TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 2, pp. 543-557, April 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

CATEGORICAL PROPERTIES OF SEQUENTIALLY DENSE MONOMORPHISMS OF SEMIGROUP ACTS

Mojgan Mahmoudi and Leila Shahbaz

Abstract. Let $\mathcal{M}$ be a class of (mono)morphisms in a category $\mathcal{A}$. To study mathematical notions, such as injectivity, tensor products, flatness, one needs to have some categorical and algebraic information about the pair ($\mathcal{A}$,$\mathcal{M}$).

In this paper we take $\mathcal{A}$ to be the category **Act-S** of acts over a semigroup S, and $\mathcal{M}_d$ to be the class of sequentially dense monomorphisms (of interest to computer scientists, too) and study the categorical properties, such as limits and colimits, of the pair ($\mathcal{A}$,$\mathcal{M}$). Injectivity with respect to this class of monomorphisms have been studied by Giuli, Ebrahimi, and the authors who used it to obtain information about injectivity relative to monomorphisms.

1. Introduction and Preliminaries

Let $\mathcal{M}$ be a class of (mono)morphisms of a category $\mathcal{A}$. To study mathematical notions, such as injectivity and flatness, one needs to have some categorical and algebraic information about the pair ($\mathcal{A}$,$\mathcal{M}$) (see [1, 3, 14]).

In this paper we take $\mathcal{A}$ to be the category **Act-S** of (right) acts over a semigroup S and $\mathcal{M}_d$ to be the class of sequentially dense monomorphisms, to be defined in Section 2, and study the categorical properties of this pair which are usually related to the behaviour of $\mathcal{M}_d$-injectivity (see [13]).

In the following we first recall some facts about the category **Act-S** needed in this paper.

Let S be a semigroup and A be a set. If we have a mapping (called the *action* of S on A)

$$\begin{aligned} \mu :\quad & A \times S \to A \\ & (a, s) \mapsto as := \mu(a, s) \end{aligned}$$

Received October 15, 2007, accepted September 7, 2009.
Communicated by Wen-Fong Ke.
2000 *Mathematics Subject Classification*: 08B25, 18A20, 18A30, 20M30, 20M50.
Key words and phrases: Sequential closure, Sequential dense.

such that $a(st) = (as)t$ for $a \in A, s, t \in S$, we call A a (*right*) S-*act* or a (*right*) *act* over S.

If S is a monoid with identity 1, we usually also require that $a1 = a$ for $a \in A$.

A subset A' of an S-act A is called a *subact* of A, written as $A' \leq A$, if $a's \in A'$ for all $s \in S$ and $a' \in A'$.

The semigroup S itself becomes an S-act by taking its operation as its action. A subact of the S-act S is a *right ideal* of the semigroup S. A subset $K \subseteq S$ is called a *left ideal* of S if $SK \subseteq K$, and an *ideal* or a *two-sided ideal* of S if $SK \subseteq K$ and $KS \subseteq K$. Also note that if S does not have an identity, one can attach an identity 1 to it to get a monoid, or an S-act, $S^1 = S \cup \{1\}$.

Also, recall that an element a of an S-act A is said to be a *fixed* or a *zero* element if $as = a$, for all $s \in S$.

A *homomorphism* (or an *equivariant map*, or an S-*map*) from an S-act A to an S-act B is a function from A to B such that for each $a \in A, s \in S$, $f(as) = f(a)s$.

Since the identity maps and the composition of two equivariant maps are equivariant, we have the category **Act-S** of all right S-acts and S-maps between them.

An S-act B containing (an isomorphic copy of) an S-act A as a subact is called an *extension* of A.

As a very interesting example of acts, used in computer science as a convenient means of algebraic specification of process algebras (see [7], [8]), consider the monoid $(\mathbb{N}^\infty, \cdot, \infty)$, where $\mathbb{N}$ is the set of natural numbers and $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$ with $n < \infty, \forall n \in \mathbb{N}$ and $m \cdot n = \min\{m, n\}$ for $m, n \in N^\infty$. Then an N^∞-act is called a projection algebra (see [7, 10, 12]).

Let A be an S-act. An equivalence relation ρ on A is called an S-*act congruence*, or simply a *congruence* on A, if $a\rho a'$ implies $as\rho a's$ for $a, a' \in A$, $s \in S$. If ρ is a congruence on A, then the factor set $A/\rho = \{[a]_\rho : a \in A\}$ is clearly an S-act, called the *factor act* of A by ρ, with the action given by $[a]_\rho s = [as]_\rho$, for $s \in S$, $a \in A$.

If $H \subseteq A \times A$ then we denote the *congruence generated by* H by $\rho(H)$; it is the smallest congruence on A containing H. One can see that $x\rho(H)y$ if and only if either $x = y$ or there exist $s_1, s_2, ..., s_n \in S^1$, $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in A$ such that $(a_i, b_i) \in H$ or $(b_i, a_i) \in H$, and

$$\begin{array}{ccccccc} x = a_1 s_1 & & b_2 s_2 = a_3 s_3 & & \cdots & & b_n s_n = y \\ & b_1 s_1 = a_2 s_2 & & b_3 s_3 = a_4 s_4 & & \cdots & \end{array}$$

Now we give some categorical ingredients of **Act-S** needed in the sequel (see also [5, 11]).

The class of S-acts is an equational class, and so the category **Act-S** is complete (has all products and equalizers). In fact, limits in this category are computed as in the category **Set** of sets and equipped with a natural action. In particular, the terminal object of **Act-S** is the singleton $\{0\}$, with the obvious S-action. Also, for S-acts

A, B, their cartesian product $A \times B$ with the S-action defined by $(a, b)s = (as, bs)$ is the *product* of A and B in **Act-S**.

The pullback of a given diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \end{array}$$

in **Act-S** is the subact $P = \{(c, a) : c \in C, a \in A, g(c) = f(a)\}$ of $C \times A$, and pullback maps $p_C : P \to C$, $p_A : P \to A$ are restrictions of the projection maps. Notice that for the case where g is an inclusion, P can be taken as $f^{-1}(C)$.

All colimits in **Act-S** exist and are calculated as in **Set** with the natural action of S on them. In particular, $\emptyset$ with the empty action of S on it, is the initial object of **Act-S**. Also, the *coproduct* of S-acts A, B is their disjoint union $A \sqcup B = (A \times \{1\}) \cup (B \times \{2\})$ with the obvious action, and coproduct injections are defined naturally.

The pushout of a given diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

in **Act-S** is the factor act $Q = (B \sqcup C)/\theta$, where θ is the congruence relation on $B \sqcup C$ generated by all pairs $(u_B f(a), u_C g(a))$, $a \in A$, where $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are the coproduct injections. Also, the pushout maps are given as $q_1 = \gamma u_C : C \to (B \sqcup C)/\theta$, $q_2 = \gamma u_B : B \to (B \sqcup C)/\theta$, where $\gamma : B \sqcup C \to (B \sqcup C)/\theta$ is the canonical epimorphism. Multiple pushouts in **Act-S** are constructed analogously.

Recall that for a family $\{A_i : i \in I\}$ of S-acts, each with a unique fixed element 0, the *direct sum* $\oplus_{i \in I} A_i$ is defined to be the subact of the product $\prod_{i \in I} A_i$ consisting of all $(a_i)_{i \in I}$ such that $a_i = 0$ for all $i \in I$ except a finite number of indices.

Free objects in **Act-S** exist. In fact $X \times S^1$, where S^1 is S with an identity adjoined, with the action $(x, t)s = (x, ts)$ is the *free S-act* on the set X.

Cofree objects exist in **Act-S**. In fact $X^{S^1} = \{f \mid f : S^1 \to X \ \text{ is a function}\}$ with the action given by $(fs)(t) = f(st)$ is the *cofree S-act* on the set X.

A morphism in **Act-S** is a *monomorphism* if and only if it is one-one (so sometimes we consider monomorphisms as inclusion maps), and *epimorphisms* in **Act-S** are exactly onto S-maps. These follow from the existence of free and cofree S-acts, respectively.

We also need to mention the construction of general limits and colimits which will be needed in the sequel.

Let $\mathcal{A} : \mathbf{I} \to$ **Act-S** be a diagram in **Act-S** ($\mathbf{I}$ is a small category and $\mathcal{A}$ is a functor) determining the acts $\mathcal{A}(\alpha) = A_\alpha$, for $\alpha \in I = Obj(\mathbf{I})$, and S-maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, for $\lambda : \alpha \to \beta$ in $Mor(\mathbf{I})$. Recall that the limit of this diagram is $\varprojlim_\alpha A_\alpha := \bigcap_{\lambda \in Mor(\mathbf{I})} E_\lambda$, where for $\lambda : \alpha \to \beta$ in $Mor(\mathbf{I})$,

$$E_\lambda = \{a = (a_\alpha)_{\alpha \in I} \in \prod_\alpha A_\alpha : g_{\alpha\beta} p_\alpha(a) = p_\beta(a)\}$$

and p_α, p_β are the α, βth projection maps of the product. Also, the limit S-maps are $q_\alpha =: p_\alpha|_{\varprojlim_\alpha A_\alpha} : \varprojlim_\alpha A_\alpha \to A_\alpha$.

Also, the colimit of the above diagram is obtained as $\varinjlim_\alpha A_\alpha =: \coprod_{\alpha \in I} A_\alpha / \theta$, where θ is the congruence generated by

$$H = \{(u_\alpha(a_\alpha), u_\beta g_{\alpha\beta}(a_\alpha)) : a_\alpha \in A_\alpha, \alpha \to \beta \in Mor(\mathbf{I})\}.$$

The colimit S-maps are $g_\alpha := \gamma_\theta u_\alpha : A_\alpha \to \varinjlim_\alpha A_\alpha$ where u_α's are the coproduct injection maps and γ_θ is the canonical epimorphism of the quotient.

Recall that a directed system of S-acts and S-maps is a family $(B_\alpha)_{\alpha \in I}$ of S-acts indexed by an updirected set I endowed by a family $(g_{\alpha\beta} : B_\alpha \to B_\beta)_{\alpha \leq \beta \in I}$ of S-maps such that given $\alpha \leq \beta \leq \gamma \in I$ we have $g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}$, and also $g_{\alpha\alpha} = id$. Note that the *directed colimit* (which is usually called the *direct limit* in literature) of a directed system $((B_\alpha)_{\alpha \in I}, (g_{\alpha\beta})_{\alpha \leq \beta \in I})$ in **Act-S** is given as $\varinjlim_\alpha B_\alpha = \coprod_\alpha B_\alpha / \rho$, where the congruence ρ is given by $(b_\alpha, b_\beta) \in \rho$ if and only if there exists $\delta \geq \alpha, \beta$ such that $u_\delta g_{\alpha\delta}(b_\alpha) = u_\delta g_{\beta\delta}(b_\beta)$, where u_α's are injection maps of the coproduct.

Notice that the family $g_\alpha = \gamma_\rho u_\alpha : B_\alpha \to \varinjlim_\alpha B_\alpha$ of S-maps satisfies $g_\beta g_{\alpha\beta} = g_\alpha$ for $\alpha \leq \beta$, where $\gamma_\rho : \coprod_\alpha B_\alpha \to \varinjlim_\alpha B_\alpha$ is the canonical epimorphism.

2. Sequential Closure Operator

In this section we introduce and briefly study a closure operator; the dense monomorphisms resulting from it are the subject of study in this paper. First note that, denoting the lattice of all subacts of an S-act B by $SubB$, following [2] for the general definition of closure operators on a category (which is not a priori assumed to be idempotent), we get:

Definition 2.1. A family $C = (C_B)_{B \in \mathbf{Act-S}}$, with $C_B : SubB \to SubB$, taking the subbact $A \leq B$ to $C_B(A)$, is called a *closure operator* on **Act-S** if it satisfies the following laws:

(c_1) (*Extension*) $A \leq C_B(A)$,

(c_2) (*Monotonicity*) $A_1 \leq A_2$ implies $C_B(A_1) \leq C_B(A_2)$,

(c_3) (*Continuity*) $f(C_B(A)) \leq C_C(f(A))$, for all morphisms $f : B \to C$.

Now, one has the usual two classes of monomorphisms related to the notion of a closure operator as follows:

Definition 2.2. Let $A \leq B$ be in **Act-S**. We say that A is C-*closed* in B if $C_B(A) = A$, and it is C-*dense* in B if $C_B(A) = B$. Also, an S-map $f : A \to B$ is said to be C-dense (C-closed) if $f(A)$ is a C-dense (C-closed) subact of B.

We take $\mathcal{M}_c$ to be the set of all C-closed, and $\mathcal{M}_d$ to be the set of all C-dense monomorphisms.

Definition 2.3. A closure operator C is said to be:

(a) *Weakly hereditary* if for every S-act B and every $A \leq B$, A is C-dense in $C_B(A)$.

(b) *Hereditary* if for every S-act B and $A_1 \leq A_2 \leq B$,

$$C_{A_2}(A_1) = C_B(A_1) \cap A_2.$$

(c) *Grounded* if for every S-act B, $C_B(\emptyset) = \emptyset$.

(d) *Additive* if for every S-act B, $C_B(A_1 \cup A_2) = C_B(A_1) \cup C_B(A_2)$.

(e) *Productive* if for every family of subacts A_i of B_i, taking $A = \prod_i A_i$ and $B = \prod_i B_i$, $C_B(A) = \prod_i C_{B_i}(A_i)$.

(f) *Idempotent* if $C_B(C_B(A)) = C_B(A)$ for all S-acts B and $A \leq B$.

(g) *Discrete* if $C_B(A) = A$ for every $A \leq B$.

(h) *Trivial* if $C_B(A) = B$ for every $A \leq B$.

Now, we introduce the *sequential closure operator* on the category of S-acts and investigate some of its properties (see also [9] and [4]).

Definition 2.4. The *sequential closure operator* $C^d = (C^d_B)_{B \in \mathbf{Act-S}}$ on **Act-S** is defined as

$$C^d_B(A) = \{b \in B : bS \subseteq A\}$$

for any subact A of an S-act B.

Notice that for the case where S is a monoid, every subact A of B is C^d-closed, and A is C^d-dense in B if and only if $A = B$. Note that, by Definition 2.2, a subact A of an S-act B is C^d-dense, which will also be called *sequentially dense* or s-*dense*, in B if $bS \subseteq A$ for each $b \in B$. An S-map $f : A \to B$ is said to be s-dense if $f(A)$ is an s-dense subact of B.

Remark 2.5. For each subact A of an S-act B, $C^d_B(A)$ is the largest subact T of B with the property $TS \subseteq A$.

We now prove some of the properties of this closure operator.

Theorem 2.6. *The closure operator C^d is hereditary, weakly hereditary, productive, grounded if $S \neq \emptyset$, discrete if and only if S is a monoid, and also trivial if and only if S is empty.*

Proof. We just prove some parts of this result; the remainder are also straight forward. For hereditariness, let $A_1 \leq A_2 \leq B$ and $a \in C^d_{A_2}(A_1)$. Then $aS \subseteq A_1, a \in A_2$. Thus $aS \subseteq A_1, a \in B$. Hence $a \in C^d_B(A_1) \cap A_2$. Conversely, let $a \in C^d_B(A_1) \cap A_2$. Then $a \in A_2, aS \subseteq A_1$. Thus $a \in C^d_{A_2}(A_1)$.

For the last part, we see that if $S = \emptyset$ then $C^d_B(A) = \{b \in B : bS \subseteq A\} = B$. If $S \neq \emptyset$, let $s \in S$. Then, taking sets $A \subset B$ as S-acts with the identity action, and $b \in B - A$, we have $bs = b \notin A$. Thus $C^d_B(A) \neq B$. ∎

Corollary 2.7. *If $A \leq B \leq C$ then $C^d_B(A) \subseteq C^d_C(A)$.*

As the following result shows, C^d is not idempotent in general.

Theorem 2.8. *The closure operator C^d is idempotent if and only if $S^2 = S$.*

Proof. Let C^d be idempotent. Since

$$S^1 = C^d_{S^1}(S) = C^d_{S^1}(C^d_S(S^2)) \subseteq C^d_{S^1}(C^d_{S^1}(S^2)) = C^d_{S^1}(S^2)$$

and $S^1S \subseteq S^2$ which means that $S \subseteq S^2$. The converse is obvious. ∎

Lemma 2.9. *A (right) ideal I of S is s-dense, that is $C^d_S(I) = S$, if and only if $S^2 \subseteq I$.*

Theorem 2.10. *The closure operator C^d is additive if and only if for every element b in an S-act B, bS is join prime in the lattice $Sub(B)$.*

Proof. Let A and D be subacts of an S-act B and $b \in C^d_B(A \cup D)$. Then, $bS \subseteq A \cup D$ and hence, bS being $\vee$-prime, $bS \subseteq A$ or $bS \subseteq D$. Thus, $b \in C^d_B(A) \cup C^d_B(D)$. This, using monotonicity of C^d, shows that each C^d_B, and hence C^d, is additive.

Conversely, let C^d, and hence each C^d_B, be additive. Let $b \in B$ and $bS \subseteq A \cup D$, where A and D are subacts of B. Then, by monotonicity and additivity,

$$C^d_B(bS) \subseteq C^d_B(A \cup D) = C^d_B(A) \cup C^d_B(D).$$

Now, since $b \in C^d_B(bS)$, $b \in C^d_B(A)$ or $b \in C^d_B(D)$. Thus, $bS \subseteq A$ or $bS \subseteq D$, proving that bS is join prime in $Sub(B)$. ∎

Theorem 2.11. *If S has a left identity element e, then C^d is additive.*

Proof. Note that, in this case, for any subact A of an S-act B, $b \in C^d_B(A)$ if and only if $be \in A$. This is because, $bs = b(es) = (be)s$, for each $s \in S$.

Now, if A and D are subacts of B and $bS \subseteq A \cup D$, for $b \in B$, then one can easily see that $bS \subseteq A$ or $bS \subseteq D$, depending on $be \in A$ or $be \in D$, respectively. ■

3. Categorical Properties of s-Dense Monomorphisms

In this final section we study some categorical and algebraic properties of the category **Act-S** with respect to sequentially dense monomorphisms. We study the composition, limit, and colimit properties in the following three subsections.

3.1. Composition properties of s-dense monomorphisms

In this subsection we investigate some properties of the class $\mathcal{M}_d$, mostly to do with the composition of dense monomorphisms. These properties and the ones given in the next two subsections are normally used to study injectivity, and of course other mathematical notions.

The class $\mathcal{M}_d$ is clearly isomorphism closed; that is, contains all isomorphisms and is closed under composition with isomorphisms. But, unfortunately $\mathcal{M}_d$ is not always closed under composition:

Lemma 3.1. *The class $\mathcal{M}_d$ is closed under composition if and only if the C^d-closure operator is idempotent.*

Proof. If the composition of s-dense monomorphisms is an s-dense monomorphism then, since the inclusion maps $S^2 \hookrightarrow S$ and $S \hookrightarrow S^1$ are clearly s-dense, we get that S^2 is s-dense in S^1. Hence, $S = 1S \subseteq S^2$ and so, by Theorem 2.8, C^d is idempotent. For the converse, let $A \leq B$ and $B \leq D$ be s-dense subacts. Then, $D = C^d_D(B) = C^d_D(C^d_B(A)) \subseteq C^d_D(C^d_D(A)) = C^d_D(A)$. Thus $D = C^d_D(A)$. ■

As the above result shows, the composition of s-dense monomorphisms need not be s-dense. For example take a semigroup S with $S^2 \neq S$ and consider the inclusion maps $S^2 \hookrightarrow S$ and $S \hookrightarrow S^1$ given in the above proof (see Theorem 2.8). But the following useful result shows that the composition of an s-dense monomorphism with a surjective morphism is s-dense.

Proposition 3.2. *The composition of an s-dense morphism with a surjective morphism is an s-dense morphism.*

Proof. Let $f : A \to B$ be an s-dense monomorphism and $g : B \to C$ be a surjective S-map. We want to show that for each $c \in C$ and $s \in S$, $cs \in \mathrm{Im}(gf)$.

Let $c \in C$. Since g is surjective, there exists $b \in B$ such that $c = g(b)$. Now, since f is s-dense, for each $s \in S, bs = f(a)$ for some $a \in A$. Thus $cs = g(b)s = g(bs) = g(f(a)) = (gf)(a) \in \mathrm{Im}(gf)$. Hence gf is s-dense.

Now, let $g : A \to B$ be a surjective S-map and $f : B \to C$ be an s-dense monomorphism. Take $c \in C, s \in S$. Since f is s-dense, there exists $b \in B$ such that $f(b) = cs$. Since g is surjective, there exists $a \in A$ such that $g(a) = b$. Now, $cs = f(b) = f(g(a)) = (fg)(a) \in \mathrm{Im}(fg)$. Hence fg is s-dense. ∎

The following result shows that $\mathcal{M}_d$ is *right* (*left*) *cancellable*, in the sense that for monomorphisms f and g if $gf \in \mathcal{M}_d$ then $g \in \mathcal{M}_d$ ($f \in \mathcal{M}_d$).

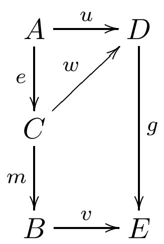
Proposition 3.3. *The class $\mathcal{M}_d$ is right and left cancellable.*

Proof. For the right cancellability, let gf be in $\mathcal{M}_d$ for monomorphisms $f : A \to B,\ g : B \to C$. Take $s \in S, c \in C$. Since gf is s-dense, there exists $a \in A$ such that $(gf)(a) = cs$. Now, $g(f(a)) = cs,\ cs \in \mathrm{Im}g$. Thus $g \in \mathcal{M}_d$. For the left cancellability, let $b \in B,\ s \in S$, and so $g(bs) \in C$. Since gf is s-dense, there exists $a \in A$ such that $gf(a) = g(bs)$. Now, since g is a monomorphism, $f(a) = bs$ and hence $bs \in \mathrm{Im}f$. ∎

Proposition 3.4. *Let $f : A \to B \in$***Act-S***. Then there are unique* (*always up to isomorphism*) *morphisms $e, m \in$***Act-S** *such that:*

(1) (*right $\mathcal{M}_d$-factorization*) *$f = me$, where $m : C \to B \in \mathcal{M}_d$, $e : A \to C$, and*

(2) (*diagonalization property*) *for every commutative diagram*



in **Act-S** *with $g : D \to E \in \mathcal{M}_d$, there is a uniquely determined morphism $w : C \to D$ with $gw = vm$ and $we = u$.*

Proof. Take $f : A \to B$, and let $C = f(A) \cup BS$. Define $e : A \to C$ by $e(a) = f(a)$ for $a \in A$, and take $m : C \to B$ to be the inclusion map. Then $f = me$. To see (2), define $w : C \to D$ by $w(f(a)) = u(a),\ w(bs) = v(bs) = v(b)s$. Then w is well-defined, for, if $bs = b's'$ then v, being well-defined, we get $w(bs) = v(bs) = v(b's') = w(b's')$; and if $f(a) = f(a')$ then $gu(a) = vf(a) = vf(a') = gu(a')$ and so $u(a) = u(a')$ since g is a monomorphism; and if $f(a) = bs$ then $gwf(a) = gu(a) = vf(a) = v(bs) = gw(bs)$ which gives $wf(a) = w(bs)$, since g is a monomorphism. It is clear, by the definition of w, that $gw = vm$ and $we = u$. Also, w is unique by this property, since having $w' : C \to D$ with $gw = vm = gw'$, we get that $w = w'$ because g is a monomorphism.

To show the uniqueness of m and e, let there also exist morphisms $m' : C' \to B$ and $e' : A \to C'$ satisfying conditions (1) and (2) above. Then, there are $w : C \to C'$, $w' : C' \to C$ such that $we = e'$, $m'w = id_B m$ and $w'e' = e$, $mw' = id_B m'$. Hence $mw'w = m$, which makes $w'w = id_C$ since m is a monomorphism. Similarly, $ww' = id_{C'}$. This means that e and e', also m and m', are isomorphic. ■

3.2. Limits of s-dense monomorphisms

In this subsection we will investigate the behaviour of dense monomorphisms with respect to limits.

Proposition 3.5. *The class $\mathcal{M}_d$ is closed under products.*

Proof. Let $(f_i : A_i \to B_i)_{i \in I}$ be a family of s-dense monomorphisms. Consider the commutative diagram

$$\begin{array}{ccc} \prod_{i\in I} A_i & \xrightarrow{f} & \prod_{i\in I} B_i \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p'_i} \\ A_i & \xrightarrow{f_i} & B_i \end{array}$$

We show that $f = (f_i)_{i\in I} : \prod_{i\in I} A_i \to \prod_{i\in I} B_i$ is an s-dense monomorphism. Let $b = (b_i)_{i\in I} \in \prod_{i\in I} B_i$ and $s \in S$. Since each f_i is s-dense, $b_i s \in \mathrm{Im} f_i$. Now $bs = (b_i s)_{i\in I} \in \mathrm{Im} f$. Hence f is s-dense. It is obvious that f is a monomorphism. So $f \in \mathcal{M}_d$. ■

Proposition 3.6. *The class $\mathcal{M}_d$ is closed under $\mathcal{M}_d$-pullbacks.*

Proof. Consider the pullback diagram

$$\begin{array}{ccc} f^{-1}(C) & \stackrel{i}{\hookrightarrow} & A \\ {\scriptstyle \overline{f}=f|_{f^{-1}(C)}}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \end{array}$$

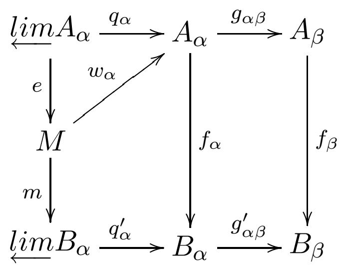
with $g, f \in \mathcal{M}_d$. For simplicity we consider g to be inclusion. We have to show that $\mathrm{Im}(g\overline{f})$ is s-dense. Let $b \in B$, $s \in S$. Since g and f are s-dense, there exist $c \in C$ and $a \in A$ such that $c = bs = f(a)$. Thus $bs = c = g(c) = g(f(a)) \in \mathrm{Im}(g\overline{f})$. ■

Proposition 3.7. *The class $\mathcal{M}_d$ is stable under $\mathcal{M}_d$-pullbacks; in the sense that pullback of any s-dense monomorphism along any morphism is again s-dense.*

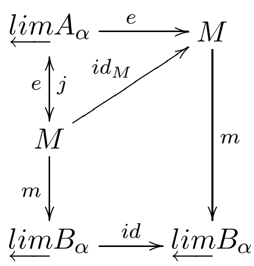
Proof. Consider the pullback diagram given in the proof of the above proposition with $g \in \mathcal{M}_d$ as inclusion, and let f be an arbitrary S-map. We have to show that i is s-dense. Let $a \in A,\ s \in S$. Since g is s-dense, $f(as) = f(a)s \in C$. Thus $as \in f^{-1}(C)$. ∎

Proposition 3.8. *The class $\mathcal{M}_d$ is closed under limits.*

Proof. Let $\mathcal{A}, \mathcal{B} : \mathbf{I} \to$ **Act-S** be diagrams in **Act-S** determining the acts A_α, B_α, for $\alpha \in I = Obj(\mathbf{I})$, and S-maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, $g'_{\alpha\beta} : B_\alpha \to B_\beta$, for $\alpha \to \beta$ in $Mor(\mathbf{I})$. Consider limits of these diagrams with limit maps $q_\alpha : \varprojlim A_\alpha \to A_\alpha$, $q'_\alpha : \varprojlim B_\alpha \to B_\alpha$. Let $\{f_\alpha : A_\alpha \to B_\alpha : \alpha \in I\}$ be a family of s-dense monomorphisms such that $g'_{\alpha\beta} f_\alpha = f_\beta g_{\alpha\beta}$. Let f denote $\varprojlim f_\alpha : \varprojlim A_\alpha \to \varprojlim B_\alpha$ which exists by the universal property of limits. We show that f belongs to $\mathcal{M}_d$. Consider the diagram



where $f = me$ is the right $\mathcal{M}_d$-factorization of f, which exists by Proposition 3.4. Since each $f_\alpha \in \mathcal{M}_d$, the diagonalization property of the factorization for each α implies that there exists $w_\alpha : M \to A_\alpha$ such that $f_\alpha w_\alpha = q'_\alpha m, w_\alpha e = q_\alpha$. Then, the uniqueness of w_α's gives that $g_{\alpha\beta} w_\alpha = w_\beta$ for each $\alpha \to \beta$. Now, by the universal property of limits, there exists $j : M \to \varprojlim A_\alpha$ with $q_\alpha j = w_\alpha$ for each α. We show that j is in fact an isomorphism. We have $q_\alpha je = w_\alpha e = q_\alpha$ for each α, and so, by the universal property of limits, $je = id_{\varprojlim A_\alpha}$. Also, considering the following diagram



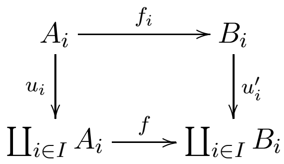
we get that $q'_\alpha mej = f_\alpha w_\alpha ej = f_\alpha q_\alpha j = f_\alpha w_\alpha = q'_\alpha m$ for each α, and hence $mej = m$ which, by the uniqueness of the diagonalization property, gives $ej = id_M$. Therefore, j, and hence e, is an isomorphism. But $f = me$ and $\mathcal{M}_d$ is closed under composition with isomorphisms, so f belongs to $\mathcal{M}_d$. ∎

3.3. Colimits of s-dense monomorphisms

This subsection is devoted to the study of the behaviour of s-dense monomorphisms with respect to colimits.

Proposition 3.9. *The class $\mathcal{M}_d$ is closed under coproducts.*

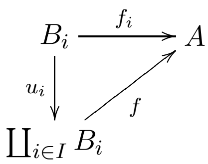
Proof. Consider the diagram



in which $\{f_i : A_i \to B_i : i \in I\}$ is a family of s-dense monomorphisms. Let $f : \coprod_{i\in I} A_i \to \coprod_{i\in I} B_i$ be the S-map satisfying $f(u_i(a_i)) = u'_i f_i(a_i)$, for $a_i \in A_i$, which exists by the universal property of coproducts; in fact, $f(a_i, i) = (f_i(a_i), i)$. We have to show that f is an s-dense monomorphism. Let $b \in \coprod_{i\in I} B_i$, $s \in S$. Then there exists $i \in I$, $b_i \in B_i$ such that $b = u'_i(b_i)$. Since f_i is s-dense, there exists $a_i \in A_i$ such that $f_i(a_i) = b_i s$, and hence $u'_i f_i(a_i) = u'_i(b_i s) = u'_i(b_i)s = bs$. Now, $bs = u'_i f_i(a_i) = f u_i(a_i) \in \mathrm{Im} f$. Thus f is s-dense. Also, f is a monomorphism, because u'_i and f_i, $i \in I$ are monomorphisms. ∎

Proposition 3.10. *Let $\{f_i : B_i \to A : i \in I\}$ be a family of s-dense S-maps. Then $f : \coprod_{i\in I} B_i \to A$ is an s-dense S-map.*

Proof. Consider the diagram



where $f : \coprod_{i\in I} B_i \to A$ is the S-map obtained by the universal property of coproducts. Then, since $fu_i = f_i$ belongs to $\mathcal{M}_d$, applying the proof of right cancellability of $\mathcal{M}_d$ (Proposition 3.3), we get $f \in \mathcal{M}_d$. ∎

Proposition 3.11. *The class $\mathcal{M}_d$ is closed under direct sums.*

Proof. Let $\{f_i : A_i \to B_i : i \in I\}$ be a family of s-dense monomorphisms. Then, using Proposition 3.5, we get that $f = \oplus_{i\in I} f_i = \prod_{i\in I} f_i : \oplus_{i\in I} A_i \to \oplus_{i\in I} B_i$ is an s-dense monomorphism. More precisely, if $(b_i)_{i\in I} \in \oplus_{i\in I} B_i$ and $s \in S$ then, by Proposition 3.5, there exists $(a_i)_{i\in I} \in \prod_{i\in I} A_i$ with $f((a_i)_{i\in I}) = (b_i)_{i\in I} s$. But,

$(a_i)_{i\in I} \in \oplus_{i\in I} A_i$, because for all i with $b_i = 0$ we have $0 = b_i s = f(a_i) = f_i(a_i)$ and so $a_i = 0$, since f_i is a monomorphism. ■

We recall the following lemma from [6]. First recall that it is said that *pushouts transfer monomorphisms* in a category if for a pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{k} & D \end{array}$$

if g is a monomorphism then so is k.

Lemma 3.12. *Pushouts transfer monomorphisms in* **Act-S**.

Proposition 3.13. *In* **Act-S**, *pushouts transfer s-dense monomorphisms.*

Proof. Consider the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h} \\ B & \xrightarrow{k} & (B \sqcup C)/\theta \end{array}$$

where $g \in \mathcal{M}_d$, $h = \gamma u_C : C \to (B \sqcup C)/\theta$, $k = \gamma u_B : B \to (B \sqcup C)/\theta$, $\gamma : B \sqcup C \to (B \sqcup C)/\theta$ is the natural epimorphism, and $u_B : B \to B \sqcup C, u_C : C \to B \sqcup C$ are coproduct injections, and θ is the congruence relation on $B \sqcup C$ generated by all pairs $H = \{(u_B f(a), u_C g(a)) : a \in A\}$. We show that k belongs to $\mathcal{M}_d$. Notice that by the above lemma, k is a monomorphism, so it is enough to show that k is s-dense. Let $[x]_\theta \in (B \sqcup C)/\theta$ and $s \in S$ be arbitrary. Then, $x = u_B(b)$ for some $b \in B$, or $x = u_C(c)$ for some $c \in C$. In the former case, we have $[x]_\theta s = k(b)s = k(bs) \in \mathrm{Im}(k)$. In the latter case, using that g is s-dense, we get $a \in A$ with $g(a) = cs$ and hence $[x]_\theta s = [u_C(c)]_\theta s = h(c)s = h(cs) = hg(a) = kf(a) \in \mathrm{Im}(k)$. ■

Proposition 3.14. *The pushout of s-dense monomorphisms belongs to $\mathcal{M}_d$.*

Proof. Applying the notations of the above proposition, with a similar argument to its proof, one gets that when f and g in the pushout diagram are s-dense monomorphisms, then so is $kf = hg$. ■

Note that if the composition of s-dense monomorphisms were s-dense, the above result would have been just a direct corollary of the last proposition.

Proposition 3.15. *The multiple pushout of s-dense monomorphisms is an s-dense monomorphism. Also, multiple pushouts transfer s-dense monomorphisms.*

Proof. Let $\{d_i : A \to B_i : i \in I\}$ be a family of s-dense monomorphisms. Recall that the multiple pushout of this family is $\coprod_{i\in I} B_i/\theta$, where θ is the congruence on $\coprod_{i\in I} B_i$ generated by all pairs $H = \{(u_i d_i(a), u_j d_j(a)) : i, j \in I, a \in A\}$, where for each $i \in I$, $u_i : B_i \to \coprod_{i\in I} B_i$ is the ith coproduct injection map. Also, the multiple pushout maps are $d'_i = \gamma u_i : B_i \to \coprod_{i\in I} B_i/\theta$ where $\gamma : \coprod_{i\in I} B_i \to \coprod_{i\in I} B_i/\theta$ is the natural epimorphism.

First, we see that for each $i \in I$, d'_i is a monomorphism. Let for $b_i, b'_i \in B_i$, $d'_i(b_i) = d'_i(b'_i)$. Then $(u_i(b_i), u_i(b'_i)) \in \theta$ and thus either $b_i = b'_i$ or there exist elements $a_1, a_2, \ldots, a_n \in A$, $k_1 \ldots k_{n+2} \in I$ such that $u_i(b_i) = u_{k_1} d_{k_1}(a_1)$, $u_{k_2} d_{k_2}(a_1) = u_{k_3} d_{k_3}(a_2)$, $\cdots$, $u_{k_{n+2}} d_{k_{n+2}}(a_n) = u_i(b'_i)$. Therefore, $k_1 = i$, $k_2 = k_3$, $k_4 = k_5$, ..., $k_{n+2} = i$, and hence, since each d_k is a monomorphism, $b_i = d_i(a_1)$, $a_1 = a_2 = a_3 = a_4 = a_5 = ... = a_{n-1} = a_n$, $d_i(a_n) = b'_i$. Thus $b_i = d_i(a_1) = d_i(a_2) = d_i(a_3) = ... = d_i(a_n) = b'_i$.

To show that each $d'_i d_i$ (and hence each d'_i) is dense, let $b \in \coprod_{i\in I} B_i/\theta$ and $s \in S$. Then, there exist $j \in I$ and $b_j \in B_j$ such that $b = [u_j(b_j)]_\theta$. Since d_j is s-dense, there exists an element $a \in A$ such that $d_j(a) = b_j s$. Now, $bs = [u_j(b_j)]_\theta s = [u_j(b_j s)]_\theta = d'_j(b_j s) = d'_j d_j(a) = d'_i d_i(a) \in \mathrm{Im}(d'_i d_i)$. ■

Definition 3.16. We say that a category $\mathcal{A}$ has $\mathcal{M}$-*bounds* if for every set indexed family $\{m_i : A \to A_i : i \in I\}$ of $\mathcal{M}$-morphisms there is an $\mathcal{M}$-morphism $m : A \to B$ which factors over all m_i's; that is there are $d_i : A_i \to B$ with $d_i m_i = m$.

Proposition 3.17. *The category* **Act-S** *has $\mathcal{M}_d$-bounds.*

Proof. Let $\{h_\alpha : A \to B_\alpha : \alpha \in I\}$ be a set indexed family in $\mathcal{M}_d$ and $h : A \to B = \coprod_\alpha B_\alpha/\theta$ be the multiple pushout of h_α's. Then h factors over all h_α's, and is an s-dense monomorphism, by Proposition 3.15. ■

Definition 3.18. We say that a category $\mathcal{A}$ has $\mathcal{M}$-*amalgamation* property, if the morphism m in the definition of $\mathcal{M}$-bounds factors over all m_i's through members of $\mathcal{M}$; that is d_i's belong to $\mathcal{M}$.

Proposition 3.19. *The category* **Act-S** *has $\mathcal{M}_d$-amalgamation property.*

Proof. Since, by Proposition 3.15, multiple pushout transfers s-dense monomorphisms, we are done. ■

Proposition 3.20. *The category* **Act-S** *has $\mathcal{M}_d$-directed colimits.*

Proof. Let $((B_\alpha)_{\alpha\in I}, (g_{\alpha\beta})_{\alpha\leq\beta\in I})$ be a directed system of S-acts and S-maps, and $g_\alpha : B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha$ are the colimit maps. Take s-dense monomorphisms $h_\alpha : A \to B_\alpha$, $\alpha \in I$, with $g_{\alpha\beta}h_\alpha = h_\beta$ for $\alpha \leq \beta \in I$. Let $h : A \to \underrightarrow{lim}_\alpha B_\alpha$ be the directed colimit of h_αs. That is, $h = \underrightarrow{lim}_\alpha h_\alpha = g_\gamma h_\gamma = g_\alpha h_\alpha = g_\beta h_\beta = \ldots$. Then since each h_α is a monomorphism, h is a monomorphism. Also, h is s-dense because for $b \in \underrightarrow{lim}_\alpha B_\alpha$ and $s \in S$, since $b \in \underrightarrow{lim}_\alpha B_\alpha$, there exists $x_\sigma \in B_\sigma$ such that $b = [x_\sigma]_\rho$ and since h_σ is s-dense, there exists an element $a_s \in A$ with $h_\sigma(a_s) = x_\sigma s$. Then $bs = [x_\sigma]_\rho s = g_\sigma(x_\sigma)s = g_\sigma(x_\sigma s) = g_\sigma h_\sigma(a_s) = h(a_s) \in \mathrm{Im}(h)$. ■

Definition 3.21. We say that a category $\mathcal{A}$ fulfills the $\mathcal{M}$*-chain* condition if for every directed system $((A_\alpha)_{\alpha\in I}, (f_{\alpha\beta})_{\alpha\leq\beta\in I})$ whose index set I is a well-ordered chain with the least element 0, and $f_{0\alpha} \in \mathcal{M}$ for all α, there is a (so called "upper bound") family $(g_\alpha : A_\alpha \to A)_{\alpha\in I}$ with $g_0 \in \mathcal{M}$ and $g_\beta f_{\alpha\beta} = g_\alpha$.

Proposition 3.22. *The category* **Act-S** *fulfills the* $\mathcal{M}_d$*-chain condition.*

Proof. Take $A = \underrightarrow{lim}_\alpha A_\alpha$ and let $g_\alpha : A_\alpha \to A$ be the colimit maps. Then, applying Proposition 3.20, we get the result. ■

Theorem 3.23. *The class* $\mathcal{M}_d$ *is closed under colimits.*

Proof. Let $\mathcal{A}, \mathcal{B} : \mathbf{I} \to$ **Act-S** be diagrams in **Act-S** determining the acts A_α, B_α for $\alpha \in I = Obj(\mathbf{I})$, and the S-maps $g_{\alpha\beta} : A_\alpha \to A_\beta$, $g'_{\alpha\beta} : B_\alpha \to B_\beta$, for $\alpha \to \beta$ in $Mor(\mathbf{I})$. Consider the colimits of these diagrams with the colimit maps $g_\alpha = \gamma_\theta u_\alpha : A_\alpha \to \underrightarrow{lim}_\alpha A_\alpha = \coprod_{\alpha\in I} A_\alpha/\theta$, $g'_\alpha = \gamma_{\theta'} u'_\alpha : B_\alpha \to \underrightarrow{lim}_\alpha B_\alpha = \coprod_{\alpha\in I} B_\alpha/\theta'$, and assume that $\{f_\alpha : A_\alpha \to B_\alpha : \alpha \in I\}$ is a family of s-dense monomorphisms such that $g'_{\alpha\beta} f_\alpha = f_\beta g_{\alpha\beta}$. We show that $f =: \underrightarrow{lim}_\alpha f_\alpha$ is an s-dense monomorphism. Recall that θ is the congruence generated by $H = \{(u_\alpha(a_\alpha), u_\beta g_{\alpha\beta}(a_\alpha)) : a_\alpha \in A_\alpha,\ \alpha \to \beta \in Mor(\mathbf{I})\}$, and θ' is the congruence generated by $H' = \{(u'_\alpha(b_\alpha), u'_\beta g'_{\alpha\beta}(b_\alpha)) : b_\alpha \in B_\alpha,\ \alpha \to \beta \in Mor(\mathbf{I})\}$, and notice that $f[u_\alpha(a_\alpha)]_\theta = [u'_\alpha f_\alpha(a_\alpha)]_{\theta'}$. Since each f_α is a monomorphism, it is not hard to check that f is a monomorphism. To see that f is s-dense, let $s \in S$, $x = [u'_\alpha(b_\alpha)]_{\theta'} \in \underrightarrow{lim}_\alpha B_\alpha$ for some $\alpha \in I$. Since f_α is s-dense, there exists $a_\alpha \in A_\alpha$ with $f_\alpha(a_\alpha) = b_\alpha s$. Then, $g_\alpha(a_\alpha) = [u_\alpha(a_\alpha)]_\theta \in \underrightarrow{lim}_\alpha A_\alpha$ and we have $xs = [u'_\alpha(b_\alpha s)]_{\theta'} = g'_\alpha(b_\alpha s) = g'_\alpha f_\alpha(a_\alpha) = f g_\alpha(a_\alpha) \in \mathrm{Im}(f)$. ■

Acknowledgments

The authors thank the referee for his/her very careful reading and useful comments. We also would like to thank Professor M. Mehdi Ebrahimi for his very good comments and helpful conversations during this research.

REFERENCES

1. B. Banaschewski, Injectivity and essential extensions in equational classes of algebras, *Queen's Papers in Pure and Applied Mathematics*, **25** (1970), 131-147.
2. D. Dikranjan and W. Tholen, *Categorical structure of closure operators, with applications to topology, algebra, and discrete mathematics*; Mathematics and Its Applications, Kluwer Academic Publ., 1995.
3. M. M. Ebrahimi, Algebra in a Grothendieck topos: Injectivity in quasi- equational classes, *J. Pure Appl. Alg.*, **26** (1982), 269-280.
4. M. M. Ebrahimi, On ideal closure operators of M-sets, *Southeast Asian Bull. of Math.*, **30** (2006), 439-444.
5. M. M. Ebrahimi and M. Mahmoudi, The category of M-sets, *Ital. J. Pure Appl. Math.*, **9** (2001), 123-132.
6. M. M. Ebrahimi, M. Mahmoudi and Gh. Moghaddasi, On the Baer criterion for acts over semigroups, *Comm. Algebra*, **35(2)** (2007), 3912-3918.
7. H. Ehrig, F. Parisi-Presicce, P. Boehm, C. Rieckhoff, C. Dimitrovici and M. Grosse-Rhode, Algebraic data type and process specifications based on projection Spaces, *Lecture Notes in Computer Sci.*, **332** (1988), 23-43.
8. H. Ehrig, F. Parisi-Presicce, P. Bohem, C. Rieckhoff, C. Dimitrovici and M. Grosse-Rhode, Combining data type and recursive process specifications using projection algebras, *Theoretical Computer Science*, **71** (1990), 347-380.
9. E. Giuli, On m-separated projection spaces, *Appl. Categ. Struc.*, **2** (1994), 91-99.
10. H. Herrlich and H. Ehrig, The construct **PRO** of projection spaces: its internal structure, *Lecture Notes in Computer Sci.*, **393** (1988), 286-293.
11. M. Kilp, U. Knauer and A. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, New York, 2000.
12. M. Mahmoudi and M. M. Ebrahimi, Purity and equational compactness of projection algebras, *Appl. Cat. Struc.*, **9(4)** (2001), 381-394.
13. M. Mahmoudi and L. Shahbaz, Characterizing semigroups by sequentially dense injective acts, *Semigroup Forum*, **75(1)** (2007), 116-128.
14. W. Tholen, Injective objects and cogenerating sets, *J. Algebra*, **73(1)** (1981), 139-155.

Mojgan Mahmoudi and Leila Shahbaz
Department of Mathematics and Center of Excellence in Algebraic
and Logical Structures in Discrete Mathematics
Shahid Beheshti University
G. C., Tehran
Iran
E-mail: m-mahmoudi@cc.sbu.ac.ir
l-shahbaz@cc.sbu.ac.ir