TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 3, pp. 1115-1128, June 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

EIGENVALUES OF THE LAPLACE OPERATOR WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. An eigenvalue problem on a bounded domain for the Laplacian with a nonlinear Robin-like boundary condition is investigated. We prove the existence, isolation and simplicity of the first two eigenvalues.

1. Introduction

Assume $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. We consider the following class of eigenvalue problems

$$
(1) \qquad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ -\dfrac{\partial u}{\partial \nu} \in \beta(u) & \text{on } \partial\Omega\,, \end{cases}
$$

where $\lambda \in \mathbb{R}$, $\beta : D(\beta) \subset \mathbb{R} \to \mathbb{R}$ is a maximal monotone mapping, and $\partial u/\partial \nu$ denotes the outward normal derivative of u. To our knowledge, such problems have not been much discussed so far in the literature. On the other hand, it is worth pointing out that eigenvalue problems are always important, particularly in analyzing more complicated equations. We just remember the recent advances in [1, 2, 4, 7, 9-14].

In this paper we consider a particular nonlinearity, $\beta(r) = \alpha r_+$, where α is a positive constant, and $r_+ := \max\{r, 0\}$ for all $r \in \mathbb{R}$. Therefore, problem (1) becomes

$$
(2) \qquad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ -\dfrac{\partial u}{\partial \nu} = \alpha u_+ & \text{on } \partial\Omega\,. \end{cases}
$$

Received November 3, 2009, accepted December 6, 2009.
Communicated by B. Ricceri.
2000 *Mathematics Subject Classification*: 35P30, 35D05, 35J60, 35J70, 58E05.
Key words and phrases: Eigenvalue problem, Nonlinear boundary condition, Isolated eigenvalue, Rayleigh principle.
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The natural space for nonlinear eigenvalue problems of the type (1) is the Sobolev space $H^1(\Omega)$. Recall that if $u \in H^1(\Omega)$ then $u_+,\ u_- \in H^1(\Omega)$ and

$$\nabla u_+ = \begin{cases} 0, & \text{if} \quad [u \le 0] \\ \nabla u, & \text{if} \quad [u > 0]\,, \end{cases} \qquad \nabla u_- = \begin{cases} 0, & \text{if} \quad [u \ge 0] \\ \nabla u, & \text{if} \quad [u < 0]\,, \end{cases}$$

(see, e.g. [6, Theorem 7.6]), where $u_\pm(x) = \max\{\pm u(x), 0\}$ for a.e. $x \in \Omega$.

We will say that $\lambda \in \mathbb{R}$ is an *eigenvalue* of problem (2) if there exists $u \in H^1(\Omega) \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla \varphi \, dx + \alpha \int_{\partial\Omega} u_+ \varphi \, d\sigma(x) = \lambda \int_\Omega u\varphi \, dx\,, \tag{3}$$

for any $\varphi \in H^1(\Omega)$. Such a function u will be called an *eigenfunction* corresponding to the eigenvalue λ. In fact, u is more regular. Indeed, it is known (see [3, Proposition 2.9, p. 63]) that $A = -\Delta$ with $D(A) = \{u \in H^2(\Omega);\ -\partial u/\partial\nu \in \beta(u)\ a.a.\ x \in \partial\Omega\}$ is a maximal (cyclically) monotone operator in $L^2(\Omega)$, and moreover there exist some constants $C_1,\ C_2 > 0$ such that

$$\|v\|_{H^2(\Omega)} \le C_1 \|v - \Delta v\|_{L^2(\Omega)} + C_2,\ \forall v \in D(A).$$

Therefore, if u is an eigenfunction of problem (2) corresponding to some λ, then it is easy to see that the (unique) solution of the equation $v + Av = f$, where $f = (1+\lambda)u$, is $v = u$, thus $u \in H^2(\Omega)$, and

$$\|u\|_{H^2(\Omega)} \le C_1 |1+\lambda| \cdot \|u\|_{L^2(\Omega)} + C_2. \tag{4}$$

Note that u satisfies problem (2) in a classical sense.

Define

$$\lambda_1 = \inf_{v \in H^1(\Omega)\setminus\{0\},\ \int_\Omega v\, dx \ge 0} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx + \alpha \int_{\partial\Omega} v_+^2 \, d\sigma(x)}{\displaystyle\int_\Omega v^2 \, dx}\,. \tag{5}$$

The main result of this paper is given by the following theorem.

Theorem 1. *The numbers $\lambda_0 = 0$ and λ_1 (defined by relation (5)) represent the first two eigenvalues of problem (2), provided that $\alpha > 0$ is small. They are isolated in the set of eigenvalues of problem (2). Moreover, the sets of eigenfunctions corresponding to λ_0 and λ_1 are positive cones (more precisely, one-dimensional half-spaces) in $H^1(\Omega)$.*

The study of problem (2) is motivated by many applications. It is worth pointing out that we obtain in the next section a Rayleigh type principle: for $\alpha > 0$ small the first nontrivial eigenvalue λ_1 is a minimum value of the Rayleigh quotient associated with the corresponding classical Robin problem.

2. Proof of the Main Result

Lemma 1. *No* $\lambda < 0$ *can be an eigenvalue of problem* (2).

Proof. Assume $\lambda \in \mathbb{R}$ is an eigenvalue of problem (2) with the corresponding eigenfunction $u \in H^1(\Omega) \setminus \{0\}$. Taking $\varphi = u$ in (3) we find

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} u_+^2 \, d\sigma(x)}{\displaystyle\int_\Omega u^2 \, dx} \geq 0 \, . \qquad \blacksquare$$

Lemma 2. $\lambda_0 = 0$ *is an eigenvalue of problem* (2) *and the set of its corresponding eigenfunctions is given by all the negative real constants.*

Proof. The first part of the lemma is obvious. Let us now consider $u \in H^1(\Omega) \setminus \{0\}$ an eigenfunction corresponding to λ_0. Taking $\varphi = u$ in relation (3) we deduce that

$$\int_\Omega |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} u_+^2 \, d\sigma(x) = 0 \, .$$

Therefore, $\int_\Omega |\nabla u|^2 \, dx = \int_{\partial\Omega} u_+^2 \, d\sigma(x) = 0$. Consequently, u should be a negative real number. $\blacksquare$

Lemma 3. λ_0 *is isolated in the set of eigenvalues of problem (2).*

Proof. Assume by contradiction that λ_0 is not isolated. Then there exists a sequence of positive eigenvalues of problem (2), say (λ_n), such that $\lambda_n \searrow 0$. For each n we denote by u_n the corresponding eigenfunction of λ_n. Since we deal with a homogeneous problem we can assume that for each n we have $\|u_n\|_{L^2(\Omega)} = 1$. Relation (3) implies that for each n we have

$$\int_\Omega \nabla u_n \nabla \varphi \, dx + \alpha \int_{\partial\Omega} (u_n)_+ \varphi \, d\sigma(x) = \lambda_n \int_\Omega u_n \varphi \, dx \, , \tag{6}$$

for any $\varphi \in H^1(\Omega)$. Taking $\varphi = u_n$ in relation (6) we find

$$\int_\Omega |\nabla u_n|^2 \, dx + \alpha \int_{\partial\Omega} (u_n)_+^2 \, d\sigma(x) = \lambda_n \int_\Omega u_n^2 \, dx = \lambda_n \, . \tag{7}$$

We deduce that (u_n) is bounded in $H^1(\Omega)$. In fact, by estimate (4) with $\lambda = \lambda_n$ and $u := u_n$, it follows that (u_n) is bounded in $H^2(\Omega)$. Consequently, there exists $u \in H^2(\Omega)$ such that, on a subsequence, u_n converges strongly to u in $H^1(\Omega)$ and in $L^2(\partial\Omega)$ as well. Furthermore, $(u_n)_+$ converges strongly to u_+ in $L^2(\partial\Omega)$.

The above pieces of information lead to

$$\begin{aligned} &\int_\Omega |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} u_+^2 \, d\sigma(x) \\ &= \lim_{n\to\infty} \left[\int_\Omega |\nabla u_n|^2 \, dx + \alpha \int_{\partial\Omega} (u_n)_+^2 \, d\sigma(x) \right] = \lim_{n\to\infty} \lambda_n = 0 \,. \end{aligned}$$

Thus, $\int_\Omega |\nabla u|^2 \, dx = 0$ and $\int_{\partial\Omega} u_+^2 \, d\sigma(x) = 0$. It follows that u is a negative constant satisfying $\|u\|_{L^2(\Omega)} = 1$. More precisely, $u = -1/|\Omega|^{1/2}$.

Turning back, relation (6) with $\varphi = u$ implies

$$\lambda_n \int_\Omega u_n u \, dx = -\alpha \frac{1}{|\Omega|^{1/2}} \int_{\partial\Omega} (u_n)_+ \, d\sigma(x) \le 0, \quad \text{for all } n \,.$$

It follows that

$$\int_\Omega u_n \, dx \ge 0, \quad \text{for all } n \,,$$

which implies

$$\int_\Omega u \, dx \ge 0 \,.$$

This contradicts the fact that u is a negative constant. Consequently, the result of Lemma 3 holds true. ∎

Remark 1. Let us assume that $\lambda > 0$ is an eigenvalue of problem (2) with the corresponding eigenfunction u. Taking $\varphi \equiv 1$ in relation (3) it follows that

$$\alpha \int_{\partial\Omega} u_+ \, d\sigma(x) = \lambda \int_\Omega u \, dx \,,$$

which implies that

$$\int_\Omega u \, dx \ge 0 \,.$$

Thus, the nonzero eigenvalues of problem (2) have the corresponding eigenfunctions in the cone

$$\mathcal{C} = \left\{ w \in H^1(\Omega); \ \int_\Omega w \, dx \ge 0 \right\} \,.$$

Consequently, the definition of λ_1 given in relation (5) is natural (we will prove later that for $\alpha > 0$ small enough λ_1 is an eigenvalue of problem (2)).

Lemma 4. *There exists* $u \in \mathcal{C} \setminus \{0\}$ *such that*

$$\lambda_1 = \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} u_+^2 \, d\sigma(x)}{\displaystyle\int_\Omega u^2 \, dx} .$$

Proof. Let $(u_n) \subset \mathcal{C} \setminus \{0\}$ be a minimizing sequence for λ_1, i.e.

$$\frac{\displaystyle\int_\Omega |\nabla u_n|^2 \, dx + \alpha \int_{\partial\Omega} (u_n)_+^2 \, d\sigma(x)}{\displaystyle\int_\Omega u_n^2 \, dx} \to \lambda_1 ,$$

as $n \to \infty$. We can assume that $\|u_n\|_{L^2(\Omega)} = 1$ for all n. It follows that u_n is bounded in $H^1(\Omega)$. Thus, there exists $u \in H^1(\Omega)$ such that (a subsequence of) u_n converges weakly to u in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$. It follows that $\|u\|_{L^2(\Omega)} = 1$, i.e. $u \neq 0$, and $\int_\Omega u \, dx \geq 0$. Thus, $u \in \mathcal{C} \setminus \{0\}$. The above pieces of information combined with the weak lower semicontinuity of the L^2-norm imply

$$\int_\Omega |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} u_+^2 \, d\sigma(x) \leq \lim_{n\to\infty} \left[\int_\Omega |\nabla u_n|^2 \, dx + \alpha \int_{\partial\Omega} (u_n)_+^2 \, d\sigma(x) \right] = \lambda_1 .$$

Since $\|u\|_{L^2(\Omega)} = 1$ the above inequality and the definition of λ_1 show that the conclusion of Lemma 4 holds true. ■

Remark 2. We point out the fact that $\lambda_1 > 0$. Indeed, assuming by contradiction that $\lambda_1 = 0$ then by Lemma 4 there exists $u \in \mathcal{C} \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} u_+^2 \, d\sigma(x) = 0 .$$

It follows that u is a negative constant with $\int_\Omega u \, dx \geq 0$, a contradiction. Consequently $0 = \lambda_0 < \lambda_1$. Moreover, it is trivial to see that no $\lambda \in (0, \lambda_1)$ can be an eigenvalue of problem (2).

In the following we show that for $\alpha > 0$ small enough λ_1 is an eigenvalue of problem (2). In order to do that we denote for $\alpha \in (-\epsilon, \infty)$, with $\epsilon > 0$ small enough,

$$\lambda_1(\alpha) = \inf_{u \in \mathcal{C}\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} u_+^2 \, d\sigma(x)}{\displaystyle\int_\Omega u^2 \, dx} ,$$

and

$$\mu_1(\alpha) = \inf_{u \in H^1(\Omega)\setminus\{0\},\, \int_\Omega u\, dx = 0} \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx + \alpha \int_{\partial\Omega} u_+^2\, d\sigma(x)}{\displaystyle\int_\Omega u^2\, dx}\,.$$

It is clear that for all $\alpha > 0$ we have $\mu_1(\alpha) \geq \lambda_1(\alpha)$, but, it is not obvious if either $\mu_1(\alpha) > \lambda_1(\alpha)$ or $\mu_1(\alpha) = \lambda_1(\alpha)$. However, we are able to prove the following result:

Lemma 5. *For any $\alpha > 0$ small enough we have $\mu_1(\alpha) > \lambda_1(\alpha)$.*

Proof. Obviously, for all $\alpha \geq 0$, both $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are finite. This property extends for $\alpha \in (-\epsilon, 0)$, with $\epsilon > 0$, small enough. Indeed, for all $u \in H^1(\Omega)$ with $\|u\|_{L^2(\Omega)} = 1$, we have (by the continuity of the trace operator)

$$\int_{\partial\Omega} u_+^2\, d\sigma(x) \leq \int_{\partial\Omega} u^2\, d\sigma(x) \leq C\left(\int_\Omega |\nabla u|^2\, dx + 1\right),$$

where C is a positive constant. Therefore,

$$\int_\Omega |\nabla u|^2\, dx + \alpha \int_{\partial\Omega} u_+^2\, d\sigma(x) \geq (1+\alpha C)\int_\Omega |\nabla u|^2\, dx + \alpha C \geq -\epsilon C\,,$$

for all $\alpha \in (-\epsilon, 0)$, $u \in H^1(\Omega)$ with $\|u\|_{L^2(\Omega)} = 1$, provided that $\epsilon > 0$ satisfies $1 - \epsilon C \geq 0$. Thus, both $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are well defined for $\alpha \in (-\epsilon, \infty)$. (Even more, a similar proof as the one used in Lemma 4 shows that both $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are attained.)

Now, let us point out the fact that the functions $\lambda_1(\alpha)$, $\mu_1(\alpha) : (-\epsilon, \infty) \to \mathbb{R}$ are concave functions. Clearly, for any $\varphi \in \mathcal{C} \setminus \{0\}$ the function

$$(-\epsilon, \infty) \ni \alpha \longrightarrow \frac{\displaystyle\int_\Omega |\nabla \varphi|^2\, dx + \alpha \int_{\partial\Omega} \varphi_+^2\, d\sigma(x)}{\displaystyle\int_\Omega \varphi^2\, dx}\,,$$

is an affine function, consequently, a concave function. Since the infimum of a family of concave functions is a concave function, it follows that $\lambda_1(\alpha)$ is concave. Similarly, $\mu_1(\alpha)$ is also concave. Thus, we deduce that $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are continuous functions for $\alpha \in (-\epsilon, \infty)$. On the other hand, $\lambda_1(0) = 0$ and $\mu_1(0) = \lambda_{1,N}$, where 0 and $\lambda_{1,N}$ are the first two eigenvalues of the Neumann problem (see, e.g. [5, Chapter 4.2.1]), i.e.

$$\text{(8)} \qquad \begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

It is well-known that $\lambda_{1,N} > 0$ (see, [5, Proposition 4.2.2 and Proposition 4.2.3]). Thus, we found $\lambda_1(0) < \mu_1(0)$. This inequality and the fact that $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ are continuous functions for $\alpha \in (-\epsilon, \infty)$ imply that $\lambda_1(\alpha) < \mu_1(\alpha)$ for any $\alpha > 0$, small enough. The proof of Lemma 5 is complete. ∎

Lemma 6. *Assume that $u \in \mathcal{C} \setminus \{0\}$ is a minimizer for the infimum given by relation (5), with $\int_\Omega u\, dx > 0$. Then λ_1 is an eigenvalue of problem (2) and u is an eigenfunction corresponding to λ_1.*

Proof. Let $\varphi \in H^1(\Omega)$ be fixed. Then for any ϵ lying in a small neighborhood of the origin, we have $\int_\Omega (u + \epsilon\varphi)\, dx > 0$, i.e. $u + \epsilon\varphi \in \mathcal{C}$. Define the function

$$f(\epsilon) = \frac{\displaystyle\int_\Omega |\nabla(u + \epsilon\varphi)|^2\, dx + \alpha \int_{\partial\Omega} (u + \epsilon\varphi)_+^2\, d\sigma(x)}{\displaystyle\int_\Omega (u + \epsilon\varphi)^2\, dx}\,.$$

Clearly, f is well defined in a small neighborhood of the origin and possesses a minimum in $\epsilon = 0$. Consequently,

$$f'(0) = 0\,,$$

or, by some simple computations,

$$\int_\Omega \nabla u \nabla\varphi\, dx + \alpha \int_{\partial\Omega} u_+\varphi\, d\sigma(x) = \lambda_1 \int_\Omega u\varphi\, dx\,.$$

Clearly the above equality holds true for any $\varphi \in H^1(\Omega)$. We deduce that u is an eigenfunction corresponding to the eigenvalue λ_1, and the proof of Lemma 6 is complete. ∎

Proposition 1. *The number λ_1, defined by relation (5), is an eigenvalue of problem (2), provided that $\alpha > 0$ is small enough.*

Proof. The conclusion of Proposition 1 is a simple consequence of Lemmas 4, 5 and 6. ∎

Lemma 7. *If λ_1 is an eigenvalue of problem (2) and $u \in H^1(\Omega) \setminus \{0\}$ is an eigenfunction corresponding to λ_1, then $u \geq 0$ in Ω (thus, $\int_\Omega u\, dx > 0$).*

Proof. Relation (3) shows that

$$\int_\Omega \nabla u \nabla\varphi\, dx + \alpha \int_{\partial\Omega} u_+\varphi\, d\sigma(x) = \lambda_1 \int_\Omega u\varphi\, dx\,, \tag{9}$$

for every $\varphi \in H^1(\Omega)$. First, we claim that $u_+ \neq 0$. Indeed, assuming the contrary, we deduce that

$$\int_{\Omega} \nabla u_- \nabla \varphi \, dx = \lambda_1 \int_{\Omega} u_- \varphi \, dx \,, \tag{10}$$

for every $\varphi \in H^1(\Omega)$. Taking $\varphi = 1$ we find

$$\int_{\Omega} u_- \, dx = 0 \,,$$

that means, $u_- = 0$ and thus $u = 0$, a contradiction. Consequently, $u_+ \neq 0$. Then, taking $\varphi = u_+$ in (9) we have

$$\lambda_1 = \frac{\displaystyle\int_{\Omega} |\nabla u_+|^2 \, dx + \alpha \int_{\partial\Omega} u_+^2 \, d\sigma(x)}{\displaystyle\int_{\Omega} u_+^2 \, dx} \,.$$

By Lemma 6 we infer that u_+ is an eigenfunction corresponding to λ_1, or

$$\int_{\Omega} \nabla u_+ \nabla \varphi \, dx + \alpha \int_{\partial\Omega} u_+ \varphi \, d\sigma(x) = \lambda_1 \int_{\Omega} u_+ \varphi \, dx \,, \tag{11}$$

for every $\varphi \in H^1(\Omega)$. Relations (9) and (11) imply that relation (10) holds true. Taking again $\varphi = 1$ in (10) we find again $\int_{\Omega} u_- \, dx = 0$ which leads to $u_- = 0$ in Ω. The proof of Lemma 7 is complete. ■

Remark 3. By Lemma 7, if λ_1 is an eigenvalue of problem (2), then it is the first eigenvalue of the following Robin problem

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega, \\ -\dfrac{\partial u}{\partial \nu} = \alpha u & \text{on } \partial\Omega \,. \end{cases} \tag{12}$$

In the following we argue that fact in detail. It is well-known that the number

$$\gamma_1 = \inf_{v \in H^1(\Omega) \setminus \{0\}} \frac{\displaystyle\int_{\Omega} |\nabla v|^2 \, dx + \alpha \int_{\partial\Omega} v^2 \, d\sigma(x)}{\displaystyle\int_{\Omega} v^2 \, dx} \,,$$

known as the Rayleigh quotient, is positive and represents the first eigenvalue of problem (12). Moreover, γ_1 is simple, that means, all the associated eigenfunctions are merely multiples of each other. It is also known that these eigenfunctions belong to $C(\overline{\Omega}) \cap C^1(\Omega)$ (see [4, Lemma 2.1]). Furthermore, an eigenfunction of γ_1 can be chosen with a single sign, particularly with positive sign (see, e.g. [7]). The definitions of γ_1 and λ_1 show that $\gamma_1 \geq \lambda_1$. Actually, by Lemma 7 we have $\lambda_1 = \gamma_1$, i.e. λ_1 is the first eigenvalue of problem (12). Thus, the set of eigenfunctions corresponding to λ_1 is a positive cone in $H^1(\Omega)$. More precisely, if u is a positive eigenfunction for the Robin problem, associated with γ_1, then the set of eigenfunctions for problem (2), associated with λ_1 ($= \gamma_1$), is the one dimensional half-space $\{tu; t > 0\}$. Hence λ_1 is simple.

Finally, we focus our attention on proving that λ_1 is isolated. We will use a technique borrowed from [2] that will be described in the following.

Lemma 8. *Assume $\lambda > 0$ is an eigenvalue of problem (2) and $u \in H^1(\Omega)\setminus\{0\}$ is an eigenfunction corresponding to λ. Define $\Omega_- = \{x \in \Omega;\ \ u(x) < 0\}$. If $|\Omega_-| > 0$ then there exists a positive constant C (independent of λ and u) such that*

$$((\lambda + 1)C)^{-N/2} \leq |\Omega_-| \, .$$

Proof. Recalling again relation (3) we have

$$\int_\Omega \nabla u \nabla \varphi \, dx + \alpha \int_{\partial\Omega} u_+ \varphi \, d\sigma(x) = \lambda \int_\Omega u\varphi \, dx \, ,$$

for every $\varphi \in H^1(\Omega)$. Taking $\varphi = u_-$ we find

$$\int_\Omega |\nabla u_-|^2 \, dx = \lambda \int_\Omega u_-^2 \, dx \, ,$$

or by taking into account that $L^{2^\star}(\Omega)$ is continuously embedded in $L^2(\Omega)$, where $2^\star = 2N/(N-2)$ is the critical Sobolev exponent, we deduce by the Hölder's inequality

$$\int_\Omega |\nabla u_-|^2 \, dx + \int_\Omega u_-^2 \, dx = (\lambda+1) \int_\Omega u_-^2 \, dx \leq (\lambda+1)\|u_-\|^2_{L^{p^\star}(\Omega)} |\Omega_-|^{1-2/2^\star} \, .$$

Next, since $H^1(\Omega)$ is continuously embedded in $L^{2^\star}(\Omega)$ we deduce that there exists a positive constant C such that

$$\|v\|^2_{L^{2^\star}(\Omega)} \leq C \left(\int_\Omega |\nabla v|^2 \, dx + \int_\Omega v^2 \, dx \right) \, ,$$

for any $v \in H^1(\Omega)$. The last two inequalities imply

$$1 \leq (\lambda+1) C |\Omega_-|^{2/N} \, .$$

The proof of Lemma 8 is complete. ∎

Lemma 9. λ_1 *is isolated in the set of eigenvalues of problem* (2).

Proof. By Remark 2 it is clear that λ_1 is isolated from the left. We show that it is also isolated from the right. Assume by contradiction that this is not the case. Then there exists a sequence of positive eigenvalues of problem (2), say (λ_n), such that $\lambda_n \searrow \lambda_1$. For each n we denote by u_n an eigenfunction corresponding to λ_n. Since we deal with a homogeneous problem we can assume that for each n we have $\|u_n\|_{L^2(\Omega)} = 1$. Relation (3) implies that for each n we have

$$\int_\Omega \nabla u_n \nabla \varphi \, dx + \alpha \int_{\partial\Omega} (u_n)_+ \varphi \, d\sigma(x) = \lambda_n \int_\Omega u_n \varphi \, dx\,, \tag{13}$$

for any $\varphi \in H^1(\Omega)$. Arguing as in the proof of Lemma 3, we deduce that (u_n) is bounded in $H^2(\Omega)$. Consequently, there exists $u \in H^2(\Omega)$ such that u_n converges, on a subsequence, to u in $H^1(\Omega)$ and in $L^2(\partial\Omega)$ as well. Furthermore, we also have $(u_n)_+$ converges strongly to u_+ in $L^2(\partial\Omega)$. Passing to the limit as $n \to \infty$ in (13) we get

$$\int_\Omega \nabla u \nabla \varphi \, dx + \alpha \int_{\partial\Omega} (u)_+ \varphi \, d\sigma(x) = \lambda_1 \int_\Omega u \varphi \, dx\,, \tag{14}$$

for any $\varphi \in H^1(\Omega)$. Since $\|u\|_{L^2(\Omega)}$=1 it follows that $u \neq 0$ and thus, it is an eigenfunction corresponding to λ_1. By Lemma 7 we deduce that $u \geq 0$ in Ω. In fact, according to Remark 3, $u \in C(\overline{\Omega}) \cap C^1(\Omega)$ and $u(x) > 0$ for all $x \in \Omega$. Let now $\epsilon > 0$ be arbitrary but fixed and let $K \subset \Omega$ be a compact such that $|\Omega \setminus K| < \epsilon/2$. Obviously, there exists a $\delta > 0$ (depending on K) such that $u(x) \geq \delta > 0$ for every $x \in K$.

On the other hand, it is clear that u_n converges to u a.e. in Ω and thus, in K. Consequently, by the Egorov's Theorem (see, e.g. [15, Théorème 2.37]) we deduce that for $\epsilon > 0$ fixed above there exists a measurable set $\omega \subset K$ with $|\omega| < \epsilon/2$ such that u_n converges uniformly to u on $K \setminus \omega$. Since $u \geq \delta > 0$ in K we deduce that for any n large enough we have $u_n \geq 0$ on $K \setminus \omega$. For each n we define $(\Omega_n)_- = \{x \in \Omega; \;\; u_n(x) < 0\}$. We can assume that for each n $|(\Omega_n)_-| > 0$. Indeed, otherwise there exists a particular n for which we have $u_n \geq 0$ (and $u_n \neq 0$) in Ω. Taking $\varphi = u$ in (13) and $\varphi = u_n$ in (14) we deduce that

$$\lambda_n \int_\Omega u_n u \, dx = \lambda_1 \int_\Omega u u_n \, dx\,.$$

Since $\int_\Omega u u_n \; dx > 0$ the above equality leads to $\lambda_n = \lambda_1$ which represents a contradiction with the fact that $\lambda_n > \lambda_1$. Consequently, we have $|(\Omega_n)_-| > 0$ for all n. It follows that for any n large enough we have $(\Omega_n)_- \subset \omega \cup (\Omega \setminus K)$. Using the above facts and Lemma 8 we have that the following inequalities hold true

$$((\lambda_n + 1)C)^{-N/2} \leq |(\Omega_n)_-| \leq |\omega| + |\Omega \setminus K| < \epsilon\,,$$

provided that n is large enough. Therefore

$$((\lambda_1 + 1)C)^{-N/2} \leq \epsilon,$$

for all $\epsilon > 0$, which is impossible. Consequently, the conclusion of Lemma 9 holds true. ∎

3. Final Comments

In this section we point out some facts that are direct consequences of the discussion presented in the above sections.

First, we highlight the fact that for any $\alpha > 0$ the number $\gamma_1 = \gamma_1(\alpha)$, introduced in Remark 3 and which represents the first eigenvalue of the Robin problem (that is problem (12)) is an eigenvalue of problem (2). The above assertion is a consequence of the fact that there exists $u \in H^1(\Omega) \setminus \{0\}$ with $u \geq 0$ a.e. in Ω such that

$$\int_\Omega \nabla u \nabla \varphi \, dx + \alpha \int_{\partial\Omega} u\varphi \, d\sigma(x) = \gamma_1 \int_\Omega u\varphi \, dx\,,$$

for all $\varphi \in H^1(\Omega)$. Since $u \geq 0$ a.e. in Ω it follows that actually relation (3) is verified with $\lambda = \gamma_1$. The definitions of $\lambda_1(\alpha)$ and $\gamma_1(\alpha)$ imply that for any $\alpha > 0$ we have $\gamma_1(\alpha) \geq \lambda_1(\alpha)$. Moreover, by Remark 3 we know that for $\alpha > 0$ *small enough* we have $\gamma_1(\alpha) = \lambda_1(\alpha)$. However, we cannot conclude that for *any* $\alpha > 0$ we have $\gamma_1(\alpha) = \lambda_1(\alpha)$.

Second, we focus our attention on the numbers $\lambda_1(\alpha)$ and $\mu_1(\alpha)$ defined after Remark 2. It is clear that for *all* $\alpha > 0$ we have $\mu_1(\alpha) \geq \lambda_1(\alpha)$. Moreover, for $\alpha > 0$ *small enough*, by Lemma 5, we have that $\mu_1(\alpha) > \lambda_1(\alpha)$ and $\lambda_1(\alpha)$ is an eigenvalue of problem (2) (see Lemma 6). On the other hand, nothing is clear if $\alpha > 0$ is far from the origin. At least theoretically it may happen that for some $\alpha > 0$ large $\mu_1(\alpha) = \lambda_1(\alpha)$. In that case the reasoning from Lemma 6 does not work and consequently we cannot state whether $\lambda_1(\alpha)$ is an eigenvalue or not. However, we can show the following result which is undoubtedly connected with the above discussion:

Proposition 2. *If there exists $\alpha > 0$ for which any minimizer $u \in \mathcal{C} \setminus \{0\}$ of $\lambda_1(\alpha)$ satisfies $\int_\Omega u \, dx = 0$ then $\lambda_1(\alpha)$ is not an eigenvalue of problem* (2).

Proof. Assume, by contradiction, that $\lambda_1(\alpha)$ is an eigenvalue of problem (2). Then, any eigenfunction u corresponding to $\lambda_1(\alpha)$ is a minimizer with $\int_\Omega u \, dx = 0$. On the other hand, by Lemma 7 we have $\int_\Omega u \, dx > 0$, a contradiction. The proof of Proposition 2 is complete. ∎

Define

$$V = \{u \in H^1(\Omega);\ \int_\Omega u \, dx = 0\}\,.$$

Clearly, $H^1(\Omega) = V \oplus \mathbb{R}$ and $V \subset \mathcal{C}$. It seems that for some $\alpha > 0$ large $\lambda_1(\alpha)$ is attained on V, i.e., $\lambda_1(\alpha) = \mu_1(\alpha)$. In this case, by Proposition 2, $\lambda_1(\alpha)$ is not an eigenvalue of problem (2). Since in general $\lambda_1 \leq \gamma_1$, we would have in this case $\lambda_1(\alpha) < \gamma_1(\alpha)$.

A similar proof as the one of Lemma 4 shows that for each $\alpha > 0$ there exists $v_\alpha \in V \setminus \{0\}$ a minimizer of $\mu_1(\alpha)$. Moreover, as in Lemma 6 it can be proved that for v_α given above we have

$$\int_\Omega \nabla v_\alpha \nabla \varphi \, dx + \alpha \int_{\partial\Omega} (v_\alpha)_+ \varphi \, d\sigma(x) = \mu_1(\alpha) \int_\Omega v_\alpha \varphi \, dx \,, \tag{15}$$

for all $\varphi \in V$. However, the above relation is not enough to state that $\mu_1(\alpha)$ is an eigenvalue of problem (2) in the sense of the definition given by relation (3).

In connection with the above discussion, let us introduce the following definition: we say that $\lambda > 0$ is an *extended eigenvalue* of problem (2) if there exists $u \in \mathcal{C} \setminus \{0\}$ such that

$$\int_\Omega \nabla u \nabla(\varphi - u) \, dx + \alpha \int_{\partial\Omega} u_+(\varphi - u) \, d\sigma(x) \geq \lambda \int_\Omega u(\varphi - u) \, dx \,, \tag{16}$$

for all $\varphi \in \mathcal{C}$. It is obvious that the *classical* eigenvalues of problem (2) (given by relation (3)) are also extended eigenvalues. On the other hand, it is also clear that $\mu_1(\alpha)$ is an extended eigenvalue of problem (2), for any $\alpha > 0$. Thus, relation (16) gives a connection between $\lambda_1(\alpha)$ and $\mu_1(\alpha)$. In fact, if $u \in \mathcal{C} \setminus \{0\}$ is an extended eigenfunction corresponding to some extended eigenvalue $\lambda > 0$ of problem (2), then either u is an interior point of $\mathcal{C}$ (i.e., $u = u_1 + c$, for some $u_1 \in V$ and $c > 0$) so that λ is a classical eigenvalue, or $u \in V \setminus \{0\}$ and $v = u$ satisfies (15).

It is also worth pointing out the fact that since problem (2) has a nonlinear boundary condition, the study of the existence of other eigenvalues (different from λ_0 and $\lambda_1(\alpha)$) is more difficult than in the case of problems involving linear boundary conditions. Methods which are usually used fail in this case. In this context, we just notice that we cannot apply the Ljusternik-Schnirelman theory in this case, since the Euler-Lagrange energetic functional associated with problem (2) is not *even*, a crucial condition required by the application of the quoted method. However, in the 1-dimensional case the existence of infinitely many eigenvalues can be easily stated. Note that problem (2) with $\Omega = (0,1)$ becomes

$$\begin{cases} -u''(t) = \lambda u(t) & \text{for } t \in (0,1)\,, \\ u'(0) = \alpha u_+(0), \quad -u'(1) = \alpha u_+(1)\,. \end{cases} \tag{17}$$

On the other hand, it is known (see, e.g., [8, p. 10]) that the 1-dimensional Neumann problem

$$\begin{cases} -u''(t) = \lambda u(t) & \text{for } t \in (0,1)\,, \\ u'(0) = u'(1) = 0\,, \end{cases} \tag{18}$$

has the eigenvalues $\mu_k = k^2\pi^2$, $k = 0, 1, ...$, with the corresponding eigenfunctions $u_k(t) = -\cos(k\pi t)$. Simple computations show that for each $k \in \mathbb{Z}_+$, μ_{2k} is an eigenvalue of problem (17) with the corresponding eigenfunction u_{2k}.

Finally, let us point out that all the discussion on problem (2) presented in this paper can be extended (by using similar arguments) to the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in} \quad \Omega, \\ -|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \alpha u_+^{p-1} & \text{on } \partial\Omega\,, \end{cases}$$

where $p \in (1, N)$ is a real number and $\Delta_p \cdot = \operatorname{div}(|\nabla \cdot|^{p-2}\nabla \cdot)$ stands for the p-Laplace operator.

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