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EXISTENCE OF HOMOCLINIC SOLUTIONS FOR THE SECOND-ORDER DISCRETE P-LAPLACIAN SYSTEMS

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Abstract. By using critical point theory, we establish some existence criteria to guarantee the second-order discrete p-Laplacian systems $\triangle(\varphi_p(\Delta u(n-1))) - a(n)|u(n)|^{p-2}u(n) + \nabla W(n, u(n)) = 0$ have at least one homoclinic orbit, where $p > 1,\ n \in \mathbb{Z},\ u \in \mathbb{R}^N$, $a \in C(\mathbb{Z}, \mathbb{R})$ and $W \in C^1(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$ are no periodic in n.

1. Introduction

Consider the second-order discrete p-Laplacian system

$$\triangle(\varphi_p(\Delta u(n-1))) - a(n)|u(n)|^{p-2}u(n) + \nabla W(n, u(n)) = 0, \tag{1.1}$$

where $p > 1, \varphi_p(s) = |s|^{p-2}s,\ n \in \mathbb{Z},\ u \in \mathbb{R}^N$, $a : \mathbb{Z} \to \mathbb{R}$ and $W : \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$. As usual, we say that a solution $u(n)$ of (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $\to \pm\infty$. In addition, if $u(n) \not\equiv 0$ then $u(n)$ is called a nontrivial homoclinic solution.

It is well-known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. If it has the smoothly connected homoclinic orbits, then it cannot stand the perturbation, its perturbed system probably produce chaotic phenomenon. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic orbits of (1.1) emanating from 0.

In general, system (1.1) may be regarded as a discrete analogue of the following second order Hamiltonian system

$$\frac{d}{dt}\left(|\dot{u}(t)|^{p-2}\dot{u}(t)\right) - a(t)|u(t)|^{p-2}u(t) + \nabla W(t, u(t)) = 0,\ t \in \mathbb{R}, u \in \mathbb{R}^N. \tag{1.2}$$

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When $p = 2$, system (1.2) reduces to second-order Hamiltonian system

$$\ddot{u}(t) - a(t)u(t) + \nabla W(t, u(t)) = 0. \tag{1.3}$$

In recent years, the existence and multiplicity of homoclinic orbits for Hamiltonian systems have been investigated in many papers via variational methods and many results were obtained based on various hypotheses on the potential functions, see, e.g., [3, 6-11, 17, 18, 25-26]. For system (1.2), if $a(t)$ and $W(t, x)$ are $T-$periodic in t, Rabinowitz [27] showed the existence of homoclinic orbits as a limit of $2kT-$periodic solutions of system (1.2). Analogous results for general Hamiltonian systems were obtained by Coti-Zelati, Ekeland and Sere [7], Felmer [12], Izydorek and Janczewska [17] and Tang and Xiao [32-35].

If $a(t)$ and $W(t, x)$ are no periodic in t, the problem of existence of homoclinic orbits for system (1.2) is quite different from the ones just described, because of lack of compactness of the Sobolev embedding. In [29], Rabinowitz and Tanaka studed (1.2) without a periodicity assumption.

In some recent papers [13-15], the authors studied the existence of periodic solutions and subharmonic solutions of some special forms of (1.1) by using the critical point theory. These papers show that the critical point method is an effective approach to the study of periodic solutions for difference equations. Along this direction, Ma and Guo [19] (with periodicity assumption) and [20] (without periodicity assumption) applied the critical point theory to prove the existence of homoclinic solutions of the following special form of (1.1) (with $N = 1$)

$$\triangle[p(n)\triangle u(n-1)] - q(n)u(n) + f(n, u(n)) = 0, \tag{1.4}$$

where $n \in \mathbb{Z},\ u \in \mathbb{R}$, $p,\ q : \mathbb{Z} \to \mathbb{R}$ and $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$.

Using the original ideas of Omana and Willem [25], Ma and Guo [20] used mountain pass theorems and compact imbedding lemma to prove following two theorems.

Theorem A. ([20]). *Assume that $p,\ q$ and f satisfy the following conditions:*

(p) $p(n) > 0$ *for all* $n \in \mathbb{Z}$;

(q) $q(n) > 0$ *for all* $n \in \mathbb{Z}$ *and* $\lim_{|n|\to+\infty} q(n) = +\infty$;

(f1) $f \in C(\mathbb{Z} \times \mathbb{R}, \mathbb{R})$ *and there is a constant* $\mu > 2$ *such that*

$$0 < \mu \int_0^x f(n, s)ds \le xf(n, x), \quad \forall\ (n, x) \in \mathbb{Z} \times (\mathbb{R} \setminus \{0\});$$

(f2) $\lim_{x\to 0} f(n, x)/x = 0$ *uniformly with respect to* $n \in \mathbb{Z}$.

Then the system (1.4) possesses a nontrivial homoclinic solution.

In fact, condition (f1) is the special form (with $N = 1$) of the following so-called global Ambrosetti-Rabinowitz condition on W due to Ambrosetti-Rabinowitz (e.g., [4]):

(AR) *For every $n \in \mathbb{Z}$, W is continuously differentiable in x, and there is a constant $\mu > 2$ such that*

$$0 < \mu W(n,x) \leq (\nabla W(n,x), x), \quad \forall\ (n,x) \in \mathbb{Z} \times (\mathbb{R}^N \setminus \{0\});$$

which implies that $W(n,x)$ is of superquadratic growth as $|x| \to +\infty$, where and in the sequel, $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{R}^N$ and $|\cdot|$ is the induced norm.

In the last decade there has been an increasing interest in the study of ordinary differential systems driven by the p-Laplacian (or the generalization of Laplacian) and with periodic boundary conditions, see [21, 23, 37] and the references cited therein. However, as the authors are aware, there are few papers discussing the existence of homoclinic solutions for the p-Laplacian systems. Just recently, Tang and Xiao [34] addressed the existence of homoclinic solutions for a kind of second-order periodic p-Laplacian systems different from system (1.1). In the present paper, we are interested in the existence of homoclinic solutions for system (1.1), where $a(n)$ and $W(n,x)$ are no periodic in n. The intention of this paper is that, under some relaxed assumptions on $W(n,x)$, we establish some existence criteria to guarantee that system (1.1) has at least one or infinitely many homoclinic solutions by using the Mountain Pass Theorem and genus properties. In particular, when $p = 2$, our results generalize Theorems A by relaxing condition (f1) and (f2).

Our main results are the following theorems.

Theorem 1.1. *Assume that a and W satisfy (A) and the following assumptions:*

(A) $a(n) : \mathbb{Z} \to (0,\infty)$ *and* $a(n) \to +\infty$ *as* $|n| \to \infty$*;*

(W1) $W(n,x) = W_1(n,x) - W_2(n,x)$, $W_1, W_2 \in C^1(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$*, and there is a bounded set $J \subset \mathbb{Z}$ such that*

$$\frac{1}{a(n)}|\nabla W(n,x)| = o(|x|^{p-1}) \quad \text{as} \quad x \to 0$$

uniformly in $n \in \mathbb{Z} \setminus J$;

(W2) *There is a constant $\mu > p$ such that*

$$0 < \mu W_1(n,x) \leq (\nabla W_1(n,x), x), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^N \setminus \{0\};$$

(W3) $W_2(n,0) \equiv 0$ *and there is a constant $\varrho \in [p,\mu)$ such that*

$$W_2(n,x) \geq 0, \quad (\nabla W_2(n,x), x) \leq \varrho W_2(n,x), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^N.$$

Then the system (1.1) *possesses a nontrivial homoclinic solution.*

Theorem 1.2. *Assume that* a *and* W *satisfy* (A), $(W2)$ *and the following assumptions:*

(W1') $W(n,x) = W_1(n,x) - W_2(n,x)$, $W_1, W_2 \in C^1(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$, *and*

$$\frac{1}{a(n)}|\nabla W(n,x)| = o(|x|^{p-1}) \quad \text{as} \quad x \to 0$$

uniformly in $n \in \mathbb{Z}$;

(W3') $W_2(n,0) \equiv 0$ *and there is a constant* $\varrho \in (p,\mu)$ *such that*

$$(\nabla W_2(n,x), x) \leq \varrho W_2(n,x), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^N.$$

Then the system (1.1) *possesses a nontrivial homoclinic solution.*

Remark 1.1. *Obviously, when* $p = 2$, *both conditions* $(W1)$ *and* $(W1')$ *are weaker than* $(f2)$.

When $W(n,x)$ is subquadratic at infinity, as far as the authors are aware, there is no research about the existence of homoclinic solutions of (1.1). In the present paper, we are interested in the case that $a(n)$ and $W(n,x)$ are neither autonomous nor periodic in n. Motivated by paper [38], the intention of this paper is that, under the assumption that $W(n,x)$ is indefinite sign and subquadratic as $|x| \to +\infty$, we will establish some existence criteria to guarantee that system (1.1) has at least one homoclinic solution by using Clark's Theorem in critical point theory.

When $W(t,x) = a(t)|x|^\gamma$, where $1 < \gamma < 2$ and $a \in C(\mathbb{R},\mathbb{R}) \cap L^{2/(2-\gamma)}(\mathbb{R},\mathbb{R})$, in the recent papers [38], Zhang and Yuan obtained the homoclinic solution by using a standard minimizing argument.

Now we present the basic hypothesis on a and W in order to announce the results in this paper.

(W4) *For every* $n \in \mathbb{Z}$, $W \in C^1(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$, *and there exist two constants* $1 < \gamma_1 < \gamma_2 < p$ *and two functions* $a_1, a_2 \in l^{p/(p-\gamma_1)}(\mathbb{Z}, [0, +\infty))$ *such that*

$$|W(n,x)| \leq a_1(n)|x|^{\gamma_1}, \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^N, \ |x| \leq 1$$

and

$$|W(n,x)| \leq a_2(n)|x|^{\gamma_2}, \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^N, \ |x| \geq 1.$$

(W5) *There exist two functions* $b \in l^{p/(p-\gamma_1)}(\mathbb{Z}, [0, +\infty))$ *and* $\varphi \in C([0, +\infty), [0, +\infty))$ *such that*

$$|\nabla W(n,x)| \leq b(n)\varphi(|x|), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}^N,$$

where $\varphi(s) = O(s^{\gamma_1 - 1})$ *as* $s \to 0^+$;

(W6) *There exist* $n_0 \in \mathbb{Z}$ *and two constants* $\eta > 0$ *and* $\gamma_3 \in (1, p)$ *such that*

$$W(n_0, x) \geq \eta |x|^{\gamma_3}, \quad \forall \ x \in \mathbb{R}^N, \ \ |x| \leq 1.$$

Then, We have the following theorem.

Theorem 1.3. *Assume that* a *and* W *satisfy* (*A*), (*W4*), (*W5*) *and* (*W6*). *Then the system* (*1.1*) *possesses at least one nontrivial homoclinic solution.*

By Theorem 1.3, we have the following corollary.

Corollary 1.1. *Assume that* a *and* W *satisfy (A) and the following conditions:*

(W7) $W(n, x) = a(n)V(x)$, *where* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ *and* $a \in l^{p/(p-\gamma_1)}(\mathbb{Z}, [0, +\infty))$, $\gamma_1 \in (1, p)$ *is a constant, such that* $a(n_0) > 0$ *for some* $n_0 \in \mathbb{Z}$;

(W8) *There exist constants* $M, M' > 0$, $\gamma_2 \in [\gamma_1, p)$ *and* $\gamma_3 \in (1, p)$ *such that*

$$M'|x|^{\gamma_3} \leq V(x) \leq M|x|^{\gamma_1}, \quad \forall \ x \in \mathbb{R}^N, \ \ |x| \leq 1$$

and

$$0 < V(x) \leq M|x|^{\gamma_2}, \quad \forall \ x \in \mathbb{R}^N, \ \ |x| \geq 1;$$

(W9) $\nabla V(x) = O(|x|^{\gamma_1 - 1})$ as $x \to 0$.

Then the system (1.1) possesses at least one nontrivial homoclinic solution.

The rest of the this paper is organized as follows. In Section 2, we introduce some notations and preliminary results. In Section 3, we complete the proofs of Theorems 1.1-1.3. In Section 4, we give some examples to to illustrate our results.

Throughout this paper, we let $q \in (1, \infty)$ such that $1/p + 1/q = 1$.

2. Preliminaries

Let

$$S = \left\{ \{u(n)\}_{n \in \mathbb{Z}} \ : \ u(n) \in \mathbb{R}^N, \ n \in \mathbb{Z} \right\},$$

$$E = \left\{ u \in S \ : \sum_{n \in \mathbb{Z}} [|\Delta u(n-1)|^p + a(n)|u(n)|^p] < +\infty \right\}$$

and for $u \in E$, let

$$\|u\| = \left\{ \sum_{n \in \mathbb{Z}} [|\Delta u(n-1)|^p + a(n)|u(n)|^p] \right\}^{1/p}. \tag{2.1}$$

Then E is a uniform convex Banach space with this norm, see [16].

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{p}\|u\|^p - \sum_{n\in\mathbb{Z}} W(n, u(n)). \tag{2.2}$$

If (A) and (W1) or (W1') or (W4) hold, then $I \in C^1(E, \mathbb{R})$ and one can easily check that

$$\begin{aligned} \langle I'(u), v\rangle = \sum_{n\in\mathbb{Z}} \big[&|\Delta u(n-1)|^{p-2}(\Delta u(n-1), \Delta v(n-1)) \\ &+a(n)|u(n)|^{p-2}(u(n), v(n)) \\ &-(\nabla W(n, u(n)), v(n))\big]. \end{aligned} \tag{2.3}$$

Furthermore, the critical points of I in E are classical solutions of (1.1) with $u(\pm\infty) = 0$.

We will obtain the critical points of I by using the Mountain Pass Theorem. We recall it and a minimization theorem as:

Lemma 2.1. ([4]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy (PS)-condition. Suppose that I satisfies the following conditions:*

(i) $I(0) = 0$*;*

(ii) *There exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;*

(iii) *There exists $e \in E \setminus \bar{B}_\rho(0)$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where $B_\rho(0)$ is an open ball in E of radius ρ centered at 0, and

$$\Gamma = \{g \in C([0, 1], E) \ : \ g(0) = 0, g(1) = e\}.$$

Lemma 2.2. ([24]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the (PS)-condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

Lemma 2.3. *For $u \in E$,*

$$a\|u\|_\infty^p \leq a\|u\|_{l^p}^p \leq \|u\|^p. \tag{2.4}$$

where $a = \min\{a(n) : n \in \mathbb{Z}\}$.

Proof. Since $u \in E$, it follows that $\lim_{|n|\to\infty} |u(n)| = 0$. Hence, there exists $n^* \in \mathbb{Z}$ such that

$$\|u\|_\infty = |u(n^*)| = \max_{n\in\mathbb{Z}} |u(n)|.$$

By (A) and (2.1), we have

$$\|u\|^p \geq \sum_{n\in\mathbb{Z}} a(n)|u(n)|^p \geq a \sum_{n\in\mathbb{Z}} |u(n)|^p \geq a\|u\|_\infty^p.$$

The proof is complete.

Lemma 2.4. *Assume that (W2) and (W3) or (W3') hold. Then for every* $(n, x) \in \mathbb{Z} \times \mathbb{R}^N$,

(i) $s^{-\mu} W_1(n, sx)$ *is nondecreasing on* $(0, +\infty)$*;*

(ii) $s^{-\varrho} W_2(n, sx)$ *is nonincreasing on* $(0, +\infty)$.

The proof of Lemma 2.4 is routine and so we omit it.

3. Proof of Theorems

Proof of Theorem 1.1. It is clear that $I(0) = 0$. We first show that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $c > 0$ such that

$$|I(u_k)| \leq c, \quad \|I'(u_k)\|_{E^*} \leq \mu c \quad \text{for } k \in \mathbb{N}. \tag{3.1}$$

From (2.1), (2.2), (3.1), (W2) and (W3), we obtain

$$\begin{aligned} & pc + pc\|u_k\| \\ \geq\ & pI(u_k) - \frac{p}{\mu}\langle I'(u_k), u_k\rangle \\ =\ & \frac{\mu - p}{\mu}\|u_k\|^p + p\sum_{n\in\mathbb{Z}}\left[W_2(n, u_k(n)) - \frac{1}{\mu}(\nabla W_2(n, u_k(n)), u_k(n))\right] \\ & -p\sum_{n\in\mathbb{Z}}\left[W_1(n, u_k(n)) - \frac{1}{\mu}(\nabla W_1(n, u_k(n)), u_k(n))\right] \\ \geq\ & \frac{\mu - p}{\mu}\|u_k\|^p, \quad k \in \mathbb{N}. \end{aligned}$$

It follows that there exists a constant $A > 0$ such that

$$\|u_k\| \leq A \quad \text{for } k \in \mathbb{N}. \tag{3.2}$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E. For any given number $\varepsilon > 0$, by (W1), we can choose $\xi > 0$ such that

$$|\nabla W(n,x)| \leq \varepsilon a(n)|x|^{p-1} \quad \text{for} \ \ n \in \mathbb{Z} \setminus J, \ \ \text{and} \ \ |x| \leq \xi. \tag{3.3}$$

Since $a(n) \to \infty$, we can also choose an integer $\Pi > \max\{|k| : k \in J\}$ such that

$$a(n) \geq \frac{A^p}{\xi^p}, \qquad |n| \geq \Pi. \tag{3.4}$$

By (2.4), (3.2) and (3.4), we have

$$\begin{aligned} |u_k(n)|^p &= \frac{1}{a(n)} a(n)|u_k(n)|^p \\ &\leq \frac{\xi^p}{A^p} \sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^p \\ &\leq \frac{\xi^p}{A^p} \|u_k\|^p \\ &\leq \xi^p \quad \text{for} \ \ |n| \geq \Pi, \ \ k \in \mathbb{N}. \end{aligned} \tag{3.5}$$

Since $u_k \rightharpoonup u_0$ in E, it is easy to verify that $u_k(t)$ converges to $u_0(t)$ pointwise for all $n \in \mathbb{Z}$, that is

$$\lim_{k\to\infty} u_k(n) = u_0(n), \qquad \forall\ n \in \mathbb{Z}, \tag{3.6}$$

Hence, we have by (3.5) and (3.6)

$$|u_0(n)| \leq \zeta \quad \text{for} \ \ |n| \geq \Pi. \tag{3.7}$$

It follows from (3.6) and the continuity of $\nabla W(n,x)$ on x that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=-\Pi}^{\Pi} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n))||u_k(n) - u_0(n)| < \varepsilon \quad \text{for} \ \ k \geq k_0. \tag{3.8}$$

On the other hand, it follows from (3.2), (3.3), (3.5), (3.6) and (3.7) that

$$\begin{aligned} &\sum_{|n|>\Pi} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n))||u_k(nt) - u_0(n)| \\ \leq &\sum_{|n|>\Pi} (|\nabla W(n,u_k(n))| + |\nabla W(n,u_0(n))|)(|u_k(n)| + |u_0(n)|) \\ \leq &\ \varepsilon \sum_{|n|>\Pi} a(n)(|u_k(n)|^{p-1} + |u_0(n)|^{p-1})(|u_k(n)| + |u_0(n)|) \\ \leq &\ 2\varepsilon \sum_{|n|>\Pi} a(n)(|u_k(n)|^p + |u_0(n)|^p) \\ \leq &\ 2\varepsilon(\|u_k\|^p + \|u_0\|^p) \\ \leq &\ 2\varepsilon(A^p + \|u_0\|^p), \qquad k \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Combining (3.8) with (3.9) we get

$$\sum_{n\in\mathbb{Z}} |\nabla W(n,u_k(n)) - \nabla W(n,u_0(n))|\,|u_k(n)-u_0(n)| \to 0 \quad \text{as} \quad k\to\infty. \tag{3.10}$$

Using the Hölder's inequality

$$ac+bd \le (a^p+b^p)^{1/p}(c^q+d^q)^{1/q},$$

where a,b,c,d are nonnegative numbers and $1/p+1/q=1, p>1$, it follows from (2.3) that

$$\begin{aligned}
&\langle I'(u_k)-I'(u_0), u_k-u_0\rangle \\
&= \sum_{n\in\mathbb{Z}} |\Delta u_k(n-1)|^{p-2}(\Delta u_k(n-1), \Delta u_k(n-1)-\Delta u_0(n-1)) \\
&\quad + \sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^{p-2}(u_k(n), u_k(n)-u_0(n)) \\
&\quad - \sum_{n\in\mathbb{Z}} |\Delta u_0(n-1)|^{p-2}(\Delta u_0(n-1), \Delta u_k(n-1)-\Delta u_0(n-1)) \\
&\quad - \sum_{n\in\mathbb{Z}} a(n)|u_0(n)|^{p-2}(u_0(n), u_k(n)-u_0(n)) \\
&\quad - \sum_{n\in\mathbb{Z}} (\nabla W(n,u_k(n)) - \nabla W(n,u_0(n)), u_k(n)-u_0(n)) \\
&= \|u_k\|^p + \|u_0\|^p - \sum_{n\in\mathbb{Z}} |\Delta u_k(n-1)|^{p-2}(\Delta u_k(n-1), \Delta u_0(n-1)) \\
&\quad - \sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^{p-2}(u_k(n), u_0(n)) \\
&\quad - \sum_{n\in\mathbb{Z}} |\Delta u_0(n-1)|^{p-2}(\Delta u_0(n-1), \Delta u_k(n-1)) \qquad (3.11)\\
&\quad - \sum_{n\in\mathbb{Z}} a(n)|u_0(n)|^{p-2}(u_0(n), u_k(n)) \\
&\quad - \sum_{n\in\mathbb{Z}} (\nabla W(n,u_k(n)) - \nabla W(n,u_0(n)), u_k(n)-u_0(n)) \\
&\ge \|u_n\|^p + \|u_0\|^p - \left(\sum_{n\in\mathbb{Z}} |\Delta u_0(n-1)|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} |\Delta u_k(n-1)|^p\right)^{1/q} \\
&\quad - \left(\sum_{n\in\mathbb{Z}} a(n)|u_0(n)|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^p\right)^{1/q} \\
&\quad - \left(\sum_{n\in\mathbb{Z}} |\Delta u_k(n-1)|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} |\Delta u_0(n-1)|^p\right)^{1/q} \\
&\quad - \left(\sum_{n\in\mathbb{Z}} a(n)|u_k(n)|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}} a(n)|u_0(n)|^p\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{n\in\mathbb{Z}}(\nabla W(n,u_k(n))-\nabla W(n,u_0(n)),u_k(n)-u_0(n))\\
\geq\ & \|u_k\|^p+\|u_0\|^p-\left(\sum_{n\in\mathbb{Z}}[|\Delta u_0(n-1)|^p+a(n)|u_0(n)|^p]\right)^{1/p}\\
&\left(\sum_{n\in\mathbb{Z}}[|\Delta u_k(n-1)|^p+a(n)|u_k(n)|^p]\right)^{1/q}\\
&-\left(\sum_{n\in\mathbb{Z}}[|\Delta u_k(n-1)|^p+a(n)|u_k(n)|^p]\right)^{1/p}\\
&\left(\sum_{n\in\mathbb{Z}}[|\Delta u_0(n-1)|^p+a(n)|u_0(n)|^p]\right)^{1/q}\\
&-\sum_{n\in\mathbb{Z}}(\nabla W(n,u_k(n))-\nabla W(n,u_0(n)),u_k(n)-u_0(n))\\
=\ & \|u_k\|^p+\|u_0\|^p-\|u_0\|\|u_k\|^{p-1}-\|u_k\|\|u_0\|^{p-1}\\
&-\sum_{n\in\mathbb{Z}}(\nabla W(n,u_k(n))-\nabla W(n,u_0(n)),u_k(n)-u_0(n))\\
=\ & \left(\|u_k\|^{p-1}-\|u_0\|^{p-1}\right)(\|u_k\|-\|u_0\|)\\
&-\sum_{n\in\mathbb{Z}}(\nabla W(n,u_k(n))-\nabla W(n,u_0(n)),u_k(n)-u_0(n)).
\end{aligned}
$$

Since $I'(u_k)\to 0$ as $k\to+\infty$ and $u_k\rightharpoonup u_0$ in E, it follows from (3.2) that

$$\langle I'(u_k)-I'(u_0),u_k-u_0\rangle\to 0 \quad \text{as} \quad k\to\infty,$$

which, together with (3.10) and (3.11), yields $\|u_k\|\to\|u\|$ as $k\to+\infty$. By the uniform convexity of E and the fact that $u_k\rightharpoonup u_0$ in E, it follows from the Kadec-Klee property [10] that $u_k\to u_0$ in E. Hence, I satisfies (PS)-condition.

We now show that there exist constants $\rho,\alpha>0$ such that I satisfies assumption (ii) of Lemma 2.1 with these constants. By (W1), there exists $\eta\in(0,1)$ such that

$$|\nabla W(n,x)|\leq\frac{1}{2}a(n)|x|^{p-1} \quad \text{for} \quad n\in\mathbb{Z}\setminus J,\quad |x|\leq\eta. \tag{3.12}$$

Since $W(n,0)=0$, it follows that

$$|W(n,x)|\leq\frac{1}{2p}a(n)|x|^{p} \quad \text{for} \quad n\in\mathbb{Z}\setminus J,\quad |x|\leq\eta. \tag{3.13}$$

Set

$$M=\sup\left\{\frac{W_1(n,x)}{a(n)}\ \middle|\ n\in J,\ x\in\mathbb{R}^N,\ |x|=1\right\}. \tag{3.14}$$

Set $\delta = \min\{1/(2pM+1)^{1/(\mu-p)}, \eta\}$. If $\|u\| = a^{1/p}\delta := \rho$, then by (2.4), $|u(n)| \leq \delta \leq \eta < 1$ for $n \in \mathbb{Z}$. By (3.14) and Lemma 2.4 (i), we have

$$\begin{aligned}\sum_{n\in J} W_1(n, u(n)) &\leq \sum_{\{n\in J,\ u(n)\neq 0\}} W_1\left(n, \frac{u(n)}{|u(n)|}\right) |u(n)|^\mu \\ &\leq M \sum_{n\in J} a(n)|u(n)|^\mu \\ &\leq M\delta^{\mu-p} \sum_{n\in J} a(n)|u(n)|^p \\ &\leq \frac{1}{2p} \sum_{n\in J} a(n)|u(n)|^p.\end{aligned} \tag{3.15}$$

Set

$$\alpha = \frac{a\delta^p}{2p}.$$

Hence, from (2.1), (3.13), (3.15) and (W3), we have

$$\begin{aligned}I(u) &= \frac{1}{p}\|u\|^p - \sum_{n\in\mathbb{Z}} W(n, u(n)) \\ &= \frac{1}{p}\|u\|^p - \sum_{n\in\mathbb{Z}\setminus J} W(n, u(n)) - \sum_{n\in J} W(n, u(n)) \\ &\geq \frac{1}{p}\|u\|^p - \frac{1}{2p} \sum_{n\in\mathbb{Z}\setminus J} a(n)|u(n)|^p - \sum_{n\in J} W_1(n, u(n)) \\ &\geq \frac{1}{p}\|u\|^p - \frac{1}{2p} \sum_{n\in\mathbb{Z}\setminus J} a(n)|u(n)|^p - \frac{1}{2p}\sum_{n\in J} a(n)|u(n)|^p \\ &= \frac{1}{p}\sum_{n\in\mathbb{Z}} |\Delta u(n-1)|^p + \frac{1}{2p}\sum_{n\in\mathbb{Z}} a(n)|u(n)|^p \\ &\geq \frac{1}{2p}\sum_{n\in\mathbb{Z}} [|\Delta u(n-1)|^p + a(n)|u(n)|^p] \\ &= \frac{1}{2p}\|u\|^p \\ &= \alpha.\end{aligned} \tag{3.16}$$

(3.16) shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$, i.e., I satisfies assumption (ii) of Lemma 2.1.

Finally, it remains to show that I satisfies assumption (iii) of Lemma 2.1. Take $\omega \in E$ such that

$$|\omega(n)| = \begin{cases} 1, & \text{for } |n| \leq 1, \\ 0, & \text{for } |n| \geq 2, \end{cases} . \tag{3.17}$$

and $|\omega(n)| \le 1$ for $|n| \in (1,2]$. For $\sigma > 1$, by Lemma 2.4 (i) and (3.17), we have

$$\sum_{n=-1}^{1} W_1(n, \sigma\omega(n)) \ge \sigma^{\mu} \sum_{n=-1}^{1} W_1(n, \omega(n)) = m\sigma^{\mu}, \tag{3.18}$$

where $m = \sum_{n=-1}^{1} W_1(n, \omega(n)) > 0$.

For any $u \in E$, it follows from (2.4) and Lemma 2.4 (ii) that

$$\begin{aligned} &\sum_{n=-2}^{2} W_2(n, u(n)) \\ &= \sum_{\{n\in[-2,2]\,:\,|u(n)|>1\}} W_2(n, u(n)) + \sum_{\{n\in[-2,2]\,:\,|u(n)|\le 1\}} W_2(n, u(n)) \\ &\le \sum_{\{n\in[-2,2]\,:\,|u(n)|>1\}} W_2\left(n, \frac{u(n)}{|u(n)|}\right) |u(n)|^{\varrho} \\ &\quad + \sum_{n=-2}^{2} \max_{|x|\le 1} |W_2(n,x)| \\ &\le \|u\|_{\infty}^{\varrho} \sum_{n=-2}^{2} \max_{|x|=1} |W_2(n,x)| + \sum_{n=-2}^{2} \max_{|x|\le 1} |W_2(n,x)| \\ &\le a^{-\frac{\varrho}{p}} \|u\|^{\varrho} \sum_{n=-2}^{2} \max_{|x|=1} |W_2(n,x)| + \sum_{n=-2}^{2} \max_{|x|\le 1} |W_2(n,x)| \\ &= M_1 \|u\|^{\varrho} + M_2, \end{aligned} \tag{3.19}$$

where

$$M_1 = a^{-\frac{\varrho}{p}} \sum_{n=-2}^{2} \max_{|x|=1} |W_2(n,x)|, \qquad M_2 = \sum_{n=-2}^{2} \max_{|x|\le 1} |W_2(n,x)|.$$

By (2.1), (3.17), (3.18) and (3.19), we have for $\sigma > 1$

$$\begin{aligned} I(\sigma\omega) &= \frac{1}{p}\|\sigma\omega\|^p + \sum_{n\in\mathbb{Z}} [W_2(n, \sigma\omega(n)) - W_1(n, \sigma\omega(n))] \\ &\le \frac{\sigma^p}{p}\|\omega\|^p + \sum_{n=-2}^{2} W_2(n, \sigma\omega(n)) - \sum_{n=-1}^{1} W_1(n, \sigma\omega(n)) \\ &\le \frac{\sigma^p}{p}\|\omega\|^p + M_1\sigma^{\varrho}\|\omega\|^{\varrho} + M_2 - m\sigma^{\mu}. \end{aligned} \tag{3.20}$$

Since $\mu > \varrho \ge p$ and $m > 0$, (3.20) implies that there exists $\sigma_0 > 1$ such that $\|\sigma_0\omega\| > \rho$ and $I(\sigma_0\omega) < 0$. Set $e = \sigma_0\omega(t)$. Then $e \in E$, $\|e\| = \|\sigma_0\omega\| > \rho$ and $I(e) = I(\sigma_0\omega) < 0$. By Lemma 2.1, I possesses a critical value $d \ge \alpha$ given by

$$d = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)), \tag{3.21}$$

where

$$\Gamma = \{g \in C([0,1], E) \; : \; g(0) = 0, \; g(1) = e\} .$$

Hence, there exists $u^* \in E$ such that

$$I(u^*) = d, \quad \text{and} \;\; I'(u^*) = 0. \tag{3.22}$$

Then function u^* is a desired classical solution of system (1.1). Since $d > 0$, u^* is a nontrivial homoclinic solution. The proof is complete.

Proof of Theorem 1.2. In the proof of Theorem 1.1, the condition that $W_2(n,x) \geq 0$ in (W3) is only used in the proofs of (3.2) and assumption (ii) of Lemma 2.1. Therefore, we only prove (3.2) and assumption (ii) of Lemma 2.1 still holds use (W1') and (W3') instead of (W1) and (W3). We first prove that (3.2) still holds. From (2.1), (2.2), (3.1), (W2) and (W3'), we obtain

$$\begin{aligned}
& pc + \frac{pc\mu}{\varrho}\|u_k\| \\
\geq\; & pI(u_k) - \frac{p}{\varrho}\langle I'(u_k), u_k\rangle \\
=\; & \frac{\varrho - p}{\varrho}\|u_k\|^p + p\sum_{n\in\mathbb{Z}}\left[W_2(n, u_k(n)) - \frac{1}{\varrho}(\nabla W_2(n, u_k(n)), u_k(n))\right] \\
& -p\sum_{n\in\mathbb{Z}}\left[W_1(n, u_k(n)) - \frac{1}{\varrho}(\nabla W_1(n, u_k(n)), u_k(n))\right] \\
\geq\; & \frac{\varrho - p}{\varrho}\|u_k\|^p, \qquad k \in \mathbb{N}.
\end{aligned}$$

It follows that there exists a constant $A > 0$ such that (3.2) holds. Next, we prove that assumption (ii) of Lemma 2.1 still holds. By (W1'), there exists $\eta \in (0,1)$ such that

$$|\nabla W(n,x)| \leq \frac{1}{2}a(n)|x|^{p-1} \quad \text{for} \;\; n \in \mathbb{Z}, \;\; |x| \leq \eta. \tag{3.23}$$

Since $W(n,0) = 0$, it follows that

$$|W(n,x)| \leq \frac{1}{2p}a(n)|x|^p \quad \text{for} \;\; n \in \mathbb{Z}, \;\; |x| \leq \eta. \tag{3.24}$$

If $\|u\| = a^{1/p}\eta := \rho$, then by (2.4), $|u(n)| \leq \eta$ for $n \in \mathbb{Z}$. Set

$$\alpha = \frac{a\eta^p}{2p}.$$

Hence, from (2.1) and (3.24), we have

$$
\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\
&\geq \frac{1}{p}\|u\|^p - \frac{1}{2p}\sum_{n\in\mathbb{Z}} a(n)|u(n)|^p \\
&= \frac{1}{p}\sum_{n\in\mathbb{Z}} |\Delta u(n-1)|^p + \frac{1}{2p}\sum_{n\in\mathbb{Z}} a(n)|u(n)|^p \\
&\geq \frac{1}{2p}\sum_{n\in\mathbb{Z}} [|\Delta u(n-1)|^p + a(n)|u(n)|^p] \\
&= \frac{1}{2p}\|u\|^p \\
&= \alpha.
\end{aligned} \tag{3.25}
$$

(3.25) shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$, i.e., assumption (ii) of Lemma 2.1 holds. The proof of Theorem 1.2 is completed.

Proof of Theorem 1.3. In view of Lemma 2.2, $I \in C^1(E, \mathbb{R})$. In what follows, we first show that I is bounded from below. By (W4), (2.4) and Hölder inequality, we have

$$
\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\
&= \frac{1}{p}\|u\|^p - \sum_{\mathbb{Z}(|u(n)|\leq 1)} W(n,u(n)) - \sum_{\mathbb{Z}(|u(n)|>1)} W(n,u(n)) \\
&\geq \frac{1}{p}\|u\|^p - \sum_{\mathbb{Z}(|u(n)|\leq 1)} a_1(n)|u(n)|^{\gamma_1} - \sum_{\mathbb{Z}(|u(n)|>1)} a_2(n)|u(n)|^{\gamma_2} \\
&\geq \frac{1}{p}\|u\|^p - a^{-\gamma_1/p}\left(\sum_{\mathbb{Z}(|u(n)|\leq 1)} |a_1(n)|^{p/(p-\gamma_1)}\right)^{(p-\gamma_1)/p} \\
&\quad \left(\sum_{\mathbb{Z}(|u(n)|\leq 1)} a(n)|u(n)|^p\right)^{\gamma_1/p} \\
&\quad -a^{-\gamma_1/p}\left(\sum_{\mathbb{Z}(|u(n)|>1)} |a_2(n)|^{p/(p-\gamma_1)}\right)^{(p-\gamma_1)/p} \\
&\quad \times \left(\sum_{\mathbb{Z}(|u(n)|>1)} |u(n)|^{p(\gamma_2-\gamma_1)/\gamma_1} a(n)|u(n)|^p\right)^{\gamma_1/p}
\end{aligned} \tag{3.26}
$$

$$\begin{aligned}
&\geq \frac{1}{p}\|u\|^p - a^{-\gamma_1/p}\left(\sum_{\mathbb{Z}(|u(n)|\leq 1)} |a_1(n)|^{p/(p-\gamma_1)}\right)^{(p-\gamma_1)/p}\|u\|^{\gamma_1}\\
&\quad -a^{-\gamma_1/p}\|u\|_\infty^{\gamma_2-\gamma_1}\left(\sum_{\mathbb{Z}(|u(n)|>1)} |a_2(n)|^{p/(p-\gamma_1)}\right)^{(p-\gamma_1)/p}\|u\|^{\gamma_1}\\
&\geq \frac{1}{p}\|u\|^p - a^{-\gamma_1/p}\left(\sum_{\mathbb{Z}(|u(n)|\leq 1)} |a_1(n)|^{p/(p-\gamma_1)}\right)^{(p-\gamma_1)/p}\|u\|^{\gamma_1}\\
&\quad -a^{-\gamma_2/p}\left(\sum_{\mathbb{Z}(|u(n)|>1)} |a_2(n)|^{p/(p-\gamma_1)}\right)^{(p-\gamma_1)/p}\|u\|^{\gamma_2}\\
&\geq \frac{1}{p}\|u\|^p - a^{-\gamma_1/p}\|a_1\|_{p/(p-\gamma_1)}\|u\|^{\gamma_1} - a^{-\gamma_2/p}\|a_2\|_{p/(p-\gamma_1)}\|u\|^{\gamma_2}.
\end{aligned}$$

Since $1 < \gamma_1 < \gamma_2 < p$, (3.26) implies that $I(u) \to +\infty$ as $\|u\| \to +\infty$. Consequently, I is bounded from below.

Next, we prove that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then by (2.4) and (3.26), there exists a constant $A > 0$ such that

(3.27) $$\|u_k\|_\infty \leq a^{-1/p}\|u_k\| \leq A, \qquad k \in \mathbb{N}.$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u_0$ in E. It is easy to verify that $u_k(n)$ converges to $u_0(n)$ pointwise for all $n \in \mathbb{Z}$, that is

(3.28) $$\lim_{k\to\infty} u_k(n) = u_0(n), \qquad \forall\, n \in \mathbb{Z}.$$

Hence, we have by (3.27)

(3.29) $$\|u_0\|_\infty \leq A.$$

By (W5), there exists $M_2 > 0$ such that

(3.30) $$\varphi(|x|) \leq M_2|x|^{\gamma_1-1}, \qquad \forall\, x \in \mathbb{R}^N, \quad |x| \leq A.$$

For any given number $\varepsilon > 0$, by (W5), we can choose an integer $\Pi > 0$ such that

(3.31) $$\left(\sum_{|n|>\Pi} (b(n))^{p/(p-\gamma_1)}\right)^{(p-\gamma_1)/p} < \varepsilon.$$

It follows from (3.28) and the continuity of $\nabla W(n, x)$ on x that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=-\Pi}^{\Pi} |\nabla W(n, u_k(n)) - \nabla W(n, u_0(n))||u_k(n) - u_0(n)| < \varepsilon \quad \text{for } k \geq k_0. \tag{3.32}$$

On the other hand, it follows from (2.4), (3.27), (3.29), (3.30), (3.31) and (W5) that

$$\begin{aligned} &\sum_{|n|>\Pi} |\nabla W(n, u_k(n)) - \nabla W(n, u_0(n))||u_k(n) - u_0(n)| \\ &\leq \sum_{|n|>\Pi} b(n)[\varphi(|u_k(n)|) + \varphi(|u_0(n)|)](|u_k(n)| + |u_0(n)|) \\ &\leq M_2 \sum_{|n|>\Pi} b(n)(|u_k(n)|^{\gamma_1 - 1} + |u_0(n)|^{\gamma_1 - 1})(|u_k(n)| + |u_0(n)|) \\ &\leq 2M_2 \sum_{|n|>\Pi} b(n)(|u_k(n)|^{\gamma_1} + |u_0(n)|^{\gamma_1}) \\ &\leq 2M_2 a^{-\gamma_1/p} \left(\sum_{|n|>\Pi} |b(n)|^{p/(p-\gamma_1)} \right)^{(p-\gamma_1)/p} (\|u_k\|^{\gamma_1} + \|u_0\|^{\gamma_1}) \\ &\leq 2M_2 a^{-\gamma_1/p} \left(\sum_{|n|>\Pi} |b(n)|^{p/(p-\gamma_1)} \right)^{(p-\gamma_1)/p} \left[a^{\gamma_1/p} A^{\gamma_1} + \|u_0\|^{\gamma_1} \right] \\ &\leq 2M_2 a^{-\gamma_1/p} \left[a^{\gamma_1/p} A^{\gamma_1} + \|u_0\|^{\gamma_1} \right] \varepsilon, \qquad k \in \mathbb{N}. \end{aligned} \tag{3.33}$$

Since ε is arbitrary, combining (3.32) with (3.33), we get

$$\sum_{n \in \mathbb{Z}} (\nabla W(n, u_k(n)) - \nabla W(n, u_0(n)), u_k(n) - u_0(n)) \to 0 \quad \text{as } k \to \infty. \tag{3.34}$$

Similar to the proof of Theorem 1.1, It follows from (3.11) that

$$\begin{aligned} &\langle I'(u_k) - I'(u_0), u_k - u_0 \rangle \\ \geq\ & \left(\|u_k\|^{p-1} - \|u_0\|^{p-1} \right) (\|u_k\| - \|u_0\|) \\ &- \sum_{n \in \mathbb{Z}} (\nabla W(n, u_k(n)) - \nabla W(n, u_0(n)), u_k(n) - u_0(n)). \end{aligned} \tag{3.35}$$

Since $\langle I'(u_k) - I'(u_0), u_k - u_0 \rangle \to 0$, it follows from (3.34) and (3.35) that $u_k \to u_0$ in E. Hence, I satisfies (PS)-condition.

By Lemma 2.2, $c = \inf_E I(u)$ is a critical value of I, that is there exists a critical point $u^* \in E$ such that $I(u^*) = c$.

Finally, we show that $u^* \neq 0$. Let $u_0(n_0) = (1, 0, \cdots, 0)^\top \in \mathbb{R}^N$ and $u_0(n) = 0$ for $n \neq n_0$. Then by (W6), we have

$$
\begin{aligned}
I(su_0) &= \frac{s^p}{p}\|u_0\|^p - \sum_{n\in\mathbb{Z}} W(n, su_0(n)) \\
&= \frac{s^p}{p}\|u_0\|^p - W(n_0, su_0(n)) \\
&\leq \frac{s^p}{p}\|u_0\|^p - \eta s^{\gamma_3}|u_0(n)|^{\gamma_3}, \qquad 0 < s < 1.
\end{aligned} \tag{3.36}
$$

Since $1 < \gamma_3 < p$, it follows from (3.36) that $I(su_0) < 0$ for $s > 0$ small enough. Hence $I(u^*) = c < 0$, therefore u^* is nontrivial critical point of I, and so $u^* = u^*(n)$ is a nontrivial homoclinic solution of (1.1). The proof is complete.

Proof of Corollary 1.1. Obviously, (W7) and (W8) imply (W4) holds, and (W7) and (W9) imply (W5) holds with $a_1(n) = a_2(n) = b(n) = |a(n)|$. In addition, by (W7) and (W8), we have

$$
W(n_0, x) = a(n_0)V(x) \geq M' a(n_0)|x|^{\gamma_3}, \qquad \forall\ x \in \mathbb{R}^N, \quad |x| \leq 1.
$$

This shows that (W7) holds also. Hence, by Theorem 1.3, the conclusion of Corollary 1.1 is true. The proof is complete.

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. Consider the second-order discrete p-Laplacian system

$$
\Delta(|\Delta u(n-1)|\Delta u(n-1)) - a(n)|u(n)|u(n) + \nabla W(n, u(n)) = 0, \tag{4.1}
$$

where $p = 3,\ n \in \mathbb{Z}, u \in \mathbb{R}^N,\ a \in C(\mathbb{Z}, (0, \infty))$ such that $a(n) \to +\infty$ as $|n| \to \infty$. Let

$$
W(n, x) = a(n)\left(\sum_{i=1}^{m} a_i|x|^{\mu_i} - \sum_{j=1}^{n} b_j|x|^{\varrho_j}\right),
$$

where $\mu_1 > \mu_2 > \cdots > \mu_m > \varrho_1 > \varrho_2 > \cdots > \varrho_n > 3,\ a_i, b_j > 0,\ i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Let $\mu = \mu_m,\ \varrho = \varrho_1$, and

$$
W_1(n, x) = a(n)\sum_{i=1}^{m} a_i|x|^{\mu_i}, \qquad W_2(n, x) = a(n)\sum_{j=1}^{n} b_j|x|^{\varrho_j}.
$$

Then it is easy to verify that all conditions of Theorem 1.1 are satisfied. By Theorem 1.1, system (1.1) possess a nontrivial homoclinic solution.

Example 4.2. Consider the second-order discrete p-Laplacian system

$$\Delta((\Delta u(n-1))^3) - a(n)(u(n))^3 + \nabla W(n, u(n)) = 0, \tag{4.2}$$

where $p = 4$, $n \in \mathbb{Z}, u \in \mathbb{R}^N$, $a \in C(\mathbb{Z}, (0, \infty))$ such that $a(n) \to +\infty$ as $|n| \to \infty$. Let

$$W(n, x) = a(n)\left[a_1|x|^{\mu_1} + a_2|x|^{\mu_2} - b_1(\sin n)|x|^{\varrho_1} - b_2|x|^{\varrho_2}\right],$$

where $\mu_1 > \mu_2 > \varrho_1 > \varrho_2 > 4$, $a_1, a_2 > 0, b_1, b_2 > 0$. Let $\mu = \mu_2$, $\varrho = \varrho_1$, and

$$W_1(n, x) = a(n)\left(a_1|x|^{\mu_1} + a_2|x|^{\mu_2}\right), \quad W_2(n, x) = a(n)\left[b_1(\sin n)|x|^{\varrho_1} + b_2|x|^{\varrho_2}\right].$$

Then it is easy to verify that all conditions of Theorem 1.2 are satisfied. By Theorem 1.2, system (1.1) possess a nontrivial homoclinic solution.

Example 4.3. Consider the second-order discrete p-Laplacian system

$$\Delta(\varphi_p(\Delta u(n-1))) - a(n)|u(n)|^{p-2}u(n) + \nabla W(n, u(n)) = 0, \tag{4.3}$$

where $p > 3/2$, $n \in \mathbb{Z}, u \in \mathbb{R}^N$, $a \in C(\mathbb{Z}, (0, \infty))$ such that $a(n) \to +\infty$ as $|n| \to \infty$. Let

$$W(n, x) = \frac{\cos n}{1 + |n|}|x|^{4/3} + \frac{\sin n}{1 + |n|}|x|^{3/2}.$$

Then

$$\nabla W(n, x) = \frac{4\cos n}{3(1 + |n|)}|x|^{-2/3}x + \frac{3\sin n}{2(1 + |n|)}|x|^{-1/2}x,$$

$$|W(n, x)| \leq \frac{2|x|^{4/3}}{1 + |n|}, \quad \forall\ (n, x) \in \mathbb{Z} \times \mathbb{R}^N, \ |x| \leq 1,$$

$$|W(n, x)| \leq \frac{2|x|^{3/2}}{1 + |n|}, \quad \forall\ (n, x) \in \mathbb{Z} \times \mathbb{R}^N, \ |x| \geq 1$$

and

$$|\nabla W(n, x)| \leq \frac{8|x|^{1/3} + 9|x|^{1/2}}{6(1 + |n|)}, \quad \forall\ (n, x) \in \mathbb{Z} \times \mathbb{R}^N.$$

We can choose n_0 such that

$$\cos n_0 > 0, \quad \sin n_0 > 0.$$

Let

$$\eta = \frac{\cos n_0 + \sin n_0}{1 + |n_0|}.$$

Then

$$W(n_0, x) \geq \eta |x|^{3/2}, \quad \forall \ x \in \mathbb{R}^N, \ \ |x| \leq 1.$$

These show that all conditions of Theorem 1.3 are satisfied, where

$$1 < \tfrac{4}{3} = \gamma_1 < \gamma_2 = \gamma_3 = \frac{3}{2} < p,$$
$$a_1(n) = a_2(n) = b(n) = \frac{2}{1+|n|}, \quad \varphi(s) = \frac{8s^{1/3} + 9s^{1/2}}{12}.$$

By Theorem 1.3, system (1.1) has at least a nontrivial homoclinic solution.

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