,R}(P,Q)d\nu_0\\ =&\ w_1(P)+w_2(P)+w_3(P),\end{aligned}$$

where $d\mu_0$ is a positive measure on $S_n(\Omega)$, dS_R is the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_R, $d\nu_0$ is the Riesz measure of $w(P)$.

First, we consider the case where $w(P)<+\infty$. Since $G^a_{\Omega,R}(P,Q)\to G^a_\Omega(P,Q)$ as $R\to\infty$, we have

$$w_1(P)\to c_nPI^a_\Omega\mu_0(P)<+\infty$$

and

$$w_3(P) \to G^a_\Omega \nu_0(P) < +\infty,$$

as $R \to \infty$.

By (6.1), we know that there exists a sequence of points $P_i = (r_i, \Theta_i) \in C_n(\Omega), r_i \to +\infty \ (i \to +\infty)$ such that

$$w(P_i) \lesssim V(r_i)\epsilon_i, \tag{6.2}$$

where $\epsilon_i \to 0$ as $i \to +\infty$.

Take $R_i = 2r_i$, then by Lemma 3 we have

$$\begin{aligned} w(P_i) &\gtrsim -\int_{S_n(\Omega;R_i)} \frac{\partial G^a_{\Omega,R_i}((r_i,\Theta_i),(R_i,\Phi_i))}{\partial R} dS_{R_i} \\ &\gtrsim V(r_i)\varphi(\Theta_i)\int_{S_n(\Omega;R_i)} \{-W'(R_i)\}\varphi(\Phi_i)dS_{R_i}, \end{aligned}$$

which, together with (6.1), gives that

$$I(R_i) \lesssim \epsilon_i, \tag{6.3}$$

where

$$I(R_i) = \int_{S_n(\Omega;R_i)} \{-W'(R_i)\}\varphi(\Phi_i)dS_{R_i}.$$

By virtue of (2.7) and (6.3), we obtain

$$w_2(P_i) \lesssim V(r_i)\varphi(\Theta_i)I(R_i) \lesssim V(r_i)\epsilon_i, \tag{6.4}$$

which converges to 0 as $r_i \to +\infty$.

Passing the limit as $i \to +\infty$, we have

$$w(P) = c_n P I^a_\Omega \mu_0(P) + G^a_\Omega \nu_0(P) \tag{6.5}$$

for all $P \in C_n(\Omega)$.

Secondly, we consider the case where $w(P) = +\infty$. In this case, the sum of $w_1(P)$ and $w_3(P)$ is infinite. We know that $w_2(P)$ is bounded by (6.4). As $R \to +\infty$, $w(P)$ remains infinite. So (6.5) is proved under the condition $w(P) = +\infty$.

It is easy to see that the quantities $d\mu_0$ and $d\nu_0$ are same for the functions $w(P)$ and $u(P)$ respectively. So we denote them by $d\mu$ and $d\nu$ respectively for simplicity. Then we complete the proof of Theorem 4.

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REFERENCES

1. A. Ancona, First eigenvalues and comparison of Green's functions for elliptic operators on manifolds or domains, *J. d'Anal. Math.*, **72** (1997), 45-92.
2. V. S. Azarin, Generalization of a theorem of Hayman on subharmonic functions in an m-dimensional cone, *Amer. Math. Soc. Translation*, **80** (1969), 119-138.
3. E. F. Beckenbach, Generalized convex functions, *Bull. Amer. Math. Soc.*, **43** (1937), 363-371.
4. L. Bragg and J. Dettman, Function theories for the Yukawa and Helmholtz equations, *Rocky Mountain J. Math.*, **25** (1995), 887-917.
5. M. Bramanti, Potential theory for stationary Schrödinger operators: a survey of results obtained with non-probabilistic methods, *Le Math.*, **47** (1992), 25-61.
6. R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. 1, Interscience Publishers, New York, 1953.
7. M. Essén, H. L. Jackson and P. J. Rippon, On minimally thin and rarefied sets in $\mathbf{R}^p,\ p \geq 2$, *Hiroshima Math. J.*, **15** (1985), 393-410.
8. D. Gilbarg and N. S. Trudinger, , *Elliptic partial differential equations of second order*, Springer Verlag, Berlin, 1977.
9. P. Hartman, *Ordinary differential equations*, Wiley, New York, London, Sydney, 1964.
10. A. I. Kheyfits, The Riesz-Herglotz formula for generalized harmonic functions and their boundary behavior (Russian), *Dokl. Akad. Nauk SSSR*, **321** (1991), 263-265; translation in *Soviet Math. Dokl.*, **44** (1992), 688-691.
11. B. Ya. Levin and A. I. Kheyfits, *Asymptotic behavior of subfunctions of the stationary Schrödinger operator*, Preprint, http://arXiv/abs/math/021132896, 2002, p. 96.
12. C. Miranda, *Partial differential equations of elliptic type*, Springer, Berlin, 1970.
13. L. Nirenberg, Existence theorems in partial differential equations, NYU Notes, 1954.
14. M. Reed and B. Simon, *Methods of modern mathematical physics*, Vol. 3, Acad Press, London-New York-San Francisco, 1970.
15. A. Russakovskii, On asymptotic behavior of subfunctions of finite lower order of the Schrödinger operator, *Sib. Math. J.*, **30** (1989), 160-170.
16. D. Siegel and E. O. Talvila, Uniqueness for the n-dimensional half space Dirichlet problem, *Pacific J. Math.*, **175** (1996), 571-587.
17. B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Soc.*, **7** (1982), 447-526.
18. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.

19. G. M. Verzhbinskii and V. G. Maz'ya, Asymptotic behavior of solutions of elliptic equations of the second order close to a boundary. I, *Sibirsk. Mat. J.*, **12** (1971), 874-899.

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