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A NOTE ON CIRCULAR COLORINGS OF EDGE-WEIGHTED DIGRAPHS

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Abstract. An edge-weighted digraph $(\vec{G}, \ell)$ is a strict digraph $\vec{G}$ together with a function ℓ assigning a real weight ℓ_{uv} to each arc uv. $(\vec{G}, \ell)$ is symmetric if uv is an arc implies that so is vu. A circular r-coloring of $(\vec{G}, \ell)$ is a function φ assigning each vertex of $\vec{G}$ a point on a circle of perimeter r such that, for each arc uv of $\vec{G}$, the length of the arc from $\varphi(u)$ to $\varphi(v)$ in the clockwise direction is at least ℓ_{uv}. The circular chromatic number $\chi_c(\vec{G}, \ell)$ of $(\vec{G}, \ell)$ is the infimum of real numbers r such that $(\vec{G}, \ell)$ has a circular r-coloring. Suppose that $(\vec{G}, \ell)$ is an edge-weighted symmetric digraph with positive weights on the arcs. Let T be a $\{0, 1\}$-function on the arcs of $\vec{G}$ with the property that $T(uv) + T(vu) = 1$ for each arc uv in $\vec{G}$. In this note we show that if $\sum_{uv\in E(\vec{C})} \ell_{uv} / \sum_{uv\in E(\vec{C})} T(uv) \le r$ for each dicycle $\vec{C}$ of $\vec{G}$ satisfying $0 < (\sum_{uv\in E(\vec{C})} \ell_{uv}) \bmod r < \max\{\ell_{xy} + \ell_{yx} : xy \in E(\vec{G})\}$, then $(\vec{G}, \ell)$ has a circular r-coloring. Our result generalizes the work of Zhu, *J. Comb. Theory, Ser. B*, 86 (2002), 109-113, and also strengthens the work of Mohar, *J. Graph Theory*, 43 (2003), 107-116.

1. Introduction

A graph G is called *k-colorable* if $V(G)$ can be colored by at most k colors so that adjacent vertices are colored by different colors. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest k such that G is k-colorable. In 1962, Minty [5] proved his celebrated theorem that G is k-colorable if and only if G has an orientation ω such that, for any cycle C of G and any traversal of C (each cycle has two different directions for traversal), at least $|C|/k$ edges of C whose direction in

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ω coincide with the direction of the traversal. Let us denote by $|C^+_\omega|$ the number of edges of C whose direction in ω coincide with the direction of the traversal. We denote by $\mathcal{M}(G)$ the set of all cycles of G (including cycles of length 2 which are the same edge taken twice). With these notations, Minty's result can be restated as follows:

Theorem 1. (Minty's Theorem [5]). *G is k-colorable if and only if G has an orientation ω such that*

$$\max_{C\in\mathcal{M}(G)} \frac{|C|}{|C^+_\omega|} \le k.$$

Here and hereafter, for a set $\mathcal{S} \subseteq \mathcal{M}(G)$, $\max_{C\in\mathcal{S}}$ means that the maximum is over all cycle C in $\mathcal{S}$ and over the two traversals of C.

Let $\mathcal{D}(G)$ (resp. $\mathcal{A}(G)$) denote the set of all (resp. acyclic) orientations of G. From Minty's theorem it follows immediately that, for a graph G,

$$\chi(G) = \left\lceil \min_{\omega\in\mathcal{D}(G)} \max_{C\in\mathcal{M}(G)} \frac{|C|}{|C^+_\omega|} \right\rceil. \tag{1}$$

We remark that equation (1) remains true, if $\mathcal{D}(G)$ is replaced by $\mathcal{A}(G)$.

In 1992, Tuza [7] showed that the statement of Theorem 1 remains true when $\mathcal{M}(G)$ is replaced by $\mathcal{T}(G,k)$, where $\mathcal{T}(G,k)$ denotes the set of all cycles C of length $|C| \equiv 1 \pmod{k}$ in G. We state Tuza's result in the following theorem which improves 'if' part of Theorem 1.

Theorem 2. (Tuza's Theorem [7]). *Suppose k is an integer ≥ 2. Then G is k-colorable if and only if G has an orientation ω such that*

$$\max_{C\in\mathcal{T}(G,k)} \frac{|C|}{|C^+_\omega|} \le k.$$

In 1988, as a natural refinement of the chromatic number $\chi(G)$, Vince [8] introduced the star chromatic number of a graph G and denoted it by $\chi^*(G)$. Later, Zhu [12] called it circular chromatic number and denoted it by $\chi_c(G)$. Let k and d be positive integers such that $k \ge 2d$. A (k,d)-*coloring* of a graph G is a mapping $f : V(G) \to \{0, 1, \ldots, k-1\}$ such that for any edge xy of G, $d \le |f(x) - f(y)| \le k - d$. If G has a (k,d)-coloring, then we say that G is (k,d)-*colorable*. The *circular chromatic number* $\chi_c(G)$ of a graph G is defined as

$$\chi_c(G) = \inf\{k/d : G \text{ is } (k,d)\text{-colorable}\}.$$

It was shown in [8] that the infimum in the definition of $\chi_c(G)$ is always attained, and hence the infimum can be replaced by minimum.

The circular chromatic number and its variations have received considerable attention in the past decade (see [9, 12, 14] and references therein). Vince [8] showed that, for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. Furthermore, Goddyn, Tarsi and Zhang [3] proved the following generalization of equation (1) for circular chromatic number:

$$\chi_c(G) = \min_{\omega \in \mathcal{D}(G)} \max_{C \in \mathcal{M}(G)} \frac{|C|}{|C^+_\omega|}. \tag{2}$$

Equation (2) can be restated as follows:

Theorem 3. (Goddyn, Tarsi and Zhang's Theorem [3]). *G is (k,d)-colorable if and only if G has an orientation ω such that*

$$\max_{C \in \mathcal{M}(G)} \frac{|C|}{|C^+_\omega|} \leq \frac{k}{d}.$$

Clearly, Theorem 1 is the special case $d = 1$ of Theorem 3. Now, a natural question arises: Is there an analogue of Tuza's Theorem for the (k,d)-coloring. This question was answered in the affirmative by Zhu, who in [13] showed that the statement of Theorem 3 remains true if $\mathcal{M}(G)$ is replaced by $\mathcal{Z}(G,k,d)$, where $\mathcal{Z}(G,k,d)$ consists of cycles C of G such that $1 \leq d|C| \bmod k \leq 2d-1$. We state Zhu's result in the following theorem. Notice that Theorem 4 improves 'if' part of Theorem 3 and generalizes Theorem 2.

Theorem 4. (Zhu's Theorem [13]). *G is (k,d)-colorable if and only if G has an orientation ω such that*

$$\max_{C \in \mathcal{Z}(G,k,d)} \frac{|C|}{|C^+_\omega|} \leq \frac{k}{d}.$$

The theory of circular coloring of graphs has become an important branch of chromatic graph theory with many interesting results and applications (see [9, 10, 11, 12, 14] and references therein). Many variants and generalizations of the circular chromatic number were introduced by different authors. One of the most natural and important generalizations is to edge-weighted digraphs, which is introduced and studied by Mohar [6] in 2003.

An *edge-weighted digraph* $(\vec{G}, \ell)$ is a strict digraph $\vec{G}$ together with a function ℓ assigning a real *weight* to each directed edge. For simplicity of notation, the directed edge (u,v) of $\vec{G}$ is written as uv and is called an *arc*, the weight of the arc uv in $(\vec{G}, \ell)$ is written as ℓ_{uv}.

For a positive real r, let S^r denote a circle with perimeter r centered at the origin of $\mathcal{R}^2$. In the obvious way, we can identify the circle S^r with the interval $[0,r)$. For $x, y \in S^r$, let $d_r(x,y)$ denote the length of the arc on S^r from x to y in the clockwise direction if $x \neq y$, and let $d_r(x,y) = 0$ if $x = y$. A *circular r-coloring* of an edge-weighted digraph $(\vec{G}, \ell)$ is a function $\varphi : V(\vec{G}) \to S^r$ such that $d_r(\varphi(u), \varphi(v)) \geq \ell_{uv}$ for each arc uv in $\vec{G}$. The *circular chromatic number* $\chi_c(\vec{G}, \ell)$ of an edge-weighted digraph $(\vec{G}, \ell)$, recently introduced by Mohar [6], is defined as

$$\chi_c(\vec{G}, \ell) = \inf\{r : (\vec{G}, \ell) \text{ has a circular } r\text{-coloring}\}.$$

It was shown in [6] that the notion of $\chi_c(\vec{G}, \ell)$ generalizes several well-known optimization problems, such as the circular chromatic number [8, 12], the weighted circular colorings [1], the linear arboricity of a graph and the metric traveling salesman problem.

A digraph $\vec{G}$ (resp. an edge-weighted digraph $(\vec{G}, \ell)$) is said to be *symmetric* if uv is an arc implies that so is vu. To each arc uv in $\vec{G}$ we may assign a number T_{uv} of *tokens*. The nonnegative integer function T is called an *initial marking* of $\vec{G}$. An initial marking T of $\vec{G}$ is said to be *good* if for each arc uv of $\vec{G}$, $T_{uv} + T_{vu} = 1$. Denote by $\mathcal{D}(\vec{G})$ the set of all good initial markings of $\vec{G}$. An edge-weighted digraph $(\vec{G}, \ell)$ equipped with an initial marking T is denoted by $(\vec{G}, \ell, T)$ and is called a *timed marked graph*. The *token count* (resp. *weight*) of a dicycle $\vec{C}$ in $(\vec{G}, \ell, T)$ is defined as the value $\sum_{uv \in E(\vec{C})} T_{uv}$ (resp. $\sum_{uv \in E(\vec{C})} \ell_{uv}$) and is denoted by $|\vec{C}|_T$ (resp. $|\vec{C}|_\ell$), where $E(\vec{C})$ is the set of all arcs in $\vec{C}$. For a dipath $\vec{P}$ in $(\vec{G}, \ell, T)$, the two values $|\vec{P}|_T$ and $|\vec{P}|_\ell$ are defined in the same way. Denote by $\mathcal{M}(\vec{G})$ the set of all dicycles in $\vec{G}$.

In 2003, Mohar [6, Theorem 5.2] proved the following generalization of equation (2) for edge-weighted symmetric digraph $\chi_c(\vec{G}, \ell)$ having positive weights on the arcs:

$$\chi_c(\vec{G}, \ell) = \min_{T \in \mathcal{D}(\vec{G})} \max_{\vec{C} \in \mathcal{M}(\vec{G})} \frac{|\vec{C}|_\ell}{|\vec{C}|_T}, \tag{3}$$

Mohar [6, the last paragraph of Section 5] pointed out that equation (3) also holds for edge-weighted symmetric digraphs $(\vec{G}, \ell)$ having the property that $\ell_{uv} \geq 0$ and $\ell_{uv} + \ell_{vu} \neq 0$ for each arc uv in $\vec{G}$.

For an edge-weighted symmetric digraph $(\vec{G}, \ell)$, denote by $L(\vec{G}, \ell)$ the maximum value of $\ell_{uv} + \ell_{vu}$ over all arcs uv in $\vec{G}$. Equation (3) can be restated in Theorem 5, which generalizes Theorem 3.

Theorem 5. (Mohar's Theorem [6]). *Let $(\vec{G}, \ell)$ be an edge-weighted symmetric digraph with positive weights on the arcs. Suppose that r is a real number with $r \geq L(\vec{G}, \ell)$. Then $(\vec{G}, \ell)$ has a circular r-coloring if and only if $\vec{G}$ has a good initial marking T such that*

$$\max_{\vec{C} \in \mathcal{M}(\vec{G})} \frac{|\vec{C}|_\ell}{|\vec{C}|_T} \leq r.$$

A certain natural question presents itself at this point: In Theorem 5, can $\mathcal{M}(\vec{G})$ be replaced by a subset of it? The purpose of this paper is to answer this question in the affirmative. For an edge-weighted digraph $(\vec{G}, \ell)$ and a real number $r \geq L(\vec{G}, \ell)$, denote by $\mathcal{U}(\vec{G}, \ell, r)$ the set of all dicycles $\vec{C}$ in $\vec{G}$ with $0 < |\vec{C}|_\ell \bmod r < L(\vec{G}, \ell)$. In Theorem 6, whose proof appears in Section , we show that the statement of Theorem 5 remains true if $\mathcal{M}(\vec{G})$ is replaced by $\mathcal{U}(\vec{G}, \ell, r)$.

Theorem 6. *Let $(\vec{G}, \ell)$ be an edge-weighted symmetric digraph with positive weights on the arcs. Suppose that r is a real number with $r \geq L(\vec{G}, \ell)$. Then $(\vec{G}, \ell)$ has a circular r-coloring if and only if $\vec{G}$ has a good initial marking T such that*

$$\max_{\vec{C} \in \mathcal{U}(\vec{G}, \ell, r)} \frac{|\vec{C}|_\ell}{|\vec{C}|_T} \leq r.$$

Clearly, Theorem 6 improves 'if' part of Theorem 5. Moreover, Theorem 6 generalizes Theorem 4. To see this, we introduce an equivalent definition for circular chromatic number of graphs. For a real number $r \geq 1$, a *circular r-coloring* of a graph G is a function $f : V(G) \rightarrow [0, r)$ such that for any edge xy of G, $1 \leq |f(x) - f(y)| \leq r - 1$. It was known [12, 14] that

$$\chi_c(G) = \inf\{r : G \text{ has a circular } r\text{-coloring}\}.$$

It can readily be seen that G is (k, d)-colorable if and only if G has a circular k/d-coloring.

Given an undirected graph G, we can define a symmetric digraph, denoted by $\vec{G}$, on the same vertex set such that uv is an edge of G if and only if uv is an arc of $\vec{G}$. We say that such $\vec{G}$ is the symmetric digraph *derived from* G. Denote by $(\vec{G}, \mathbf{1})$ the edge-weighted digraph with $\mathbf{1}_{uv} = 1$ for each arc uv of $\vec{G}$. Notice that $L(\vec{G}, \mathbf{1}) = 2$, and there is a natural bijection between cycles C of G (including cycles of length 2 which are the same edge taken twice) and dicycles $\vec{C}$ of $\vec{G}$. Clearly, $0 < |\vec{C}|_\mathbf{1} \bmod \frac{k}{d} < L(\vec{G}, \mathbf{1})$ if and only if $0 < d|C| \bmod k < 2d$. For each orientation ω of G, we can associate a good initial marking T^ω of $\vec{G}$ such that $T^\omega_{uv} = 1$ for each arc uv of ω. Conversely, for each good initial marking T of the symmetric digraph $\vec{G}$, we can associate an orientation ω^T of G such that uv is an arc of ω^T if and only if $T_{uv} = 1$. From our discussion above, it can readily be seen that Theorem 6 generalizes Theorem 4.

In 1996, Deuber and Zhu [1] introduced another natural generalization of circular chromatic number to vertex-weighted graphs. A *vertex-weighted graph* (G, λ) is a graph G with positive weight function λ on $V(G)$. A *circular r-coloring* of (G, λ) is a function $\phi : V(G) \rightarrow S^r$ which assigns each vertex of G an open arc of S^r such that $\phi(x) \cap \phi(y) = \emptyset$ for any edge xy in G, and $\phi(v)$ has length at least $\lambda(v)$ for each vertex v of G. The *circular chromatic number* $\chi_c(G, \lambda)$ of a vertex-weighted graph (G, λ) is defined as

$$\chi_c(G, \lambda) = \inf\{r : (G, \lambda) \text{ has a circular } r\text{-coloring}\}.$$

It is clear that $\chi_c(G) = \chi_c(G, \mathbf{1})$, where $\mathbf{1}(v) = 1$ for each vertex v of G. From the results in [1], one can conclude that

$$\chi_c(G,\lambda) = \min_{\omega\in\mathcal{D}(G)} \max_{C\in\mathcal{M}(G)} \frac{\sum_{v\in V(C)} \lambda(v)}{|C^+_\omega|}. \tag{4}$$

Given a vertex-weighted graph (G, λ), we construct an edge-weighted digraph $(\vec{G}, \ell)$ such that $\vec{G}$ is the symmetric digraph derived from G and $\ell(uv) = \lambda(v)$ for each arc uv of $\vec{G}$. From equations (3) and (4), we see that

$$\chi_c(\vec{G},\ell) = \min_{T\in\mathcal{D}(\vec{G})} \max_{\vec{C}\in\mathcal{M}(\vec{G})} \frac{|\vec{C}|_\ell}{|\vec{C}|_T} = \min_{\omega\in\mathcal{D}(G)} \max_{C\in\mathcal{M}(G)} \frac{\sum_{v\in V(C)} \lambda(v)}{|C^+_\omega|} = \chi_c(G,\lambda).$$

Notice that our construction above of $(\vec{G}, \ell)$ paralleled to the one given by Mohar in [6, page 108]. Equations (3) and (4) also give the following nice observation whose proof is straightforward, and we omit it.

Observation 7. *Let (G, λ) be a vertex-weighted graph with positive weights on the vertices. Suppose that r is a real number with $r \geq L(G, \lambda)$. Then (G, λ) has a circular r-coloring if and only if G has an orientation ω such that*

$$\max_{C\in\mathcal{U}(G,\lambda,r)} \frac{|C|_\lambda}{|C^+_\omega|} \leq r,$$

where $|C|_\lambda = \sum_{v\in V(C)} \lambda(v)$, $L(G, \lambda) = \max\{\lambda(u) + \lambda(v) : uv \in E(G)\}$ and $\mathcal{U}(G, \lambda, r) = \{C \in \mathcal{M}(G) : 0 < |C|_\lambda \bmod r < L(G, \lambda)\}$.

2. The Proof of Theorem 6

In this section, we prove the main result of this note. As you will see in the proof below, our approach in fact gives a new proof of Theorem 5 (see [11] for another new proof) which was originally proved by Mohar [6] using a linear programming duality result of Hoffman [4] and Ghouila-Houri [2].

Proof of the 'if' part of Theorem 6. Suppose that $(\vec{G}, \ell)$ has a good initial marking T such that

$$\max_{\vec{C}\in\mathcal{U}(\vec{G},\ell,r)} \frac{|\vec{C}|_\ell}{|\vec{C}|_T} \leq r. \tag{5}$$

Let G be the underlying graph of $\vec{G}$ with a spanning tree $\mathcal{T}$. For two vertices x, y of G, clearly there is a unique (x, y)-path $v_1v_2 \ldots v_k$ in $\mathcal{T}$. The (x, y)-dipath $(v_1, v_2, \ldots, v_k)$ in $\vec{G}$ generated in this way is called *the dipath of $\vec{G}$ from x to y in $\mathcal{T}$*. Fix a vertex s in G. We define a function $f_{\mathcal{T}} : V(\vec{G}) \to \mathcal{R}$ as follows:

- $f_{\mathcal{T}}(s) = 0$;
- If x is a vertex other than s then $f_{\mathcal{T}}(x) = \sum_e (\ell_e - r \cdot T_e)$, where the summation is taken over all arcs e in the dipath of $\vec{G}$ from s to x in $\mathcal{T}$.

The *weight of* $\mathcal{T}$ is defined to be $\sum_{v \in V(\vec{G})} f_{\mathcal{T}}(v)$ and is denoted by $f(\mathcal{T})$. In the following, let $\mathcal{T}$ be a spanning tree of G with the maximum weight.

Let φ be a function which assigns to each vertex v of $\vec{G}$ a color $f_{\mathcal{T}}(v) \bmod r$ in $[0, r)$. For an arbitrary arc xy of $\vec{G}$, we want to show that $d_r(\varphi(x), \varphi(y)) \geq \ell_{xy}$ and $d_r(\varphi(y), \varphi(x)) \geq \ell_{yx}$. In the following cases, we view $\mathcal{T}$ as a rooted tree with root s. In this rooted tree, let x' and y' be the fathers of vertices x and y, respectively.

Case I. Suppose that x is not on the (s, y)-path of $\mathcal{T}$ and y is not on the (s, x)-path of $\mathcal{T}$. Let $\mathcal{T}'$ be the spanning tree of G obtained from $\mathcal{T}$ by deleting the edge $x'x$ and adding the edge xy. Then, by the maximality of $\mathcal{T}$, we have $f(\mathcal{T}') \leq f(\mathcal{T})$ which gives $f_{\mathcal{T}'}(x) \leq f_{\mathcal{T}}(x)$, and hence $f_{\mathcal{T}}(y) + \ell_{yx} - r \cdot T_{yx} \leq f_{\mathcal{T}}(x)$ because y is the father of x in $\mathcal{T}'$. By symmetry we also see that $f_{\mathcal{T}}(x) + \ell_{xy} - r \cdot T_{xy} \leq f_{\mathcal{T}}(y)$. Therefore

$$\ell_{yx} - r \cdot T_{yx} \leq f_{\mathcal{T}}(x) - f_{\mathcal{T}}(y) \leq r \cdot T_{xy} - \ell_{xy}. \tag{6}$$

If $T_{xy} = 1$ then we have $\ell_{yx} \leq f_{\mathcal{T}}(x) - f_{\mathcal{T}}(y) \leq r - \ell_{xy}$. If $T_{xy} = 0$ then we have $\ell_{xy} \leq f_{\mathcal{T}}(y) - f_{\mathcal{T}}(x) \leq r - \ell_{yx}$. In either case, clearly we have $d_r(\varphi(x), \varphi(y)) \geq \ell_{xy}$ and $d_r(\varphi(y), \varphi(x)) \geq \ell_{yx}$.

Case II. Suppose that either the (s, y)-path of $\mathcal{T}$ contains x or the (s, x)-path of $\mathcal{T}$ contains y. It suffices to consider the case that y is on the (s, x)-path of $\mathcal{T}$. Let $\vec{P}$ be the dipath of $\vec{G}$ from y to x in $\mathcal{T}$ and $\vec{C} = \vec{P} + xy$ be the dicycle of $\vec{G}$ consisting of $\vec{P}$ and the arc xy. Using the same method as in the previous case, we have

$$\ell_{yx} - r \cdot T_{yx} \leq f_{\mathcal{T}}(x) - f_{\mathcal{T}}(y). \tag{7}$$

Clearly we also have $f_{\mathcal{T}}(x) - f_{\mathcal{T}}(y) = |\vec{P}|_\ell - r|\vec{P}|_T$. Denote by ρ the value $|\vec{P}|_\ell \bmod r$. Let us consider the following two subcases.

Subcase II(a). If $\rho < \ell_{yx}$ or $\rho > r - \ell_{xy}$, since $|\vec{C}|_\ell = |\vec{P}|_\ell + \ell_{xy}$, then we have $0 < |\vec{C}|_\ell \bmod r = (\rho + \ell_{xy}) \bmod r < \ell_{xy} + \ell_{yx} \leq L(\vec{G}, \ell)$. By inequality (5), we have $|\vec{C}|_\ell / |\vec{C}|_T \leq r$ which is equivalent to $|\vec{P}|_\ell - r|\vec{P}|_T \leq r \cdot T_{xy} - \ell_{xy}$, and hence $f_{\mathcal{T}}(x) - f_{\mathcal{T}}(y) \leq r \cdot T_{xy} - \ell_{xy}$. Putting this together with inequality (7), we arrive at inequalities (6), and hence $d_r(\varphi(x), \varphi(y)) \geq \ell_{xy}$ and $d_r(\varphi(y), \varphi(x)) \geq \ell_{yx}$.

Subcase II(b). If $\ell_{yx} \leq \rho \leq r - \ell_{xy}$, we still have $d_r(\varphi(x), \varphi(y)) = (f_{\mathcal{T}}(y) - f_{\mathcal{T}}(x)) \bmod r = r - \rho \geq \ell_{xy}$ and $d_r(\varphi(y), \varphi(x)) = (f_{\mathcal{T}}(x) - f_{\mathcal{T}}(y)) \bmod r = \rho \geq \ell_{yx}$.

This completes the proof of the 'if' part. ■

Proof of the 'only if' part of Theorem 6. Suppose that $(\vec{G}, \ell)$ has a circular r-coloring $\varphi : V(\vec{G}) \to [0, r)$. We will show that $\vec{G}$ has a good initial marking T such that $\max_{\vec{C}\in\mathcal{M}(\vec{G})} |\vec{C}|_\ell/|\vec{C}|_T \leq r$, which is a stronger result than what we state in Theorem 6. Define a mapping T which assigns to each arc xy of $\vec{G}$ a value from $\{0, 1\}$ such that $T(xy) = 1$ as $\varphi(x) > \varphi(y)$, and $T(xy) = 0$ as $\varphi(x) < \varphi(y)$. Clearly, T is a good initial marking of $\vec{G}$ such that $|\vec{C}|_T > 0$ for each dicycle $\vec{C}$ in $\vec{G}$ and $\varphi(x) + \ell_{xy} \leq \varphi(y) + r \cdot T_{xy}$ for each arc xy in $\vec{G}$.

Let $\hat{C} = (v_1, v_2, \ldots, v_k, v_{k+1}) \in \mathcal{M}(\vec{G})$ such that $|\hat{C}|_\ell/|\hat{C}|_T = \max_{\vec{C}\in\mathcal{M}(\vec{G})} |\vec{C}|_\ell/|\vec{C}|_T$, where $v_{k+1} = v_1$ and v_iv_{i+1} is an arc for $i = 1, 2, \ldots, k$. From the result proved in the previous paragraph, we see that $\varphi(v_i)+\ell_{v_iv_{i+1}} \leq \varphi(v_{i+1})+r\cdot T_{v_iv_{i+1}}$ for $i = 1, 2, \ldots, k$. Adding up both side of the k inequalities separately, we get

$$|\hat{C}|_\ell = \sum_{i=1}^{k} \ell_{v_iv_{i+1}} \leq \varphi(v_{k+1}) - \varphi(v_1) + r \cdot \sum_{i=1}^{k} T_{v_iv_{i+1}} = r \cdot |\hat{C}|_T,$$

and hence $\max_{\vec{C}\in\mathcal{M}(\vec{G})} |\vec{C}|_\ell/|\vec{C}|_T \leq r$, that completes the proof of the 'only if' part. ■

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