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BLOW-UP FOR PARABOLIC EQUATIONS AND SYSTEMS WITH NONNEGATIVE POTENTIAL

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Abstract. We study the blow-up behaviors of two parabolic problems on a bounded domain. One is the heat equation with nonlinear memory and the other is a parabolic system with power nonlinearity in which the coefficients of the reaction terms (potentials) are nonnegative and spatially inhomogeneous. Our aim is to show that any zero of the potential, where there is no reaction, is not a blow-up point, if the solution is monotone in time. We also give sufficient conditions for the time monotonicity of solutions.

1. Introduction

In this paper, we consider the nonlocal parabolic problem:

$$\begin{cases} u_t = \Delta u + \mu(x) \displaystyle\int_0^t u^p(x,s)\,ds, & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \bar{\Omega}, \\ u(x,t) = 0, & x \in \partial\Omega,\, t > 0 \end{cases} \tag{1.1}$$

and the following parabolic system:

$$\begin{cases} u_t = \Delta u + \mu(x) v^p & x \in \Omega, t > 0, \\ v_t = \Delta v + \mu(x) u^q & x \in \Omega, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & x \in \bar{\Omega}, \\ u(x,t) = 0,\ v(x,t) = 0, & x \in \partial\Omega,\, t > 0, \end{cases} \tag{1.2}$$

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where Ω is a bounded smooth domain, μ is Hölder continuous in $\bar{\Omega}$, $\mu(x) \geq 0$, $\mu(x) \not\equiv 0$, $p > 1$, $q > 1$ and $u_0, v_0 \geq 0$, $u_0, v_0 \not\equiv 0$ are smooth functions with $u_0, v_0|_{\partial\Omega} = 0$. We also assume that all zeros of $\mu(x)$ are included in Ω.

It is known that for each initial datum u_0 as above, (1.1) has a nonnegative classical solution u for $t \in [0, T)$ for some $T \in (0, \infty]$. If $T < \infty$, then we have

$$\limsup_{t \to T} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$$

and we say that the solution u *blows up* in finite time with the *blow-up time* T. For a given solution u that blows up at $t = T < \infty$, a point $a \in \bar{\Omega}$ is called a blow-up point if there exists a sequence $\{(x_n, t_n)\}$ in $Q_T := \Omega \times (0, T)$ such that $x_n \to a$, $t_n \uparrow T$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$. The set of all blow-up points is called the *blow-up set*.

Let us recall known results about the problem with nonlinear memory. For the equation without spatially dependent coefficient, i.e., when $\mu(x) \equiv 1$, Li and Xie [6] proved that if u blows up at $T < \infty$ and there exists $t_0 \in [0, T)$ such that

$$u_t(x, t_0) \geq 0 \quad \text{for all} \quad x \in \Omega, \tag{1.3}$$

then u satisfies

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1 (T - t)^{-\frac{2}{p-1}} \qquad \text{for all } t \in (0, T) \tag{1.4}$$

for some constant $C_1 > 0$. The characterization of the monotonicity condition (1.3) was given by Souplet in [10]. Results of blow-up points and blow-up profile of spatially homogeneous nonlinear memory were obtained by Bellout [1]. For various problems with nonlinear memory for which finite time blow-up occurs, we refer the reader to the paper by Souplet [9] and the references therein.

Now, we turn to the parabolic system (1.2). The local existence and uniqueness of solution (u, v) to (1.2) is well-known. Here a solution (u, v) blows up in finite time T if there holds

$$\limsup_{t \to T} \{\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}\} = \infty.$$

For blow-up of semilinear parabolic system, we refer the reader to, e.g., [2, 7, 11, 12, 13, 4]. The result of blow-up rate for (1.2) was proved by [2] for parabolic system of spatially homogeneous equations based on the idea of [3] for single equation.

Our first aim of this paper is to show that any solution u of (1.1) blows up in finite time and satisfies (1.4) provided that (1.3) holds. Among other things, any zero point of $\mu(x)$ is not a blow-up point under the condition (1.3). Our next aim is to consider the similar problem for the parabolic system (1.2). More precisely, we shall also prove that the time monotone nondecreasing solution blows up in finite time and it does not blow up at any zero of $\mu(x)$. A similar question for the equation $u_t = \Delta u + \mu(x)u^p$ was studied in [5].

This paper is organized as follows. In Section 2, we shall give a sufficient condition so that the time monotonicity condition holds for certain solutions of (1.1) and (1.2). Then we give some blow-up criteria for the problems (1.1) and (1.2) in Section 3. Moreover, we prove that any zero of $\mu(x)$ cannot be a blow-up point for both problems (1.1) and (1.2).

2. A Sufficient Condition for Monotonicity

In this section, we give a simple sufficient condition on the initial data so that the solution of either (1.1) or (1.2) becomes monotone in time after a certain time $t_0 \in [0, T)$. In the sequel, we let $[0, T)$ be the maximum existence time interval of solutions to (1.1) and (1.2).

For the problem (1.2), the condition

$$u_t(x, 0) \geq 0,\ v_t(x, 0) \geq 0 \quad \text{for all} \quad x \in \Omega \tag{2.1}$$

is valid if we assume that

$$\Delta u_0 + \mu(x)v_0^p \geq 0,\ \Delta v_0 + \mu(x)u_0^q \geq 0 \quad \text{ in } \Omega.$$

Moreover, under the assumption (2.1), it follows from the maximum principle that $u_t, v_t > 0$ in Q_T.

For the problem (1.1), the above simple criterion is not possible, since both conditions $u_0(x) = 0$ for all $x \in \partial\Omega$ and $u_t(x, 0) = \Delta u_0(x) \geq 0$ for all $x \in \Omega$ cannot hold at the same time. For this, we shall use an idea of Souplet [10] to derive the monotonicity condition (1.3) for the problem (1.1).

In the following, we denote

$$d(x) := \operatorname{dist}(x, \partial\Omega), \quad x \in \Omega.$$

Proposition 2.1. *Assume $\mu^{-\frac{1}{p-1}} \in L^1(\Omega)$. Let $\Phi \in C^2(\bar{\Omega})$ be positive in Ω and zero on the boundary. Suppose that there exist positive constants ε_0, η_0 such that*

$$\Delta\Phi(x) \geq \varepsilon_0 d(x) \quad \textit{for all} \quad x \in \Omega \quad \textit{with} \quad d(x) \leq \eta_0. \tag{2.2}$$

Then, for all $\lambda > 0$ large enough, the solution u of the problem (1.1) *with initial value $\lambda\Phi$ satisfies* (1.3).

Proof. The argument is very similar to that of [10] with a slightly modification. First, the function $v = u_t$ satisfies

$$\begin{cases} v_t = \Delta v + \mu(x) u^p, & x \in \Omega,\ t \in (0,T), \\ v(x,0) = \Delta u_0(x), & x \in \overline{\Omega}, \\ v(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T). \end{cases} \tag{2.3}$$

and so

$$v(x,t) = \int_\Omega G(x-y,t)\Delta u_0(y)\,dy + \int_0^t \int_\Omega G(x-y,t-s)\mu(y)u^p(y,s)\,dyds,$$

where $G(x,t)$ is the Green function of the heat operator in Ω with zero Dirichlet boundary condition. Since

$$u(x,s) \geq \int_\Omega G(x-y,s)u_0(y)\,dy =: w(x,s), \quad s \in [0,T),$$

we have

$$\begin{aligned} &\int_\Omega G(x-y,t-s)\mu(y)u^p(y,s)\,dy \geq \int_\Omega G(x-y,t-s)\mu(y)w^p(y,s)\,dy \\ &\geq \Big(\int_\Omega G(x-y,t-s)w(y,s)\,dy\Big)^p \Big(\int_\Omega G(x-y,t-s)\mu(y)^{-\frac{1}{p-1}}\,dy\Big)^{1-p} \\ &\geq C\,w^p(x,t), \qquad t \in [s,T), \end{aligned}$$

for some finite constant $C \in (0,\infty)$. Thus we have

$$v(x,t) \geq \int_\Omega G(x-y,t)\Delta u_0(y)\,dy + Ct\Big(\int_\Omega G(x-y,t)u_0(y)\,dy\Big)^p.$$

Therefore, for all $\lambda > 0$, $v_\lambda := (u_\lambda)_t$ satisfies

$$\frac{v_\lambda(x,t)}{\lambda} \geq \int_\Omega G(x-y,t)\Delta\Phi(y)\,dy + C\lambda^{p-1}t\Big(\int_\Omega G(x-y,t)\Phi(y)\,dy\Big)^p \tag{2.4}$$

for all $0 < t < T(\lambda\Phi)$, where u_λ is the solution of (1.1) with initial value $\lambda\Phi$ and $[0, T(\lambda\Phi))$ is the maximum existence time interval of u_λ.

Next, we show that there exists a positive constant η_1 such that

$$\int_\Omega G(x-y,t)\Delta\Phi(y)\,dy > 0 \quad \text{for} \quad (x,t) \quad \text{with} \quad d(x) \leq \eta_1,\ t \in [0,\eta_1]. \tag{2.5}$$

Let (λ_1, φ_1) be the first eigen-pair of $-\Delta$ in Ω with zero Dirichlet boundary condition such that $\max_\Omega \varphi_1 = 1$. Recall that there are positive constants c_1, c_2 such that $c_1\varphi_1(x) \leq d(x) \leq c_2\varphi_1(x)$ for all $x \in \Omega$. Hence, by the assumption (2.2), there exist a nonnegative smooth function ρ with support contained in $\{d(x) > \eta_1\}$

for some positive constant η_1 and a positive constant γ such that $\Delta\Phi \geq \gamma\varphi_1 - \rho$ in Ω. This yields

$$\int_\Omega G(x-y,t)\Delta\Phi(y)\,dy \geq \gamma e^{-\lambda_1 t}\varphi_1(x) - z(x,t),$$

where

$$z(x,t) := \int_\Omega G(x-y,t)\rho(y)\,dy.$$

Note that z is the solution of

$$\begin{cases} z_t = \Delta z \quad \text{in } \ \Omega\times(0,\infty), \\ z(x,t) = 0, \quad x\in\partial\Omega,\ t>0, \\ z(x,0) = \rho(x), \quad x\in\bar{\Omega}. \end{cases}$$

Hence $z(x,t)\to 0$ as $t\downarrow 0$ for $x\in\Omega$ with $d(x)\leq\eta_1$. Since ∇z is bounded, there is a small positive constant η_2 such that $z(x,t)\leq\eta_2 d(x)$ for (x,t) with $d(x)\leq\eta_1$ and $0\leq t\leq\eta_1$. Hence (2.5) follows by taking the constant $\eta_1>0$ sufficiently small.

We can easily check that the following function is a supersolution of (1.1):

$$U(t) = M^{-\frac{1}{p-1}}\left(M^{-\frac{1}{2}}\|u_0\|_{L^\infty(\Omega)}^{-\frac{p-1}{2}} - kt\right)^{-\frac{2}{p-1}},$$

where $M = \|\mu\|_{L^\infty(\Omega)}$ and $k = (p-1)(2(p+1))^{-1/2}$. Since $U(0) = \|u_0\|_{L^\infty(\Omega)}$, by the comparison principle, we conclude that

$$T(u_0) \geq \|u_0\|_{L^\infty(\Omega)}^{-\frac{p-1}{2}}/(kM^{\frac{1}{2}}).$$

Let us define

$$t_\lambda := \|\lambda\Phi\|_{L^\infty(\Omega)}^{-\frac{p-1}{2}}/(2kM^{\frac{1}{2}}).$$

Then $t_\lambda<\eta_1$ if $\lambda>\lambda_0$ for some sufficiently large constant λ_0. Therefore, it follows from (2.4), $\Phi>0$ in Ω and (2.5) that $(u_\lambda)_t(x,t_\lambda)>0$ for all x with $d(x)\leq\eta_1$, if $\lambda>\lambda_0$.

On the other hand, since $\Phi>0$ in Ω, there exists a constant $\alpha>0$ such that $\Phi(x)\geq 2\alpha$ for all $x\in\Omega$ with $d(x)\geq\eta_1$. Note that

$$\Psi(x,t) := \int_\Omega G(x-y,t_\lambda)\Phi(y)\,dy \to \Phi(x) \text{ as } t\to 0^+$$

uniformly on $\{x\in\Omega : d(x)\geq\eta_1\}$. Hence, by taking λ_0 sufficiently large, we have $\Psi(x,t_\lambda)\geq\alpha$ for all $x\in\Omega$ with $d(x)\geq\eta_1$, if $\lambda>\lambda_0$. Also, we can easily check from (2.4) that

$$\frac{v_\lambda(x,t_\lambda)}{\lambda} \geq -\|\Delta\Phi(y)\|_{L^\infty(\Omega)} + \frac{C\alpha^p\lambda^{\frac{p-1}{2}}\|\Phi\|_{L^\infty(\Omega)}^{-\frac{p-1}{2}}}{2kM^{\frac{1}{2}}} > 0$$

for all $x \in \Omega$ with $d(x) \geq \eta_1$, if λ is sufficiently large. Therefore, we conclude that $(u_\lambda)_t(x, t_\lambda) \geq 0$ in Ω for λ sufficiently large. The proposition follows. ∎

3. Blow-up Criteria and Blow-up Points

3.1. Nonlocal problem

First we show that the condition (1.3) implies (1.4) for the problem (1.1). A similar result was originally proved by [1] for spatially homogeneous equation (i.e., $\mu(x) \equiv 1$) based on an idea of [3].

Proposition 3.1. *Assume* (1.3). *Then u blows up in a finite time T and u satisfies* (1.4) *for some constant $C_1 > 0$.*

Remark 3.1. The condition of monotonicity in time implies the finite time blow-up for the homogeneous equation can be found in [8, Theorem 46.4].

Proof. Recall that $v := u_t$ is a nontrivial solution of (2.3). The Hopf lemma, the maximum principle and (1.3) imply that $u_t > 0$ in $\Omega \times (t_0, T)$ and $\frac{\partial}{\partial \nu} u_t < 0$ on $\partial\Omega \times (t_0, T)$, where ν is the unit outward normal on $\partial\Omega$.

Now we define $J := u_t - \varepsilon u^\alpha$, $\alpha := (p+1)/2$ and ε is a positive constant to be determined. Then by a simple calculation, we have

$$\begin{aligned}
J_t - \Delta J &= (u_t - \Delta u)_t - \varepsilon \alpha u^{\alpha-1}(u_t - \Delta u) + \varepsilon\alpha(\alpha-1)u^{\alpha-2}|\nabla u|^2 \\
&\geq \mu(x)u^p - \varepsilon\mu(x)\alpha u^{\alpha-1}\int_0^t u^p(x,s)\,ds \\
&= \mu(x)u^p - \varepsilon\mu(x)\alpha u^{\alpha-1}\int_0^t u^{\alpha-1}\Big(\frac{u_s - J}{\varepsilon}\Big)(x,s)\,ds \\
&= \mu(x)u_0^\alpha u^{\alpha-1} + \alpha\mu(x)u^{\alpha-1}\int_0^t u^{\alpha-1}J\,ds \\
&\geq \alpha\mu(x)u^{\alpha-1}\int_0^t u^{\alpha-1}J\,ds
\end{aligned}$$

Fix $t_1 \in (t_0, T)$ arbitrary. Then we can choose $\varepsilon > 0$ small enough such that $u_t(x, t_1) \geq \varepsilon u^\alpha(x, t_1)$ for all $x \in \Omega$. Hence $J(x, t_1) \geq 0$ for all $x \in \Omega$. Note that $J = 0$ on $\partial\Omega \times (t_1, T)$. It follows from the maximum principle for nonlocal problems (cf. [8, Proposition 52.24]) that $J \geq 0$ in $\Omega \times (t_0, T)$. Consequently, we have

$$u_t - \varepsilon u^\alpha \geq 0 \quad \text{in} \quad \Omega \times (t_1, T).$$

By an integration in t, we obtain

$$u^{1-\alpha}(x,t) \geq \varepsilon(\alpha - 1)(T - t), \quad t \in (t_1, T).$$

This means that $T < \infty$ and (1.4) follows immediately. The proof is completed. ■

Next we show a sufficient condition that assures any zero point of $\mu(x)$ is not a blow-up point. The proof, which was first given in [5] for the equation $u_t = \Delta u + \mu(x)u^p$, is based on the comparison principle as follows.

Theorem 3.1. *Assume* (1.4) *holds for some constant* $C_1 > 0$. *Then any zero point of* $\mu(x)$ *is not a blow-up point. In particular, if* (1.3) *holds, then any zero of* $\mu(x)$ *is not a blow-up point.*

Proof. Let us construct a strict supersolution in the following form:

$$w(x,t) = \frac{A}{[v(x) + (T-t)]^{\frac{2}{p-1}}},$$

where the constant $A > C_1$ and $v(x)$ will be determined later. Here w is a strict supersolution if w satisfies the inequality

$$w_t > \Delta w + \mu(x) \int_0^t w^p(x,s)\, ds. \tag{3.1}$$

Let x_0 be a zero point of $\mu(x)$. There exists $r_0 > 0$ such that $\{x : |x - x_0| \leq 2r_0\} \subset \Omega$. Under this condition, we define the following function:

$$v(x) = \delta \cos^2\left(\frac{\pi|x-x_0|}{2r_0}\right),\ B_0 := \{x \in \Omega\, :\, |x - x_0| \leq r_0\},$$

where δ is a positive constant. Note that $w(x,t) > u(x,t)$ for $x \in \partial B_0$ and $t \in (0,T)$. Also, for sufficiently large A, we have

$$w(x,0) = \frac{A}{[v(x) + T]^{\frac{2}{p-1}}} > u_0(x), \quad x \in B_0.$$

Moreover, the inequality (3.1) holds in $B_0 \times (0,T)$ if

$$1 + \Delta v(x) - \frac{p-1}{p+1} A^{p-1}\mu(x) - \frac{p+1}{p-1}\frac{|\nabla v(x)|^2}{v(x)} > 0 \tag{3.2}$$

is valid for all $x \in B_0$. It is easy to see that Δv and $|\nabla v|^2/v$ are bounded in B_0 and linear in δ. By fixing A, we first take $r_0 > 0$ small enough so that $A^{p-1}\mu(x) < 1/3$ for all $x \in B_0$. For this r_0, we then take $\delta > 0$ sufficiently small so that the inequality (3.2) holds in B_0. It follows from (3.1) that the function $z := w - u$ satisfies

$$z_t - \Delta z > \mu(x) \int_0^t b(x,s) z(x,s) ds \tag{3.3}$$

for some nonnegative function b.

We now claim that $z > 0$ in $B_0 \times [0,T)$. Otherwise, there exists the first time $t_0 > 0$ such that $z > 0$ in $B_0 \times [0,t_0)$, $z(x,t_0) \geq 0$ for all $x \in B_0$ and $z(x_0,t_0) = 0$ for some $x_0 \in B_0$. Then $(z_t - \Delta z)(x_0,t_0) \leq 0$ and

$$\mu(x_0) \int_0^{t_0} b(x_0,s) z(x_0,s) ds \geq 0,$$

this contradicts with (3.3). Hence we have the inequality

$$u(x,t) < \frac{A}{[v(x) + (T-t)]^{\frac{2}{p-1}}}, \quad x \in B_0,\ t \in (0,T).$$

Thus x_0 cannot be a blow-up point. The theorem is proved. ∎

3.2. Parabolic system

In this subsection, we shall first prove that (u,v) blows up in a finite time T and satisfies

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C_2(T-t)^{-\frac{p+1}{pq-1}}, \quad \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C_2(T-t)^{-\frac{q+1}{pq-1}} \tag{3.4}$$

for all $t \in (0,T)$ and for some $C_2 > 0$. Then we show that any zero of $\mu(x)$ is not a blow-up point.

More precisely, we prove the following theorem.

Theorem 3.2. *Assume* (2.1)*. Then (u,v) blows up in a finite time T and (u,v) satisfies* (3.4) *for all $t \in (0,T)$ for some $C_2 > 0$. Moreover, any zero point of $\mu(x)$ is not a blow-up point.*

Proof. We first define

$$J := u_t - \varepsilon v^p, \quad K := v_t - \varepsilon u^q$$

where ε is a positive constant to be determined. By a simple calculation, we have

$$\begin{aligned} J_t - \Delta J &= \mu(x) f'(v) K + \varepsilon f''(v) |\nabla v|^2 \geq \mu(x) f'(v) K, \\ K_t - \Delta K &= \mu(x) g'(u) J + \varepsilon g''(u) |\nabla u|^2 \geq \mu(x) g'(u) J \end{aligned}$$

with $f(v) = v^p$, $g(u) = u^q$. Since $(U,V) := (u_t, v_t)$ is a nontrivial solution of

$$\begin{cases} U_t = \Delta U + \mu(x) f'(v) V, & x \in \Omega,\ t > 0, \\ V_t = \Delta V + \mu(x) g'(u) U, & x \in \Omega,\ t > 0, \\ U(x,0) = u_t(x,0),\ V(x,0) = v_t(x,0), & x \in \Omega, \\ U(x,t) = 0,\ V(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases}$$

by the Hopf lemma and the maximum principle, we have $u_t, v_t > 0$ in $\Omega \times (0,T)$ and $\frac{\partial}{\partial \nu} u_t, \frac{\partial}{\partial \nu} v_t < 0$ on $\partial\Omega \times (0,T)$, where ν is the outward unit normal vector on $\partial\Omega$. Set $t_0 = T/2$. Then we can choose $\varepsilon > 0$ small enough such that

$$u_t(x,t_0) \geq \varepsilon v^p(x,t_0),\ v_t(x,t_0) \geq \varepsilon u^q(x,t_0)$$

for all $x \in \Omega$. Thus $J \geq 0$ and $K \geq 0$ on the parabolic boundary of $\Omega \times (t_0, T)$ if $\varepsilon > 0$ is sufficiently small. It follows from the maximum principle that $J \geq 0$ and $K \geq 0$ in $\Omega \times (t_0, T)$. Consequently, we have

$$u_t - \varepsilon v^p \geq 0, \quad v_t - \varepsilon u^q \geq 0 \quad \text{in} \quad \Omega \times (t_0, T).$$

Applying [8, Lemma 32.10], we conclude that $T < \infty$ and (3.4) holds for some positive constant C_2.

Next, for the blow-up points, we define

$$w(x,t) = \frac{A}{[h(x) + (T-t)]^{\frac{p+1}{pq-1}}}, \quad z(x,t) = \frac{A}{[h(x) + (T-t)]^{\frac{q+1}{pq-1}}},$$

where the constant $A > C_2$ and $h(x)$ will be determined later.

Let x_0 be any zero point of $\mu(x)$. We may assume that $\{x : |x - x_0| \leq 2r_0\} \subset \Omega$ for some $r_0 > 0$. We define

$$h(x) = \delta \cos^2\left(\frac{\pi|x - x_0|}{2r_0}\right),\ B_0 := \{x : |x - x_0| \leq r_0\},$$

where δ is a positive constant.

Note that $w(x,t) \geq u(x,t)$ and $z(x,t) \geq v(x,t)$ for $x \in \partial B_0$ and $t \in (0,T)$, by (3.4). Also,

$$w(x,0) = \frac{A}{[h(x) + T]^{\frac{p+1}{pq-1}}} \geq u_0(x) \quad \text{in } B_0,$$
$$z(x,0) = \frac{A}{[h(x) + T]^{\frac{q+1}{pq-1}}} \geq v_0(x) \quad \text{in } B_0,$$

if A is chosen sufficiently large.

The inequalities

$$w_t - \Delta w - \mu(x) z^p \geq 0, \quad z_t - \Delta z - \mu(x) w^q \geq 0 \quad \text{in } B_0 \times (0,T)$$

are equivalent to

$$1 - \frac{pq-1}{p+1} A^{p-1} \mu(x) + \Delta h(x) - \frac{p(q+1)}{pq-1} \frac{|\nabla h|^2}{h + (T-t)} \geq 0,$$

$$1 - \frac{pq-1}{q+1} A^{q-1} \mu(x) + \Delta h(x) - \frac{q(p+1)}{pq-1} \frac{|\nabla h|^2}{h + (T-t)} \geq 0.$$

We have these inequalities, if

$$1-\frac{pq-1}{p+1}A^{p-1}\mu(x)+\Delta h(x)-\frac{p(q+1)}{pq-1}\frac{|\nabla h|^2}{h}\geq 0, \tag{3.5}$$

$$1-\frac{pq-1}{q+1}A^{q-1}\mu(x)+\Delta h(x)-\frac{q(p+1)}{pq-1}\frac{|\nabla h|^2}{h}\geq 0. \tag{3.6}$$

It is easy to see that Δh and $|\nabla h|^2/h$ are bounded in B_0 and linear in δ. Thus, by taking δ and r_0 sufficiently small, we have the inequalities (3.5) and (3.6) in $B_0\times(0,T)$. Hence, by the comparison principle, we conclude that

$$w(x,t)=\frac{A}{[h(x)+(T-t)]^{\frac{p+1}{pq-1}}}\geq u(x,t),$$
$$z(x,t)=\frac{A}{[h(x)+(T-t)]^{\frac{q+1}{pq-1}}}\geq v(x,t)$$

on $B_0\times(0,T)$. In particular, $x=x_0$ is not a blow-up point of u and v. The theorem is proved. ■

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