CYCLE ADJACENCY OF PLANAR GRAPHS AND 3-COLOURABILITY

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Abstract. Suppose $G$ is a planar graph. Let $H_G$ be the graph with vertex set $V(H_G) = \{C : C$ is a cycle of $G$ with $|C| \in \{4, 6, 7\}\}$ and $E(H_G) = \{C_iC_j : C_i$ and $C_j$ are adjacent in $G\}$. We prove that if any 3-cycles and 5-cycles are not adjacent to $i$-cycles for $3 \leq i \leq 7$, and $H_G$ is a forest, then $G$ is 3-colourable.

1. Introduction

As every planar graph is 4-colourable, a natural question is which planar graphs are 3-colourable. It is known [10] that to decide whether a planar graph is 3-colourable is NP-complete. So attention is concentrated in finding sufficient conditions for planar graphs to be 3-colourable. By Grötzsch Theorem, triangle-free planar graphs are 3-colourable. In 1976, Steinberg conjectured that every planar graph without 4- and 5-cycles is 3-colourable (see [11]). This conjecture has received a lot of attention and there are many partial results and related open problems. Erdős (see [13]) suggested the following relaxation of Steinberg’s conjecture: Determine the minimum integer $k$, if it exists, such that every planar graph without cycles of length $l$ for $4 \leq l \leq k$ is 3-colourable. Abbott and Zhou [1] proved that such a $k$ exists and $k \leq 11$. This result was improved to $k \leq 10$ in [2], then to $k \leq 9$ in [3, 12], and to $k \leq 7$ in [7].

The following theorems were proved by Borodin et al. in [7].

**Theorem 1.1.** Every planar graph without cycles of length from 4 to 7 is 3-colourable.

For the purpose of using induction, instead of proving Theorem 1.1 directly, they proved the following stronger statement.
Theorem 1.2. Suppose $G$ is a planar graph without cycles of length 4 to 7 and $f_0$ is a face of $G$ of length $8 \leq i \leq 11$. Then every proper 3-colouring of the vertices of $f_0$ can be extended to a proper 3-colouring of $G$.

The distance between two cycles $C, C'$ of a graph $G$ is the shortest distance between vertices of $C$ and $C'$. Two cycles are adjacent if they have at least one edge in common. Havel asked in 1969 the question whether there is a constant $C$ such that every planar graph with minimum distance between triangles at least $C$ is 3-colourable. This question also remains open. However, it was proved in [9] that if a planar graph $G$ has no 5-cycles and every two triangles have distance at least 4, then $G$ is 3-colourable. This distance requirement between triangles is reduced to 3 in [4, 14] and then to 2 in [5]. These results motivated the following two conjectures:

Conjecture 1.3. ([9]). Every planar graph without 5-cycles and without adjacent triangles is 3-colourable.

Conjecture 1.4. ([6]). Every planar graph without triangles adjacent to cycles of length 3 or 5 is 3-colourable.

Conjecture 1.4 is stronger than Conjecture 1.3, and Conjecture 1.3 is stronger than Steinberg’s conjecture. These conjectures remain unsettled and stimulate the study of 3-colourability of planar graphs which satisfy specific adjacency relations among short cycles. In [8], it was proved that if $G$ is a planar graph in which no $i$-cycle is adjacent to a $j$-cycle whenever $3 \leq i \leq j \leq 7$, then $G$ is 3-colourable.

In this paper, we consider planar graphs in which cycles of lengths 4, 6, 7 may be adjacent to each other, but the adjacency is rather limited. For a planar graph $G$, let $H_G$ be the graph with vertex set $V(H_G) = \{C : C$ is a cycle of $G$ with $|C| \in \{4, 6, 7\}\}$ and $E(H_G) = \{C_iC_j : C_i$ and $C_j$ are adjacent in $G\}$. We prove the following result:

Theorem 1.5. For a planar graph $G$, if any 3-cycles and 5-cycles are not adjacent to $i$-cycles whenever $3 \leq i \leq 7$, and $H_G$ is a forest, then $G$ is 3-colourable.

2. Proof of Theorem 1.5

For a face $f$, denote by $b(f)$ the set of edges on the boundary of $f$. A $k$-vertex is a vertex of degree $k$. A $k$-face is a face $f$ with $|b(f)| = k$. For a vertex $v$, $N(v)$ denotes the set of neighbors of $v$. For a cycle $C$ of $G$, $int(C)$ and $ext(C)$ denote the sets of vertices lie in the interior and exterior of $C$, respectively. A cycle $C$ is called a separating cycle if $int(C) \neq \emptyset$ and $ext(C) \neq \emptyset$. Let $c_i(G)$ be the number of cycles of length $i$ in $G$. If $u, v$ are two vertices on $C$, we use $C[u, v]$ to denote the path of $C$ clockwisely from $u$ to $v$, and let $C(u, v) = C[u, v] \setminus \{u, v\}$.
\[ C[u, v] = C[u, v] \setminus \{v\}, \ C(u, v) = C[u, v] \setminus \{u\}. \] For each path \( P \) and cycle \( C \), we denote by \(|P|\) and \(|C|\) the number of vertices of \( P \) and \( C \). Let \( \Omega \) be the set of connected planar graphs satisfying the assumption of Theorem 1.5.

Theorem 1.5 follows from the following lemma:

**Lemma 2.1.** Suppose \( G \in \Omega \) and \( f_0 \) is an \( i \)-face of \( G \) with \( 3 \leq i \leq 11 \). Then every proper 3-colouring of the vertices of \( f_0 \) can be extended to the whole \( G \).

If Lemma 2.1 is true, then for any \( G \in \Omega \), either \( G \) has no triangles, and hence by Grötzsch theorem, \( G \) is 3-colourable, or \( G \) has a triangle \( C \), and it follows from Lemma 2.1 that any proper 3-colouring of \( C \) can be extended to a proper 3-colouring of the interior as well as of the exterior of \( C \). So it remains to prove Lemma 2.1.

Assume the lemma is not true and \( G \) is a counterexample with

1. \( c(G) = c_4(G) + c_5(G) + c_6(G) + c_7(G) \) is minimum.
2. subject to (1), \(|V(G)| + |E(G)|\) is minimum.

Assume the unbounded face \( f^* \) is an \( i \)-face with \( 3 \leq i \leq 11 \) and \( \phi \) is a proper 3-colouring of the vertices of \( f^* \) which cannot be extended to \( G \). Let \( C^* \) be the boundary cycle of \( f^* \).

By the minimality of \( G \), \( G \) is 2-connected, and hence each face is a cycle. Moreover, each vertex \( v \in \text{int}(C^*) \) has degree at least 3, for otherwise, one can first extend the colouring of \( C^* \) to \( G - v \), and then extend it to \( v \). Also \( G \) has no separating cycles of length \( 3 \) to \( 11 \), because if \( C \) is such a cycle, then we can first extend \( \phi \) to \( G \setminus \text{int}(C) \). Then extend this colouring to \( G \setminus \text{ext}(C) \). Therefore, \( G \) has a proper 3-colouring.

Observe that \( C^* \) has no chord, because if \( e = uv \) is a chord of \( C^* \), then \( G - e \) is a smaller counterexample. Moreover, any cycle of \( G \) of length \( 4 \leq i \leq 7 \) has no chord, for otherwise, we either have a 3-cycle or a 5-cycle adjacent to an \( i \)-cycle for some \( 3 \leq i \leq 7 \), or we have two 4-cycles and a 6-cycle that are pairwise adjacent (so these three cycles form a cycle in \( H_G \), contrary to our assumption).

If \( 4 \leq |C^*| \leq 7 \), then let \( G' \) be the graph obtained from \( G \) by adding \( 11 - |C^*| \) vertices on one edge of \( C^* \). Then \( c(G') < c(G) \) and \( G' \in \Omega \). The colouring of \( C^* \) can be easily extended to the added degree 2 vertices. By the minimality of \( G \), the colouring of the outer cycle of \( G' \) can be extended to a 3-colouring of \( G' \). Hence, \( G \) is 3-colourable, contrary to our assumption. Thus we may assume that \(|C^*| \neq 4, 5, 6, 7\).

**Claim 1.** For each internal face \( f \), there exists another internal face \( f' \) such that \( f \) and \( f' \) have exactly one edge in common. Moreover, any two internal \( k \)-faces with \( 4 \leq k \leq 7 \) have at most one edge in common.

**Proof.** Let \( f \) be an internal face of \( G \) and let \( C \) be the boundary cycle of \( f \). Certainly there is another internal face adjacent to \( f \). Assume for each internal face
Suppose 4 ≤ i, j ≤ 7 and there exist an internal i-face f and an internal j-face f′ such that e1, e2 ∈ b(f) ∩ b(f′). If e1 ∩ e2 = ∅, then e1 ∩ e2 is an internal 2-vertex. If e1 ∩ e2 = ∅, then there are three cycles of length between 3 and 7 adjacent to each other, again contrary to our assumption.

Claim 2. Suppose f is an internal k-face with 4 ≤ k ≤ 7 and C = b(f). If |V(f) ∩ C*| ≥ 2 and u, v ∈ V(f) ∩ C*, then either C[u, v] or C[v, u] is a segment of C*.

Proof. Suppose none of C[u, v] and C[v, u] is a segment of C*. Then C[u, v] ∪ C*[v, u] and C[v, u] ∪ C*[u, v] are separating cycles. Let q = |C(u, v)|, p = |C(v, u)|. Since any separating cycle has length at least 12, it follows that |C*| ≥ (12 − p) + (12 − q) − 2 = 22 − (p + q) > 11, contrary to our assumption.

Claim 3. G contains no internal k-faces with 4 ≤ k ≤ 7.

Proof. Suppose G contains an internal k-face for some k ∈ {4, 5, 6, 7}. Since H(G) is acyclic, there is an internal k1-face f1 with k1 ∈ {4, 5, 6, 7} such that f1 is adjacent to at most one face of length 4 to 7.

If f1 is adjacent to a face of length 4 to 7, then let f2 to be the unique face adjacent to f1 of length k2 ∈ {4, 5, 6, 7}. Otherwise let f2 to be a face which has exactly one edge in common with f1. Let C1, C2 be the boundary cycles of f1, f2, respectively.

By Claim 1, C1 ∩ C2 contains exactly one edge xy. For i = 1, 2, let ui be the other neighbour of x in Ci, and let vi be the other neighbour of y in Ci.

Since C* has no chord, at most one of x, y belong to C*. First we consider the case that one of x, y, say x, lies on C*. If u1 /∈ C* or N(y) ∩ C* = {x}, then let G′ be the graph obtained from G by identifying u1 and y into a vertex u*. It is easy to see that G′ ∈ Ω, and c(G′) ≤ c(G) and |V(G′)| + |E(G′)| < |V(G)| + |E(G)|. By the minimality of G, the colouring of C* can be extended to a proper 3-colouring φ of G′. By assigning the colour of u* to u1 and y, we obtain a proper 3-colouring of G that is an extension of the colouring of C*. This is in contrary to our assumption. So we have u1 ∈ C* and N(y) ∩ C* − {x} = ∅.
If \( v_1 \in C^* \), then by Claim 2, \( C_1[v_1, u_1] = C^*[v_1, u_1] \). If \( C_2(x, y) \not\subset C^* \), then \( C^* = C^*[x, v_1] \cup v_1yx \) is a separating cycle. But \( |C^*| \leq |C^*| \leq 11 \), which is a contradiction. If \( C_2(x, y) \subset C^* \), then \( v_2 \in C^* \). Since \( f_1 \) is adjacent to at most one face of length 4 to 7, so \( |C^*(v_2, v_1)| \geq 5 \). If each of \( f_1, f_2 \) has length at least 6, then \( |C^*[v_1, v_2]| \geq 9 \). If \( f_1 \) has length 4, then \( f_2 \) has length at least 6; If \( f_1 \) has length 5, then \( f_2 \) has length at least 8; If \( f_1 \) has length 6, then \( f_2 \) has length at least 4, for otherwise we would have two 4-cycles and a 6-cycle that are pairwise adjacent, in contrary to our assumption. This implies that \( |C^*[v_1, v_2]| \geq 7 \). In any case, this is a contradiction as \( |C^*| \leq 11 \). Thus we assume that \( v_1 \not\in C^* \).

Let \( t \in N(y) \cap C^* \setminus \{x\} \). Since \( v_1 \not\in C^* \), \( C^*[t, x] \cup xyt \) is a separating cycle. This implies that \( |C^*[t, x]| \geq 11 \). Since \( f_1 \) is not adjacent to a 3-cycle, \( |C^*[x, t]| \geq 3 \), contrary to the assumption that \( |C^*| \leq 11 \).

Suppose \( C^* \cap \{x, y\} = \emptyset \). If \( u_1 \not\in C^* \), then identify \( u_1 \) and \( y \). If \( v_1 \not\in C^* \), then identify \( v_1 \) and \( x \). By the minimality of \( G \), the resulting graph \( G' \) has a proper 3-colouring which is an extension of the colouring of \( C^* \). This induces a proper 3-colouring of \( G \) which is an extension of the colouring of \( C^* \). Thus we assume \( u_1, v_1 \in C^* \).

If there exists \( t \in C^* \cap N(x) \setminus \{u_1\} \), then \( |C^*[u_1, t]| \geq 7 \) and \( |C^*[t, v_1]| \geq 6 \), otherwise \( f_1 \) is adjacent to another cycle of length at most 7. Similarly, if there exists \( t \in C^* \cap N(y) \setminus \{v_1\} \), then \( |C^*[u_1, t]| \geq 6 \) and \( |C^*[t, v_1]| \geq 7 \). In both cases we have \( |C^*| \geq 12 \), which is a contradiction. So we assume \( C^* \cap N(x) = \{u_1\} \) and \( C^* \cap N(y) = \{v_1\} \). In particular, \( u_2 \not\in C^* \) and \( v_2 \not\in C^* \). If \( |f_1| \geq 6 \), then \( C^*[u_1, v_1] \cup v_1yxu_1 \) is a separating cycle. This implies that \( |C^*[u_1, v_1]| \geq 10 \) and \( |C^*| \geq 12 \), which is a contradiction. If \( |f_1| = 4 \), then we identify \( u_1 \) and \( y \). Hence \( G \) has a proper 3-colouring by minimality. If \( |f_1| = 5 \), let \( C_1 \setminus \{u_1, v_1, x, y\} = \{t\} \), then we identify \( t \) and \( x \). Hence \( G \) has a proper 3-colouring by minimality, this is a contradiction. This complete the proof of Claim 3.

Since \( |C^*| \neq 4, 5, 6, 7 \), and \( G \) has no separating cycles of length 3 to 11. Claim 3 implies that \( G \) has no cycles of length 4 to 7. If \( 8 \leq |C^*| \leq 11 \), then by applying Theorem 1.2, we can extend the 3-colouring of \( C^* \) to the whole \( G \). If \( |C^*| = 3 \), then by applying Theorem 1.1, \( G \) is 3-colourable, and we can extend the 3-colouring of \( C^* \) to the whole \( G \) by permuting the colours. Hence this means that there is no counterexample. This complete the proof of Lemma 2.1.

REFERENCES

3. O. V. Borodin, Structural properties of plane graphs without adjacent triangles and application to 3-colourings, J. Graph Theory, 21 (1996), 183-186.


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