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NON-ISOTROPIC FLAG SINGULAR INTEGRALS ON MULTI-PARAMETER HARDY SPACES

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Abstract. Recently, Han and Lu [HL] developed a discrete Littlewood-Paley-Stein analysis and multi-parameter Hardy space theory associated with isotropic flag singular integral operators initially studied by Muller-Ricci-Stein [MRS] and Nagel-Ricci-Stein [NRS]. The purpose of this paper is to carry out the multi-parameter Hardy space theory associated with non-isotropic flag singular integrals. The boundedness of such flag singular integral operator T from H^p_F to H^p_F and from H^p_F to L^p are established in this paper. Discrete Calderon's identity and Min-Max principle derived here are the main tools used to establish the non-isotropic multi-parameter Hardy space theory.

1. Introduction and Statement of Results

In the works of Muller-Ricci-Stein [MRS] and Nagel-Ricci-Stein [NRS], they established the L^p theory of flag singular integrals of both isotropic and non-isotropic types with applications to analysis on quadratic CR manifolds and Marcinkiewitz multipliers on the Heisenberg group. In this paper, we extend results in [HL] on multi-parameter Hardy space associated with isotropic flag singular integrals to those under a family of non-isotropic dilations with homogeneous norms. Precisely, we define a dilation by $\delta_r(x, y) = (rx, r^2y)$ $(r > 0)$ with the norm $|(x, y)| = (|x|^2 + |y|)^{\frac{1}{2}}$ for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ and obtain the boundedness of flag singular integral operator T by using a similar idea due to Han and Lu. In [HL], they first establish a discrete Calder*ó*n reproducing formula and a Min-Max type inequality in test function spaces and then develop the implicit multi-parameter Hardy space theory and finally get the boundedness of T on H^P_F and from H^P_F

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to L^p. In this paper, we will carry out the corresponding multi-parameter theory associated to the non-isotropic dilations.

We will use a lifting method to establish test functions on $\mathbb{R}^n \times \mathbb{R}^m$ with the following convolution construction similar to that in [HL].

Definition 1.1. We define a non-standard convolution $*_2$ by

$$\psi(x,y) = \psi^{(1)} *_2 \psi^{(2)}(x,y) = \int_{\mathbb{R}^m} \psi^{(1)}(x, y-z)\psi^{(2)}(z)dz,$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m}), \psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$, and satisfy

$$\sum_j |\widehat{\psi^{(1)}}(2^{-j}\xi_1, 2^{-2j}\xi_2)|^2 = 1, \quad \text{for all } (\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^m \backslash \{(0,0)\},$$

and

$$\sum_k |\widehat{\psi}^{(2)}(2^{-k}\eta)|^2 = 1, \quad \text{for all } \eta \in \mathbb{R}^m \backslash \{0\},$$

and the cancellation conditions

$$\int_{\mathbb{R}^{n+m}} \psi^{(1)}(x,y)x^\alpha y^\beta dxdy = 0, \int_{\mathbb{R}^m} \psi^{(2)}(z)z^\gamma dz = 0, \text{ for all multi-indices } \alpha, \beta, \text{ and } \gamma.$$

Using this convolution, we then define the non-isotropic Littlewood-Paley-Stein square function.

Definition 1.2. For $f \in L^p, 1 < p < \infty$, we define $S(f)$, the Littlewood-Paley-Stein square function of f by

$$S(f)(x,y) = \left\{ \sum_j \sum_k |\psi_{j,k} * f(x,y)|^2 \right\}^{\frac{1}{2}}, \tag{1.1}$$

where functions

$$\psi_{j,k}(x,y) = \psi_j^{(1)} *_2 \psi_k^{(2)}(x,y),$$

$$\psi_j^{(1)}(x,y) = 2^{(n+2m)j}\psi^{(1)}(2^j x, 2^{2j}y) \text{ and } \psi_k^{(2)}(z) = 2^{mk}\psi^{(2)}(2^k z).$$

From the Fourier transform, it is easy to get the following continuous Calderon reproducing formula

$$f(x,y) = \sum_{j,k} \psi_{j,k} * \psi_{j,k} * f(x,y), \text{ for any } f \in L^2(\mathbb{R}^n \times \mathbb{R}^m). \tag{1.2}$$

Following the definition of Nagel-Ricci-Stein [NRS], we have the definitions of product kernel and flag kernel associated with the non-isotropic dilations below.

Definition 1.3. A distribution $K^\sharp$ on $\mathbb{R}^{n+m}\times\mathbb{R}^m$ is said to be a product kernel on $\mathbb{R}^{n+m}\times\mathbb{R}^m$ if $K^\sharp$ is a C^∞ function away from the coordinate subspaces $\{(0,0,z)|(0,0)\in\mathbb{R}^{n+m}, z\in\mathbb{R}^m\}$ and $\{(x,y,0)|(x,y)\in\mathbb{R}^{n+m}, 0\in\mathbb{R}^m\}$, and for all $(x,y,z)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m$ with $|x|+|y|\neq 0$ and $|z|\neq 0$ satisfies

(1) (Differential Inequalities) For any multi-indices $\alpha=(\alpha_1,\cdots,\alpha_n)$, $\beta=(\beta_1,\cdots,\beta_m)$ and $\gamma_m=(\gamma_1,\cdots,\gamma_m)$

$$\left|\partial_x^\alpha\partial_y^\beta\partial_z^\gamma K^\sharp(x,y,z)\right|\le C_{\alpha,\beta,\gamma}|(x,y)|^{-(n+2m+|\alpha|+2|\beta|)}\cdot|z|^{-m-|\gamma|}.$$

(2) (Cancellation Condition)

$$\left|\int_{\mathbb{R}^m}\partial_x^\alpha\partial_y^\beta K^\sharp(x,y,z)\phi_1(\delta z)dz\right|\le C_{\alpha,\beta}|(x,y)|^{-(n+2m+|\alpha|+2|\beta|)}$$

for all multi-indices α,β and every normalized bump function ϕ_1 on $\mathbb{R}^m$ and every $\delta>0$;

$$\left|\int_{\mathbb{R}^{n+m}}\partial_z^\gamma K^\sharp(x,y,z)\phi_2(\delta x,\delta^2 y)dxdy\right|\le C_\gamma|z|^{-m-|\gamma|}$$

for every multi-index γ and every normalized bump function ϕ_2 on $\mathbb{R}^{n+m}$ and every $\delta>0$; and

$$\left|\int_{\mathbb{R}^{n+m+m}}K^\sharp(x,y,z)\phi_3(\delta_1 x,\delta_1^2 y,\delta_2 z)dxdydz\right|\le C$$

for every normalized bump function ϕ_3 on $\mathbb{R}^{n+m+m}$ and every $\delta_1>0$ and $\delta_2>0$.

Definition 1.4. A distribution on $\mathbb{R}^{n+m}$ is said to be a flag kernel on $\mathbb{R}^n\times\mathbb{R}^m$ if K is a C^∞ function away from the coordinate subspace $\{(0,y)|0\in\mathbb{R}^n, y\in\mathbb{R}^m\}$, and for all $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$ with $|x|\neq 0$ satisfies

(1) (Differential Inequalities)

$$\left|\partial_x^\alpha\partial_y^\beta K(x,y)\right|\le C_{\alpha,\beta}|x|^{-n-|\alpha|}\,|(x,y)|^{-2m-2|\beta|}$$

for any multi-indices $\alpha=(\alpha_1,\cdots,\alpha_n)$, $\beta=(\beta_1,\cdots,\beta_m)$

(2) (Cancellation Conditions)

$$\left|\int_{\mathbb{R}^m}\partial_x^\alpha K(x,y)\phi_1(\delta y)dy\right|\le C_\alpha|x|^{-n-|\alpha|}$$

for every multi-index α and every normalized bump function ϕ_1 on $\mathbb{R}^m$ and every $\delta > 0$;

$$\Big| \int_{\mathbb{R}^n} \partial_y^\beta K(x,y)\phi_2(\delta x)dx \Big| \leq C_\gamma |y|^{-m-|\beta|}$$

for every multi-index β and every normalized bump function ϕ_2 on $\mathbb{R}^n$ and every $\delta > 0$; and

$$\Big| \int_{\mathbb{R}^{n+m}} K(x,y)\phi_3(\delta_1 x, \delta_2 y)dxdy \Big| \leq C$$

for every normalized bump function ϕ_3 on $\mathbb{R}^{n+m}$ and every $\delta_1 > 0$ and $\delta_2 > 0$.

We give an alternative proof of the fact that flag singular integral operators are bounded on L^p for $1 < p < \infty$, which was initially proved in [NRS].

Theorem 1.1. *Let $T(f)(x,y) = K * f(x,y)$ be a flag singular integral on $\mathbb{R}^n \times \mathbb{R}^m$, where K is a flag kernel. Then T is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^m)$ for $1 < p < \infty$, that is,*

$$\|T(f)\|_p \leq C\|f\|_p, \text{ for } f \in L^p(\mathbb{R}^n \times \mathbb{R}^m),$$

where the constant C is dependent only on p.

In order to define the discrete Littlewood-Paley-Stein square function on a appropriate distribution space, we need to introduce the test functions of order M, $S_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$, where M is a positive integer.

Definition 1.5. We say $f(x,y,z) \in S_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ if f is a Schwartz test function and satisfies the following conditions:

(i) For $|\alpha|, |\beta|, |\gamma| \leq M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z)| \leq C \frac{1}{(1+|(x,y)|)^{n+2m+3M+|\alpha|+2|\beta|}} \frac{1}{(1+|z|)^{m+M+|\gamma|}},$$

(ii) For $|x-x'| \leq \frac{1}{2}(1+|x|)$ and $|y-y'| \leq \frac{1}{2}(1+|y|), |\alpha| = |\beta| = M$ and $|\gamma| \leq M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x',y',z)|$$
$$\leq C \frac{|(x-x', y-y')|}{(1+|(x,y)|)^{n+2m+6M}} \frac{1}{(1+|z|)^{m+M+|\gamma|}},$$

(iii) For $|z-z'| \le \frac{1}{2}(1+|z|), |\gamma| = M$ and $|\alpha|, |\beta| \le M-1$,

$$|D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z) - D_x^\alpha D_y^\beta D_z^\gamma f(x,y,z')|$$
$$\le C \frac{1}{(1+|(x,y)|)^{n+2m+3M+|\alpha|+2|\beta|}} \frac{|z-z'|}{(1+|z|)^{m+2M}},$$

(iv) For $|x-x'| \le \frac{1}{2}(1+|x|), |y-y'| \le \frac{1}{2}(1+|y|), |z-z'| \le \frac{1}{2}(1+|z|)$ and $|\nu| = M$,

$$|D_x^\nu D_y^\nu D_z^\nu f(x,y,z) - D_x^\nu D_y^\nu D_z^\nu f(x',y',z) - D_x^\nu D_y^\nu D_z^\nu f(x,y,z')$$
$$+D_x^\nu D_y^\nu D_z^\nu f(x',y',z')| \le C \frac{|(x-x',y-y')|}{(1+|(x,y)|)^{n+2m+6M}} \frac{|z-z'|}{(1+|z|)^{m+2M}},$$

(v) For $|\alpha|, |\beta|, |\gamma| \le M-1$,

$$\int_{\mathbb{R}^{n+m}} f(x,y,z) x^\alpha y^\beta dx dy = \int_{\mathbb{R}^m} f(x,y,z) z^\gamma dz = 0.$$

If $f \in \mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$, the norm of f in $\mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ then is defined by

$$\|f\|_{\mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)} = \inf\{C : (i)-(iv)\text{hold}\}.$$

It is easy to check that $\mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ with this norm is a Banach spaces.

We now define the test function space $\mathcal{S}_{F,M}$ on $\mathbb{R}^n \times \mathbb{R}^m$ associated with the flag structure.

Definition 1.6. A function $f(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ is called to be a test function in $\mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m)$ if there exists a function $f^\sharp \in \mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ such that

$$f(x,y) = \int_{\mathbb{R}^m} f^\sharp(x, y-z, z) dz. \tag{1.3}$$

The norm of f in $\mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m)$ is defined by

$$\|f\|_{\mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m)} = \inf\{\|f^\sharp\|_{\mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)} : \text{ for all representations of } f \text{ in (1.3)}\}.$$

The dual space of $\mathcal{S}_{F,M}$ is denoted by $(\mathcal{S}_{F,M})'$.

For simplicity, in the following we denote $\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$ by $\mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m)$ for a given positive integer M.

Our main tool to characterize the flag Hardy space is the discrete Littlewood-Paley-Stein analysis. Thus, we need the following discrete Calderon reproducing formula to construct another Littlewood-Paley-Stein square function.

Theorem 1.2. *If $\psi_{j,k}$ are the same as in (1.1). Then*

$$f(x,y)=\sum_j\sum_k\sum_J\sum_I |I||J|\widetilde{\psi}_{j,k}(x,y,x_I,y_J)\psi_{j,k}*f(x_I,y_J) \tag{1.4}$$

where $\widetilde{\psi}_{j,k}(x,y,x_I,y_J)\in\mathcal{S}_F(\mathbb{R}^n\times\mathbb{R}^m), I\subset\mathbb{R}^n, J\subset\mathbb{R}^m$, are dyadic cubes with side-length $\ell(I)=2^{-j-N}$ and $\ell(J)=2^{-k-N}+2^{-2j-N}$ for a fixed large integer N, x_I, y_J are any fixed points in I, J, respectively, and the series in (1.4) converges in the norm of $\mathcal{S}_F(\mathbb{R}^n\times\mathbb{R}^m)$ and in the dual space $(\mathcal{S}_F)'$.

The discrete Calderón reproducing formula (1.4) provides the following Min-Max type inequalities.

Theorem 1.3. *Suppose $\psi^{(1)},\phi^{(1)}\in\mathcal{S}(\mathbb{R}^{n+m}),\psi^{(2)},\phi^{(2)}\in\mathcal{S}(\mathbb{R}^m)$ and*

$$\psi(x,y)=\psi^{(1)}*_2\psi^{(2)}(x,y),\quad \phi(x,y)=\phi^{(1)}*_2\psi^{(2)}(x,y),$$

and ψ_{jk}, ϕ_{jk} satisfy the conditions in (1.1). Then for $f\in(\mathcal{S}_F)'$ and $0<p<\infty$,

$$\begin{aligned}&\left\|\left\{\sum_j\sum_k\sum_J\sum_I \sup_{u\in I,v\in J}|\psi_{j,k}*f(u,v)|^2\chi_I(x)\chi_J(y)\right\}^{\frac12}\right\|_p\\ &\approx\left\|\left\{\sum_j\sum_k\sum_J\sum_I \inf_{u\in I,v\in J}|\phi_{j,k}*f(u,v)|^2\chi_I(x)\chi_J(y)\right\}^{\frac12}\right\|_p\end{aligned} \tag{1.5}$$

where $\psi_{j,k}(x,y)$ and $\phi_{j,k}(x,y)$ are defined as in (1.4), $I\subset\mathbb{R}^n, J\subset\mathbb{R}^m$, are dyadic cubes with side-length $\ell(I)=2^{-j-N}$ and $\ell(J)=2^{-k-N}+2^{-2j-N}$ for a fixed large integer N,χ_I and χ_J are indicator functions of I and J, respectively.

This Min-Max type inequality ensures that the following non-isotropic flag Hardy Space H^p_F is independent of the choice of the function ψ and thus this definition is well defined.

Definition 1.7. For $0<p\le 1$, $H^p_F(\mathbb{R}^n\times\mathbb{R}^m)=\{f\in(\mathcal{S}_F)':S_d(f)\in L^p(\mathbb{R}^n\times\mathbb{R}^m)\}$ with the norm $\|f\|_{H^p_F}\approx\|S_d(f)\|_p$, where S_d is the non-isotropic discrete

Littlewood-Paley square function defined by

$$S_d(f)(x,y) = \left\{ \sum_{j,k,I,J} |\psi_{j,k} * f(x_I, y_J)|^2 \chi_I(x)\chi_J(y) \right\}^{\frac{1}{2}},$$

where j, k, I, J are the same as in Theorem 1.2.

Using the discrete Calderoń identity and Min-Max principle we can prove the following

Theorem 1.4. *If T is a flag singular integral with the flag kernel $K(x,y)$. Then for any $0 < p \le 1$ there exists a constant $C = C(p)$ such that*

$$\|T(f)\|_{H^p_F} \le C\|f\|_{H^p_F}.$$

Theorem 1.5. *Let $0 < p \le 1$. If a linear operator T is bounded on $L^2(\mathbb{R}^{n+m})$ and $H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$, then T is bounded from $H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ to $L^p(\mathbb{R}^{n+m})$.*

2. Proof of Theorems 1.1

Lemma 2.1. *Suppose that $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m})$ is supported in the unit ball of $\mathbb{R}^{n+m}$ and $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ is supported in the unit ball of $\mathbb{R}^m$, and satisfy*

$$\int_0^\infty |\widehat{\psi}^{(1)}(t\xi_1, t^2\xi_2)|^2 \frac{dt}{t} = 1, \quad \text{for all } (\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^m \backslash \{(0,0)\},$$

and

$$\int_0^\infty |\widehat{\psi}^{(2)}(s\eta)|^2 \frac{ds}{s} = 1, \quad \text{for all } \eta \in \mathbb{R}^m \backslash \{0\}.$$

We set $\psi_{t,s}(x,y) = \int_{\mathbb{R}^m} \psi^{(1)}_t(x, y-z)\psi^{(2)}_s(z)dz$, where $\psi^{(1)}_t(x,y) = t^{-n-2m}\psi^{(1)}(\frac{x}{t}, \frac{y}{t^2})$ and $\psi^{(2)}_s(z) = s^{-m}\psi^{(2)}(\frac{z}{s})$. Then, for $1 < p < \infty$,

$$\left\| \left\{ \int_0^\infty \int_0^\infty |\psi_{t,s} * f(x,y)|^2 \frac{dt}{t}\frac{ds}{s} \right\}^{\frac{1}{2}} \right\|_p \approx \|f\|_p, \tag{2.1}$$

Proof of Lemma 2.1. Define $F : \mathbb{R}^n \times \mathbb{R}^m \to H = L^2((0,\infty), \frac{dt}{t})$ by $F(x,y) = f * \psi^{(1)}_t(x,y)$ with the norm

$$\|F\|_H = \int_0^\infty |f * \psi^{(1)}_t(x,y)|^2 \frac{dt}{t},$$

and set

$$T(F)(x,y)=\left\{\int_0^\infty \|F *_2 \psi_s^{(2)}(x,y)\|_H^2 \frac{ds}{s}\right\}^{\frac{1}{2}}.$$

We notice that $T(F)(x,y)=\int\limits_0^\infty\int\limits_0^\infty |\psi_{t,s}*f(x,y)|^2\frac{dt}{t}\frac{ds}{s}\}^{\frac{1}{2}}$.
By the vector-valued Littlewood-Paley-Stein inequality,

$$\left\|\left\{\int_0^\infty \|F *_2 \psi_s^{(2)}(x,y)\|_H^2 \frac{ds}{s}\right\}^{\frac{1}{2}}\right\|_p \le C\,\|\|F\|_H\|_p,$$

and by the standard Littlewood-Paley-Stein inequality,

$$\left\|\left\{\int_0^\infty |f*\psi_t^{(1)}(x,y)|^2 \frac{dt}{t}\right\}^{\frac{1}{2}}\right\|_p \le C\|f\|_p,$$

thus,

$$\left\|\left\{\int\limits_0^\infty\int\limits_0^\infty |\psi_{t,s}*f(x,y)|^2\frac{dt}{t}\frac{ds}{s}\right\}^{\frac{1}{2}}\right\|_p \le C\|f\|_p.$$

To show the estimate $\|f\|_p\le C\|\{\int\limits_0^\infty\int\limits_0^\infty |\psi_{t,s}*f(x,y)|^2\frac{dt}{t}\frac{ds}{s}\}^{\frac{1}{2}}\|_p$, we first suppose that $f\in L^2(\mathbb{R}^n\times\mathbb{R}^m)\cap L^p(\mathbb{R}^n\times\mathbb{R}^m)$.
From the Fourier transform, we have

$$f(x,y)=\int_0^\infty\int_0^\infty \psi_{t,s}*\psi_{t,s}*f(x,y)\frac{dt}{t}\frac{ds}{s},\ \text{for any } f\in L^2,$$

then for $f\in L^2(\mathbb{R}^n\times\mathbb{R}^m)\cap L^p(\mathbb{R}^n\times\mathbb{R}^m)$,

$$\begin{aligned}\|f\|_p &= \sup_{\|g\|_{p'}\le 1}\left|<\int_0^\infty\int_0^\infty \psi_{t,s}*\psi_{t,s}*f(x,y)\frac{dt}{t}\frac{ds}{s}, g(x,y)>\right|\\ &= \sup_{\|g\|_{p'}\le 1}\left|\int_0^\infty\int_0^\infty <\psi_{t,s}*f(x,y),\psi_{t,s}*g(x,y)>\frac{dt}{t}\frac{ds}{s}\right|\\ &\le \sup_{\|g\|_{p'}\le 1}\left\|\left(\int_0^\infty\int_0^\infty |\psi_{t,s}*f|^2\frac{dt}{t}\frac{ds}{s}\right)^{\frac{1}{2}}\right\|_p\left\|\left(\int_0^\infty\int_0^\infty |\psi_{t,s}*g|^2\frac{dt}{t}\frac{ds}{s}\right)^{\frac{1}{2}}\right\|_{p'}\\ &\le C\left\|\left\{\int\limits_0^\infty\int\limits_0^\infty |\psi_{t,s}*f(x,y)|^2\frac{dt}{t}\frac{ds}{s}\right\}^{\frac{1}{2}}\right\|_p,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$

Since $L^2 \cap L^p$ is dense in L^p, Lemma 2.1 is completed through limiting argument.■

Following a similar proof as that in the above Lemma, we have

Lemma 2.2. *For* $1 < p < \infty$, $\|f\|_p \approx \|S_a(f)\|_p$.

Proof of Theorem 1.1. We may always assume that K is the integrable function by using a smooth truncation argument. From Lemma 2.1, we also have

$$\|f\|_p \approx \|\{\int_0^\infty \int_0^\infty |\psi_{t,s} * \psi_{t,s} * f(x,y)|^2 \frac{dt}{t}\frac{ds}{s}\}^{\frac{1}{2}}\|_p. \tag{2.2}$$

Thus,

$$\|T(f)\|_p \leq C\|\{\int_0^\infty \int_0^\infty |\psi_{t,s} * \psi_{t,s} * K * f(x,y)|^2 \frac{dt}{t}\frac{ds}{s}\}^{\frac{1}{2}}\|_p. \tag{2.3}$$

For $f \in L^p, 1 < p < \infty$, we claim that

$$|\psi_{t,s} * K * f(x,y)| \leq CM_s(f)(x,y), \tag{2.4}$$

where C is a constant independent of the L^1 norm of K and $M_s(f)$ is the strong maximal function of f.

If the claim is true, then from (2.1) and Fefferman-Stein vector-valued maximal theorem, Theorem 1.1 is obtained.

To show the claim, we note that $\psi_{t,s} * K(x,y) = \int \psi^\sharp_{t,s} * K^\sharp(x, y-z, z)dz$, where $\psi^\sharp_{t,s}(x,y,z) = \psi^{(1)}_t(x, y-z)\psi^{(2)}_s(z)$ and $K(x,y) = \int K^\sharp(x, y-z, z)dz$, $K^\sharp(x,y,z)$ is a product kernel.

Let

$$\triangle^1_{u,v}K^\sharp(x,y,z) = K^\sharp(x-u, y-v, z) - K^\sharp(x,y,z),$$

and

$$\triangle^2_w K^\sharp(x,y,z) = K^\sharp(x, y-w, z-w) - K^\sharp(x,y,z),$$

then

$$\begin{aligned} & \triangle^1_{u,v}(\triangle^2_w K^\sharp)(x,y,z) \\ = \; & K^\sharp(x-u, y-v-w, z-w) - K^\sharp(x-u, y-v, z) \\ & -K^\sharp(x, y-w, z-w) + K^\sharp(x,y,z). \end{aligned}$$

Since

$$\int_0^\infty |\widehat{\psi^{(1)}}(t\xi_1, t^2\xi_2)|^2 \frac{dt}{t} = 1, \quad \text{for all } (\xi_1,\xi_2) \in \mathbb{R}^n \times \mathbb{R}^m \backslash \{(0,0)\},$$

and
$$\int_0^\infty |\widehat{\psi^{(2)}}(s\eta)|^2 \frac{ds}{s} = 1, \quad \text{for all} \quad \eta \in \mathbb{R}^m \backslash \{0\}.$$

We have from the continuity of $\widehat{\psi^{(1)}}$ and $\widehat{\psi^{(2)}}$
$$\int_{\mathbb{R}^n \times \mathbb{R}^m} \psi^{(1)}(u,v)dudv = \widehat{\psi^{(1)}}(0,0) = 0, \quad \text{and} \int_{\mathbb{R}^m} \psi^{(2)}(w)dw = \widehat{\psi^{(2)}}(0) = 0,$$

thus,
$$\begin{aligned}
&|\psi^\sharp_{t,s} * K^\sharp(x,y,z)| \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} K^\sharp(x-u, y-v-w, z-w)\psi_t^{(1)}(u,v)\psi_s^{(2)}(w)dudvdw \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} \triangle^1_{u,v}(\triangle^2_w K^\sharp)(x,y,z)\psi_t^{(1)}(u,v)\psi_s^{(2)}(w)dudvdw
\end{aligned}$$

Since $\psi_t^{(1)}, \psi_s^{(2)}$ have size conditions and $K^\sharp$ satisfies conditions of Definition 1.2, we have the following estimates
$$|\psi^\sharp_{t,s} * K^\sharp(x,y,z)| \le C \frac{t}{(t+|(x,y)|)^{n+2m+1}} \frac{s}{(s+|z|)^{m+1}}, \tag{2.5}$$

where the constant C is independent of the L^1 norm of K.
For $f \in L^p(\mathbb{R}^n)$, since the Lebesgue differentiation theorem holds, we have $f(x) \le M_s(f)(x)$.
Therefore,
$$\begin{aligned}
&|\psi_{t,s} * K * f(x,y)| \\
\le\ & C\left| \int_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} f(x-u, y-v) \frac{t}{(t+|(u,v-w)|)^{n+2m+1}} \frac{s}{(s+|w|)^{m+1}} dudvdw \right| \\
\le\ & CM_s(f)(x,y).
\end{aligned}$$
■

3. Proofs of Theorems 1.2 and 1.3

In this section, we use the the continuous version of the Calderón reproducing formula on S_F and almost orthogonality estimates to derive the discrete Calderón reproducing formula and the Min-Max type inequalities on S_F.

We suppose that $\psi^\sharp(x,y,z,u,v,w)$ for $(x,y,z), (u,v,w) \in \mathbb{R}^{n+m} \times \mathbb{R}^m$ is a smooth function, satisfying the differential inequalities
$$|\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_z^{\gamma_1}\partial_u^{\alpha_2}\partial_v^{\beta_2}\partial_w^{\gamma_2}\psi^\sharp(x,y,z,u,v,w)|$$

$$(3.1)\qquad \leq A_{N,M,\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2}(1+|(x-u,y-v)|)^{-N}(1+|z-w|)^{-M}$$

and the cancellation conditions

$$(3.2)\qquad \begin{aligned}&\int_{\mathbb{R}^{n+m}}\psi^\sharp(x,y,z,u,v,w)x^{\alpha_1}y^{\beta_1}dxdy\\ =&\int_{\mathbb{R}^m}\psi^\sharp(x,y,z,u,v,w)z^{\gamma_1}dz\\ =&\int_{\mathbb{R}^{n+m}}\psi^\sharp(x,y,z,u,v,w)u^{\alpha_2}v^{\beta_2}dudv\\ =&\int_{\mathbb{R}^m}\psi^\sharp(x,y,z,u,v,w)w^{\gamma_2}dw=0,\end{aligned}$$

and for any fixed $x_0\in\mathbb{R}^n, y_0\in\mathbb{R}^m, \phi^\sharp(x,y,z,x_0,y_0)\in\mathcal{S}_\infty(\mathbb{R}^{n+m}\times\mathbb{R}^m)$ and satisfies

$$(3.3)\qquad \begin{aligned}&|\partial_x^{\alpha_1}\partial_y^{\beta_1}\partial_z^{\gamma_1}\phi^\sharp(x,y,z,x_0,y_0)|\\ \leq\ & B_{N,M,\alpha_1,\beta_1,\gamma_1,}(1+|(x-x_0,y-y_0)|)^{-N}(1+|z|)^{-M},\end{aligned}$$

for all positive integers N,M and multi-indices $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2$ of nonnegative integers. Then we have the following almost orthogonality estimate:

Lemma 3.1. *For any given positive integers L_1,L_2 and K_1,K_2, there exists a constant C depending only on L_1,L_2,K_1,K_2 and the constants in* (3.1) *and* (3.3) *such that for all positive numbers t,s,t',s'*

$$(3.4)\qquad \begin{aligned}&\left|\int_{\mathbb{R}^{n+m+m}}\psi^\sharp_{t,s}(x,y,z,u,v,w)\phi^\sharp_{t',s'}(u,v,w,x_0,y_0)dudvdw\right|\\ \leq\ & C\Big(\frac{t}{t'}\wedge\frac{t'}{t}\Big)^{L_1}\Big(\frac{s}{s'}\wedge\frac{s'}{s}\Big)^{L_2}\\ &\frac{(t\vee t')^{K_1}}{(t\vee t'+|(x-x_0,y-y_0)|^{(n+2m+K_1)}}\frac{(s\vee s')^{K_2}}{(s\vee s'+|z|)^{(m+K_2)}},\end{aligned}$$

where $\psi^\sharp_{t,s}(x,y,z,u,v,w)=t^{-n-2m}s^{-m}\psi^\sharp(\frac{x}{t},\frac{y}{t^2},\frac{z}{s},\frac{u}{t},\frac{v}{t^2},\frac{w}{s})$ and

$$\phi^\sharp_{t,s}(x,y,z,x_0,y_0)=t^{-n-2m}s^{-m}\phi^\sharp(\frac{x}{t},\frac{y}{t^2},\frac{z}{s},\frac{x_0}{t},\frac{y_0}{t^2},).$$

Proof of Lemma 3.1. Without loss of generality we may assume that $t\geq t'$ and $s'\geq s$. We first consider the case $L_1=L_2=K_1=K_2=1$.

Since

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} \phi^\sharp_{t',s'}(u,v,w,x_0,y_0)dudv = 0,\ \int_{\mathbb{R}^m} \psi^\sharp_{t,s}(x,y,z,u,v,w)dw = 0$$

we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m} \psi^\sharp_{t,s}(x,y,z,u,v,w)\phi^\sharp_{t',s'}(u,v,w,x_0,y_0)dudvdw\right|\\ =&\left|\int_{\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m} A\cdot B\ dudvdw\right|\end{aligned}$$

where

$$A = \psi^\sharp_{t,s}(x,y,z,u,v,w) - \psi^\sharp_{t,s}(x,y,z,x_0,y_0,w)$$

and

$$B = \phi^\sharp_{t',s'}(u,v,w,x_0,y_0) - \phi^\sharp_{t',s'}(u,v,z,x_0,y_0).$$

Denote

$$\begin{aligned}\Omega_1 &= \{(u,v,w)\in\mathbb{R}^{n+m}\times\mathbb{R}^m : |(u-x_0,v-y_0)|\\ &\leq \frac{1}{2}\left(t+|(x-x_0,y-y_0)|\right),\ |w-z|\leq\frac{1}{2}(s'+|z|)\},\\ \Omega_2 &= \{(u,v,w)\in\mathbb{R}^{n+m}\times\mathbb{R}^m : |(u-x_0,v-y_0)|\\ &\leq \frac{1}{2}\left(t+|(x-x_0,y-y_0)|\right),\ |w-z|>\frac{1}{2}(s'+|z|)\},\\ \Omega_3 &= \{(u,v,w)\in\mathbb{R}^{n+m}\times\mathbb{R}^m : |(u-x_0,v-y_0)|\\ &> \frac{1}{2}\left(t+|(x-x_0,y-y_0)|\right),\ |w-z|\leq\frac{1}{2}(s'+|z|)\},\\ \Omega_4 &= \{(u,v,w)\in\mathbb{R}^{n+m}\times\mathbb{R}^m : |(u-x_0,v-y_0)|\\ &> \frac{1}{2}\left(t+|(x-x_0,y-y_0)|\right),\ |w-z|>\frac{1}{2}(s'+|z|)\},\end{aligned}$$

and then,

$$\int_{\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m} \psi^\sharp_{t,s}(x,y,z,u,v,w)\phi^\sharp_{t',s'}(u,v,w,x_0,y_0)dudvdw = \int_{\Omega_1}+\int_{\Omega_2}+\int_{\Omega_3}+\int_{\Omega_4}.$$

For the first term, we use the smoothness conditions of both $\psi_{t,s}$ and $\phi_{t',s'}$,

$$\begin{aligned}
&\int_{\Omega_1} A\cdot B\,dudvdw\\
&\le \int_{\Omega_1}\left\{\frac{|(u-x_0,v-y_0)|}{t+|(x-x_0,y-y_0)|}\frac{t}{(t+|(x-x_0,y-y_0)|)^{n+2m+1}}\frac{s^2}{(s+|z-w|)^{m+2}}\right.\\
&\qquad\left.\frac{(t')^2}{(t'+|(u-x_0,v-y_0)|)^{n+2m+2}}\frac{|w-z|}{s'+|z|}\frac{s'}{(s'+|z|)^{m+1}}\right\}dudvdw\\
&\le C\quad\frac{t'}{t}\frac{s}{s'}\frac{t}{(t+|(x-x_0,y-y_0)|)^{n+2m+1}}\frac{s'}{(s'+|z|)^{m+1}}.
\end{aligned}$$

For the second term , we use the smoothness conditions of $\psi_{t,s}$ the size condition of $\phi_{t',s'}$ and the fact that $|w-z|\sim|w|$ if $|w-z|>\frac{1}{2}(s'+|z|)$,

$$\begin{aligned}
&\int_{\Omega_2} A\cdot B\,dudvdw\\
&\le \int_{\Omega_2}\left\{\frac{|(u-x_0,v-y_0)|}{t+|(x-x_0,y-y_0)|}\frac{t}{(t+|(x-x_0,y-y_0)|)^{n+2m+1}}\frac{s^2}{(s+|z-w|)^{m+2}}\right.\\
&\qquad\left.\frac{(t')^2}{(t'+|(u-x_0,v-y_0)|)^{n+2m+2}}\left(\frac{s'}{(s'+|z|)^{m+1}}+\frac{s'}{(s'+|w|)^{m+1}}\right)\right\}dudvdw\\
&\le C\quad\frac{t'}{t}\frac{s}{s'}\frac{t}{(t+|(x-x_0,y-y_0)|)^{n+2m+1}}\frac{s'}{(s'+|z|)^{m+1}}
\end{aligned}$$

For the third term, we use the size condition of $\psi_{t,s}$ and the smooth condition of $\phi_{t',s'}$,

$$\begin{aligned}
&\int_{\Omega_3} A\cdot B\,dudvdw\\
&\le \int_{\Omega_3}\left\{\left(\frac{t}{(t+|(u-x_0,v-y_0)|)^{n+2m+1}}+\frac{t}{(t+|(x-x_0,y-y_0)|)^{n+2m+1}}\right)\right.\\
&\qquad\frac{s^2}{(s+|z-w|)^{m+2}}\frac{(t')^2}{(t'+|(u-x_0,v-y_0)|)^{n+2m+2}}\\
&\qquad\left.\frac{|w-z|}{s'+|z|}\frac{s'}{(s'+|z|)^{m+1}}\right\}dudvdw\\
&\le C\quad\frac{t'}{t}\frac{s}{s'}\frac{t}{(t+|(x-x_0,y-y_0)|)^{n+2m+1}}\frac{s'}{(s'+|z|)^{m+1}}
\end{aligned}$$

For the fourth term, we use the size conditions of both $\psi_{t,s}$ and $\phi_{t',s'}$ and the fact that $|w-z| \sim |w|$ if $|w-z| > \frac{1}{2}(s'+|z|)$

$$\begin{aligned}
&\int_{\Omega_4} A \cdot B \, dudvdw \\
\leq &\int_{\Omega_4} \Bigg\{ \left(\frac{t}{(t+|(x-x_0, y-y_0)|)^{n+2m+1}} + \frac{t}{(t+|(u-x_0, v-y_0)|)^{n+2m+1}} \right) \\
&\quad \frac{s^2}{(s+|z-w|)^{m+2}} \; \frac{(t')^2}{(t'+|(u-x_0, v-y_0)|)^{n+2m+2}} \\
&\quad \left(\frac{s'}{(s'+|z|)^{m+1}} + \frac{s'}{(s'+|w|)^{m+1}} \right) \Bigg\} dudvdw \\
\leq C &\quad \frac{t'}{t} \, \frac{s}{s'} \, \frac{t}{(t+|(x-x_0, y-y_0)|)^{n+2m+1}} \; \frac{s'}{(s'+|z|)^{m+1}} .
\end{aligned}$$

For the case $\max\{L_1, L_2, K_1, K_2\} > 1$, since $\phi^\sharp_{t',s'}$ and $\psi^\sharp_{t,s}$ satisfy cancelation conditions, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} \psi^\sharp_{t,s}(x, y, z, u, v, w) \phi_{t',s'}(u, v, w, x_0, y_0) dudvdw \\
= &\int_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} E \cdot F dudvdw
\end{aligned}$$

where

$$\begin{aligned}
E &= \psi^\sharp_{t,s}(x, y, z, u, v, w) - \sum_{|\alpha|+|\beta| \leq L_1 - 1} \partial_u^\alpha \partial_v^\beta \psi^\sharp_{t,s}(x, y, z, x_0, y_0, w) \\
F &= \phi^\sharp_{t',s'}(u, v, w, x_0, y_0) - \sum_{|\gamma| \leq L_2 - 1} \partial_w^\gamma \phi^\sharp_{t',s'}(u, v, z, x_0, y_0)
\end{aligned}$$

and then using a similar argument to the first case, we could also obtain (3.4). ■

Moreover, the almost orthogonality estimate still holds in S_F. But we need to use the following relationship.

Lemma 3.2. *If* $\psi, \phi \in \mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$, *and* $\psi^\sharp, \phi^\sharp \in \mathcal{S}_\infty(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ *such that*

$$\psi(x, y) = \int_{\mathbb{R}^m} \psi^\sharp(x, y-z, z) dz, \;\; \phi(x, y) = \int_{\mathbb{R}^m} \phi^\sharp(x, y-z, z) dz.$$

Then

$$(\psi * \phi)(x, y) = \int_{\mathbb{R}^m} \left(\psi^\sharp * \phi^\sharp \right) (x, y-z, z) dz.$$

It is quite easy to prove Lemma 3.2, so we omit the proof here.

Lemma 3.3. *For any given positive integers L_1, L_2 and K_1, K_2, there exists a constant C depending only on L_1, L_2, K_1, K_2 such that if $t \vee t' \leq \sqrt{s \vee s'}$, then*

$$|\psi_{t,s} * \phi_{t',s'}(x,y)| \leq C(\frac{t}{t'} \wedge \frac{t'}{t})^{L_1}(\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \cdot \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |y|)^{(m+K_2)}},$$

and if $t \vee t' \geq \sqrt{s \vee s'}$, then

$$|\psi_{t,s} * \phi_{t',s'}(x,y)| \leq C(\frac{t}{t'} \wedge \frac{t'}{t})^{L_1}(\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \cdot \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \frac{(t \vee t')^{K_2}}{(t \vee t' + \sqrt{|y|})^{(2m+K_2)}}.$$

Proof of Lemma 3.3. Note that for all $\psi^\sharp_{t,s}, \phi^\sharp_{t',s'} \in \mathcal{S}_\infty(\mathbb{R}^{n+m} \times \mathbb{R}^m)$,

$$\psi_{t,s} * \phi_{t',s'}(x,y) = \int_{\mathbb{R}^m} \psi^\sharp_{t,s} * \phi^\sharp_{t',s'}(x, y-z, z)dz,$$

and

$$\psi^\sharp_{t,s} * \phi^\sharp_{t',s'}(x,y,z) = \int_{\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m} \psi^\sharp_{t,s}(x-u, y-v, z-w)\phi^\sharp_{t',s'}(u,v,w)dudvdw,$$

Then by (3.4), for any given positive integers L_1, L_2 and K_1, K_2, there exists a constant C depending only on L_1, L_2, K_1, K_2 such that

$$|\psi_{t,s} * \phi_{t',s'}(x,y)| \leq C(\frac{t}{t'} \wedge \frac{t'}{t})^{L_1}(\frac{s}{s'} \wedge \frac{s'}{s})^{L_2} \int_{\mathbb{R}^m} \frac{(t \vee t')^{K_1}}{(t \vee t' + |(x, y-z)|)^{(n+2m+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |z|)^{(m+K_2)}} dz.$$

Case 1. If $t \vee t' \leq \sqrt{s \vee s'}$ and $|y| \geq s \vee s'$, then

$$\int_{\mathbb{R}^m} \frac{(t \vee t')^{K_1}}{(t \vee t' + |(x, y-z)|)^{(n+2m+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |z|)^{(m+K_2)}} dz = \int_{|z| \leq \frac{1}{2}|y|, \text{ or } |z| \geq 2|y|} + \int_{\frac{1}{2}|y| \leq |z| \leq 2|y|} = I + II$$

Since if $|z| \leq \frac{1}{2}|y|$, then $|y-z| \sim |y|$, and if $|z| \geq 2|y|$, then $|y-z| \geq |y|$, we have

$$\begin{aligned}|I| &\leq C\frac{(t\vee t')^{K_1}}{(t\vee t'+|(x,y)|)^{(n+2m+K_1)}}\\ &\leq C\frac{(t\vee t')^{K_1'}}{(t\vee t'+|x|)^{(n+K_1')}}\frac{(t\vee t')^{K_2'}}{|y|^{m+K_2'/2}}\\ &\leq C\frac{(t\vee t')^{K_1'}}{(t\vee t'+|x|)^{(n+K_1')}}\cdot\frac{(s\vee s')^{K_2'/2}}{(s\vee s'+|y|)^{m+K_2'/2}}\end{aligned}$$

where we have taken $K_1 = K_1' + K_2' \geq K_1', t\vee t' \leq \sqrt{s\vee s'}$ and $|y| \geq s\vee s'$.

To estimate the term II, we have

$$\begin{aligned}|II| \leq& \frac{(s\vee s')^{K_2}}{(s\vee s'+|y|)^{(m+K_2)}}\int\limits_{\frac{1}{2}|y|\leq|z|\leq 2|y|}\frac{(t\vee t')^{K_1}}{(t\vee t'+|(x,y-z)|)^{(n+2m+K_1)}}dz\\ \leq& C\frac{(s\vee s')^{K_2}}{(s\vee s'+|y|)^{(m+K_2)}}\cdot\frac{(t\vee t')^{K_1}}{(t\vee t'+|x|)^{(n+K_1)}}\\ \leq& C\frac{(s\vee s')^{K_2'/2}}{(s\vee s'+|y|)^{(m+K_2'/2)}}\cdot\frac{(t\vee t')^{K_1'}}{(t\vee t'+|x|)^{(n+K_1')}}\end{aligned}$$

where we have used $K_2 \geq K_2'/2$ and $K_1 \geq K_1'$.

Case 2. If $t\vee t' \leq \sqrt{s\vee s'}$ and $|y| \leq s\vee s'$, then

$$\begin{aligned}&\int\limits_{\mathbb{R}^m}\frac{(t\vee t')^{K_1}}{(t\vee t'+|(x,y-z)|)^{(n+2m+K_1)}}\frac{(s\vee s')^{K_2}}{(s\vee s'+|z|)^{(m+K_2)}}dz\\ \leq& \frac{1}{(s\vee s')^m}\int\limits_{R^m}\frac{(t\vee t')^{K_1}}{(t\vee t'+|(x,y-z)|)^{(n+2m+K_1)}}dz\\ \leq& C\frac{(s\vee s')^{K_2}}{(s\vee s'+|y|)^{m+K_2}}\frac{(t\vee t')^{K_1}}{(t\vee t'+|x|)^{(n+K_1)}}.\end{aligned}$$

Case 3. If $t\vee t' \geq \sqrt{s\vee s'}$ and $\sqrt{|y|} \leq t\vee t'$. Then

$$\begin{aligned}&\int\limits_{\mathbb{R}^m}\frac{(t\vee t')^{K_1}}{(t\vee t'+|(x,y-z)|)^{(n+2m+K_1)}}\frac{(s\vee s')^{K_2}}{(s\vee s'+|z|)^{(m+K_2)}}dz\\ \leq& C\tfrac{(t\vee t')^{K_1}}{(t\vee t'+|x|)^{(n+2m+K_1)}}\\ \leq& \frac{(t\vee t')^{K_1'}}{(t\vee t'+|x|)^{(n+K_1')}}\cdot\frac{(t\vee t')^{K_2'}}{(t\vee t'+\sqrt{|y|})^{(2m+K_2')}}.\end{aligned}$$

by noticing that $K_1 = K_1' + K_2'$.

Case 4. If $t \vee t' \geq \sqrt{s \vee s'}$ and $\sqrt{|y|} \geq t \vee t'$, then

$$\begin{aligned}
&\int\limits_{\mathbb{R}^m} \frac{(t \vee t')^{K_1}}{(t \vee t' + |(x, y-z)|)^{(n+2m+K_1)}} \frac{(s \vee s')^{K_2}}{(s \vee s' + |z|)^{(m+K_2)}} dz \\
&= \int_{|z| \leq \frac{1}{2}|y| \text{ or } |z| \geq 2|y|} + \int_{\frac{1}{2}|y| \leq |z| \leq 2|y|} = I + II
\end{aligned}$$

and

$$\begin{aligned}
|I| &\leq \frac{(t \vee t')^{K_1}}{(t \vee t' + |(x, y)|)^{(n+2m+K_1)}} \\
&\leq C \frac{(t \vee t')^{K_1'}}{(t \vee t' + |x|)^{(n+K_1')}} \cdot \frac{(t \vee t')^{K_2'}}{(t \vee t' + \sqrt{|y|}\ )^{(2m+K_2')}}
\end{aligned}$$

where we have taken $K_1 = K_1' + K_2'$.

Next, we estimate

$$\begin{aligned}
|II| &\leq C \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \cdot \frac{(s \vee s')^{K_2}}{(s \vee s' + |y|)^{(m+K_2)}} \\
&\leq C \frac{(t \vee t')^{K_1}}{(t \vee t' + |x|)^{(n+K_1)}} \cdot \frac{(t \vee t')^{2K_2}}{(t \vee t' + \sqrt{|y|})^{(2m+2K_2)}} \\
&\leq C \frac{(t \vee t')^{K_1'}}{(t \vee t' + |x|)^{(n+K_1')}} \cdot \frac{(t \vee t')^{K_2'}}{(t \vee t' + \sqrt{|y|})^{(2m+K_2')}}
\end{aligned}$$

where we used $K_1 \geq K_1'$ and $2K_2 \geq K_2'$ ■

Now we can obtain the following continuous version of the Calderón reproducing formula on test function space $\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$ and its dual space $(\mathcal{S}_F)'$.

Lemma 3.4. *Assume that $\psi_{j,k}$ is the same as in (1.4). Then*

$$f(x, y) = \sum_j \sum_k \psi_{j,k} * \psi_{j,k} * f(x, y), \tag{3.5}$$

where the series converges in the norm of $\mathcal{S}_F$ and in dual space $(\mathcal{S}_F)'$.

Proof. If $f \in \mathcal{S}_F$, then there exists $f^\sharp \in \mathcal{S}_M$, such that $f(x, y) = \int\limits_{\mathbb{R}^m} f^\sharp(x, y - z, z) dz$. Because of the continuous Calderón reproducing formula of $f^\sharp$ on L^2, to show the series in (3.5) converges to S_F, it is sufficient to prove three summations below

$$\sum_{|j|>N} \sum_{|k| \leq M} \psi_{j,k}^\sharp * \psi_{j,k}^\sharp * f^\sharp, \sum_{|j| \leq N} \sum_{|k|>M} \psi_{j,k}^\sharp * \psi_{j,k}^\sharp * f^\sharp, \sum_{|j|>N} \sum_{|k|>M} \psi_{j,k}^\sharp * \psi_{j,k}^\sharp * f^\sharp$$

tend to zero in $\mathcal{S}_M(\mathbb{R}^{n+m}\times\mathbb{R}^m)$ as N and M go to infinity. We apply the almost orthogonality estimate in (3.4) by choosing $t=2^{-j}, s=2^{-k}, t'=s'=1$ and $x_0=y_0=0$ to show it. And by the duality argument, we obtain that the series in (3.5) converges in dual space. We omit details here. ∎

By Lemma 3.4, we can develop the discrete Calderón reproducing formula.

Proof of Theorem 1.2. For $f\in\mathcal{S}_F$, we discrete f from (3.5) such that

$$\begin{aligned} f(x,y) &= \sum_{j,k}\sum_{I,J}\int_J\int_I \psi_{j,k}(x-u,y-w)\left(\psi_{j,k}*f\right)(u,w)dudw \\ &= \sum_{j,k}\sum_{I,J}\left[\int_J\int_I \psi_{j,k}(x-u,y-w)dudw\right]\left(\psi_{j,k}*f\right)(x_I,y_J)+\mathcal{R}(f)(x,y) \\ &:= T(f)(x,y)+\mathcal{R}(f)(x,y) \end{aligned}$$

where I and J are dyadic cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ with side length 2^{-j-N} and $2^{-k-N}+2^{-2j-N}$ respectively, and N is large enough.

Following a similar proof as that in [HL],we get that $\mathcal{R}(f)\in\mathcal{S}_F(\mathbb{R}^n\times\mathbb{R}^m)$, and

$$||\mathcal{R}(f)||_{\mathcal{S}_F(\mathbb{R}^n\times\mathbb{R}^m)}\le C2^{-N}||f||_{\mathcal{S}_F(\mathbb{R}^n\times\mathbb{R}^m)}. \tag{3.6}$$

Then $T^{-1}=(I-\mathcal{R})^{-1}$ exists and

$$\begin{aligned} f(x,y) &= T^{-1}T(f)(x,y) \\ &= \sum_{i=0}^{\infty}\mathcal{R}^iT(f)(x,y)\sum_j\sum_k\sum_J\sum_I \\ &\quad \left[\sum_{i=0}^{\infty}\mathcal{R}^i\int_J\int_I \psi_{j,k}(\cdot-u,\cdot-v)dudv\right](x,y)\left(\psi_{j,k}*f\right)(x_I,y_J). \end{aligned}$$

Set

$$\left[\sum_{i=0}^{\infty}\mathcal{R}^i\int_J\int_I \psi_{j,k}(\cdot-u,\cdot-v)dudv\right](x,y)=|I||J|\widetilde{\psi}_{j,k}(x,y,x_I,y_J).$$

And from (3.6), it is easy to show that $\widetilde{\psi}_{j,k}(x,y,x_I,y_J)\in\mathcal{S}_F$.

Theorem 1.2 is concluded. ∎

To prove the Min-Max type inequality, we need to get the following lemma first.

Lemma 3.5. *Let I, I' and J, J' be dyadic cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively such that $\ell(I) = 2^{-j-N}$, $\ell(J) = 2^{-2j-N} + 2^{-k-N}$, $\ell(I') = 2^{-j'-N}$ and $\ell(J') = 2^{-2j'-N} + 2^{-k'-N}$. Then for any $u, u^* \in I$ and $v, v^* \in J$, we have when $j \wedge j' \geq \frac{k \wedge k'}{2}$,*

$$\sum_{I',J'} \frac{2^{-|j-j'|L_1-|k-k'|L_2} 2^{-(j\wedge j')K_1-(k\wedge k')K_2}|I'||J'|}{(2^{-j\wedge j'} + |u - x_{I'}|)^{n+K_1} (2^{-k\wedge k'} + |v - y_{J'}|)^{m+K_2}} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$

$$\leq C\ 2^{-|j-j'|L_1} \cdot 2^{-|k-k'|L_2} \left\{ M_s \left[\left(\sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \right)^r \right] \right\}^{\frac{1}{r}} (u^*, v^*)$$

and when $j \wedge j' \leq \frac{k \wedge k'}{2}$,

$$\sum_{I',J'} \frac{2^{-|j-j'|L_1-|k-k'|L_2} 2^{-(j\wedge j')K_1-(j\wedge j')K_2}|I'||J'|}{(2^{-j\wedge j'} + |u - x_{I'}|)^{n+K_1} \left(2^{-j\wedge j'} + \sqrt{|v - y_{J'}|}\right)^{2m+K_2}} |\phi_{j',k'} * f(x_{I'}, y_{J'})|$$

$$\leq C\ 2^{-|j-j'|L_1} \cdot 2^{-|k-k'|L_2} \left\{ M \left[\left(\sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \right)^r \right] \right\}^{\frac{1}{r}} (u^*, v^*)$$

where M is the Hardy-Littlewood maximal function on $\mathbb{R}^n \times \mathbb{R}^m$, and M_s is the strong maximal function on $\mathbb{R}^n \times \mathbb{R}^m$.

Proof. Since the proof is similar to that in [HL], we omit it here. ■

Now we are ready to give the

Proof of Theorem 1.3. By the discrete Calderȯn reproducing formula on S_F,

$$f(x, y) = \sum_{j',k'} \sum_{J',I'} |J'|\ |I'|\ |\tilde{\phi}_{j',k'}(x, y, x_{I'}, y_{J'})(\phi_{j',k'} * f)(x_{I'}, y_{J'})|$$

We have from the almost orthogonality estimates in Lemma 3.3 together with Lemma 3,5 that,

$$|\psi_{j,k} * f(u, v)|$$
$$\leq C \sum_{j',k'} 2^{-|j-j'|L_1} \cdot 2^{-|k-k'|L_2}$$
$$\left\{ M_s \left[\left(\sum_{J'} \sum_{I'} |\phi_{j',k'} * f(x_{I'}, y_{J'})| \chi_{J'} \chi_{I'} \right)^r \right] \right\}^{\frac{1}{r}} (u^*, v^*)$$

for any $u, u^* \in I$, $x_{I'} \in I'$, $v, v^* \in J$ and $y_{J'} \in J'$.

Using the Holder's inequality and summing over j, k, I, J, we have

$$\left\{\sum_{j,k}\sum_{I,J}\sup_{u\in I,v\in J}|\psi_{j,k}*f(u,v)|^2\chi_I\chi_J\right\}^{\frac{1}{2}}$$
$$\leq C\left\{\sum_{j',k'}\left\{M_s(\sum_{I',J'}|\phi_{j',k'}*f(x_{I'},y_{J'})|\chi_{I'}\chi_{J'})^r\right\}^{\frac{2}{r}}\right\}^{\frac{1}{2}},$$

and since $x_{I'}$ and $y_{J'}$ are arbitrary in I' and J' respectively, we could derive Theorem 1.3. by using the Fefferman-Stein vector-valued maximal theorem [FS] with $r < p$. ■

4. Proofs of Theorems 1.4 and 1.5

In this section, we use non-isotropic discrete Littlewood-Paley square function and discrete Calderón reproducing formula and Corollary 4.6 to get the boundedness of the flag singular integral operator T in H^p_F. Together with the stopping time argument of Chang and R. Fefferman [CF1-2] we then obtain the operator T is bounded from H^p_F to L^p. The H^p to L^p boundedness for singular integral operators in the pure product setting was proved by R. Fefferman [F] using atomic decomposition and Journe's type covering lemma (see [J1-2], [P]). The method employed by Han and Lu [HL] avoids the use of Journe type covering lemma. We will adapt ideas from [HL] in isotropic case to prove our theorems in non-isotropic case.

From Theorem 1.3, we can use the norm of discrete Littlewood-Paley square function to characterize the space H^p_F.

Lemma 4.1. *For $0 < p \leq 1$, we have*

$$\|f\|_{H^p_F} \approx \| \left\{\sum_j\sum_k\sum_J\sum_I|\psi_{j,k}*f(x_I,y_J)|^2\chi_I(x)\chi_J(y)\right\}^{\frac{1}{2}}\|_p \tag{4.1}$$

where $j, k, \psi, \chi_I, \chi_J, x_I, y_J$ are the same as in Theorem 1.3.

We define $\phi^{(1)}$ and $\phi^{(2)}$ to be the same as in Definition 1.1 except that $\phi^{(1)} \in C_0^\infty(\mathbb{R}^{n+m})$, $\phi^{(2)} \in C_0^\infty(\mathbb{R}^m)$, and satisfy the cancellation conditions of finite order

$$\int_{\mathbb{R}^{n+m}}\phi^{(1)}(x,y)x^\alpha y^\beta dxdy = 0, \text{ for all } \alpha,\beta \text{ satisfying } 0\leq|\alpha|\leq M_0,\ 0\leq|\beta|\leq M_0,$$

$$\int_{\mathbb{R}^m}\phi^{(2)}(z)z^\gamma dz = 0 \text{ for all } 0\leq|\gamma|\leq M_0,$$

Moreover, we may assume that $\phi^{(1)}$ and $\phi^{(2)}$ are radial functions supported in the unit balls of $\mathbb{R}^{n+m}$ and $\mathbb{R}^m$ respectively.

Lemma 4.2. *There are functions $\widetilde{\phi}_{j,k}$ and an operator T_N^{-1} such that*

$$f(x,y)=\sum_j\sum_k\sum_J\sum_I |I||J|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)$$

where functions $\widetilde{\phi}_{j,k}(x-x_I,y-y_J)$ satisfy the estimate (3.3) with $\alpha_1,\beta_1,\gamma_1,N,M$ depending on M_0, $x_0=x_I$ and $y_0=y_J$. We also have that the series converges in $L^2(\mathbb{R}^{n+m})$ and moreover T_N^{-1} is bounded on $L^2(\mathbb{R}^{n+m})$ and $H_F^p(\mathbb{R}^n\times\mathbb{R}^m)$.

Proof of Lemma 4.2. From the continuous version of Calderón reproducing formula on L^2,

$$f(x,y)=\sum_j\sum_k\sum_J\sum_I\left[\int_J\int_I\phi_{j,k}(x-u,y-v)dudv\right]\left(\phi_{j,k}*f\right)(x_I,y_J)+\mathcal{R}(f)(x,y). \tag{4.2}$$

where I,J,j,k and $\mathcal{R}$ are the same as in Theorem 1.2.

We claim that for $0<p\le 1$, if M_0 is large enough, then there is a constant $C>0$ such that $\|\mathcal{R}(f)\|_2\le C2^{-N}\|f\|_2$, and $\|\mathcal{R}(f)\|_{H_F^p(\mathbb{R}^n\times\mathbb{R}^m)}\le C2^{-N}\|f\|_{H_F^p(\mathbb{R}^n\times\mathbb{R}^m)}$.

Assume the claim for the moment and set that

$$T_N(f)=\sum_j\sum_k\sum_J\sum_I[\int_J\int_I\phi_{j,k}(x-u,y-v)dudvd]\left(\phi_{j,k}*f\right)(x_I,y_J).$$

Then both T_N and $(T_N)^{-1}=\sum_{i=1}^{\infty}\mathcal{R}^i$ are bounded on $L^2(\mathbb{R}^{n+m})\cap H_F^p(\mathbb{R}^n\times\mathbb{R}^m)$. Thus,

$$\begin{aligned}f(x,y)&=T_NT_N^{-1}(f)(x,y)\\&=\sum_j\sum_k\sum_J\sum_I|I||J|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)\end{aligned}$$

where $\widetilde{\phi}_{jk}(x-x_I,y-y_J)=\frac{1}{|I|}\frac{1}{|J|}\int_J\int_I\phi_{jk}(x-x_I-(u-x_I),y-y_J-(v-y_J))dudv$ satisfies the estimate in (3.3) and the series converges in $L^2(\mathbb{R}^{n+m})$. We use Lemma 2.2, discrete Calder*ó*n reproducing formula and almost orthogonality to show the claim. Since it is similar to that in [HL], we omit the details here. ∎

Because of Lemma 4.2 we have

Lemma 4.3. *For* $0 < p \le 1$, *if* $f \in L^2(\mathbb{R}^{n+m}) \cap H_F^p(\mathbb{R}^n \times \mathbb{R}^m)$, *then*

$$\|f\|_{H_F^p} \approx \left\| \left(\sum_j \sum_k \sum_J \sum_I |\phi_{jk} * \left(T_N^{-1}(f)\right)(x_I, y_J)|^2 \chi_I(x)\chi_J(y) \right)^{\frac{1}{2}} \right\|_p$$

where the constants are independent of the L^2 *norm of* f.

Proof of Lemma 4.3. By Lemma 4.2,

$$f(x,y) = \sum_j \sum_k \sum_J \sum_I |I||J| \widetilde{\phi}_{j,k}(x - x_I, y - y_J)\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J)$$

and $T_N^{-1}(f)$ is bounded on $L^2(\mathbb{R}^{n+m})$, thus we have from Theorem 1.2 that

$$\begin{aligned} &\phi_{j,k} * T_N^{-1}(f)(u,v) \\ &= \sum_{j',k',I',J'} |I'|\ |J'|\ \phi_{j,k} * \widetilde{\phi}_{j',k'}(\cdot,\cdot,x_{I'},y_{J'})(u,v)\ \phi_{j',k'} * T_N^{-1}(f)(u,v) \end{aligned}$$

then following the same proof in Theorem 1.3, we obtain this Corollary. We omit the details here. ∎

Lemma 4.4. $\mathcal{S}_F(\mathbb{R}^n \times \mathbb{R}^m)$ *is dense in* $H_F^p(\mathbb{R}^n \times \mathbb{R}^m)$.

Proof. The proof is almost the same as that in [HL] except that we suppose dyadic cubes J in $\mathbb{R}^m$ with side length $2^{-k-N} + 2^{-2j-N}$, so we omit the proof here. ∎

Therefore, we have from Lemma 4.4 that $L^2(\mathbb{R}^{n+m})$ is dense in $H_F^p(R^n \times R^m)$.

Proof of Theorem 1.4. For $f \in L^2(\mathbb{R}^{n+m}) \cap H_F^p(\mathbb{R}^n \times \mathbb{R}^m)$, $0 < p < 1$, Definition 1.7 and the discrete Calderón reproducing formula in Lemma 4.2 imply that

$$\begin{aligned} \|T(f)\|_{H_F^p} &\le C\|\{\sum_{j,k}\sum_{I,J} |\phi_{j,k} * K * f(x,y)|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_{L^p} \\ &= C\|\{\sum |J'|\ |I'|\ |\phi_{j,k} * K * \widetilde{\phi}_{j',k'}(\cdot - x_{I'}, \cdot - y_{J'})(x,y)\phi_{j',k'} \\ &\quad \cdot \left(T_N^{-1}(f)\right)(x_{I'}, y_{J'})|^2 \chi_I(x)\chi_J(y)\}^{\frac{1}{2}}\|_{L^p}, \end{aligned}$$

where the last sum above is over $j, k, I, J, j', k', I', J'$.

By a similar argument in (2.5), it is easy to see that

$$\begin{aligned}&|\phi_{j,k} * K * \widetilde{\phi}_{j',k'}(\cdot - x_{I'}, \cdot - y_{J'})(x,y)|\\ \leq\ & C2^{-|j-j'|M}2^{-|k-k'|M}\int\limits_{\mathbb{R}^m}\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|(x-x_{I'}, y-z-y_{J'})|^{n+2m+M}}\\ &\cdot\frac{2^{-(k\wedge k')M}}{(2^{-(k\wedge k')}+|z|)^{m+M}}dz.\end{aligned}$$

Then following a similar proof in Theorem 1.3 together with Lemma 4.3,

$$\begin{aligned}\|Tf\|_{H^p_F} &\leq C\left\|\left\{\sum_{j'}\sum_{k'}\left\{M_s\left(\sum_{J'}\sum_{I'}|\phi_{j',k'} * \left(T_N^{-1}(f)\right)(x_{I'}, y_{J'})|\chi_{J'}\chi_{I'}\right)^r\right\}^{\frac{2}{r}}(x,y)\right\}^{\frac{1}{2}}\right\|_p\\ &\leq C\left\|\left\{\sum_{j'}\sum_{k'}\sum_{J'}\sum_{I'}|\phi_{j',k'} * \left(T_N^{-1}(f)\right)(x_{I'}, y_{J'})|^2\chi_{J'}(y)\chi_{I'}(x)\right\}^{\frac{1}{2}}\right\|_p\\ &\leq C\|f\|_{H^p_F},\end{aligned}$$

Since $L^2(\mathbb{R}^{n+m}) \cap H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ is dense in $H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$, using a limiting argument, we could complete Theorem 1.4. ■

Lemma 4.5. Let $f \in L^2(\mathbb{R}^{n+m}) \cap H^p_F(\mathbb{R}^n \times \mathbb{R}^m), 0 < p \leq 1$, then $f \in L^p(\mathbb{R}^{n+m})$ and there exists a constant $C_p > 0$ independent of the L^2 norm of f such that

$$\|f\|_p \leq C\|f\|_{H^p_F}. \tag{4.3}$$

Proof of Lemma 4.5.

Denote $\widetilde{S}(f)(x,y) = \{\sum_j\sum_k\sum_J\sum_I|\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I, y_J)|^2\chi_I(x)\chi_J(y)\}^{\frac{1}{2}}$.

For $f \in L^2(\mathbb{R}^{n+m}) \cap H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ we have from Corollary 4.6 that

$$||\widetilde{S}(f)||_{L^p(\mathbb{R}^{n+m})} \leq C||f||_{H^p_F(\mathbb{R}^n\times\mathbb{R}^m)}.$$

In the following, we use the stopping time argument. Suppose $f \in L^2(\mathbb{R}^{n+m}) \cap H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ and set

$$\Omega_i = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : \widetilde{S}(f)(x,y) > 2^i\}.$$

Define

$$\mathcal{B}_i = \{(j,k,I,J) : |(I\times J)\cap\Omega_i| > \frac{1}{2}|I\times J|, |(I\times J)\cap\Omega_{i+1}| \leq \frac{1}{2}|I\times J|\},$$

where I, J are dyadic cubes in R^n, R^m with side length $2^{-j-N}, 2^{-k-N}+2^{-2j-N}$, respectively. It is worthwhile to point out that each (j,k,I,J) belongs to precisely one $\mathcal{B}_i$.
Applying the discrete Calderón reproducing formula in Lemma 4.3,

$$\begin{aligned} f(x,y) &= \sum_j\sum_k\sum_J\sum_I \widetilde{\phi}_{j,k}(x-x_I,y-y_J)|I||J|\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)\\ &= \sum_i \sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J), \end{aligned}$$

where the series converges in the L^2 norm, and it thus also converges almost everywhere.

We **Claim.**

$$\left\|\sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)\right\|_p^p \le C2^{ip}|\Omega_i|,$$

Assuming this claim for the moment, then for $0<p\le 1$

$$\begin{aligned} ||f||_p^p &\le \sum_i \|\sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)\|_p^p\\ &\le C\sum_i 2^{ip}|\Omega_i|\\ &= C\sum_i 2^{i(p-1)}|\Omega_i|2^i\\ &\le \quad C\int_0^\infty p\ \lambda^{p-1}|\{(x,y)\in R^n\times R^m\}:\widetilde{S}(f)(x,y)>\lambda\}|\ d\lambda\\ &= C\ \|\widetilde{S}(f)\|_p^p \le C\ ||f||^p_{H_F^p}. \end{aligned}$$

Now we prove the claim. For $(j,k,I,J)\in\mathcal{B}_i$, if $(x,y)\in I\times J$, then $M_s(\chi_{\Omega_i})(x,y) \ge \frac{1}{2}$. Notice that $\phi^{(1)}$ and $\phi^{(2)}$ are radial functions supported in unit balls, thus if $(j,k,I,J)_i\in\mathcal{B}_i$, then $\phi_{j,k}(x-x_I,y-y_J)$ are supported in

$$\widetilde{\Omega}_i = \{(x,y): M_s(\chi_{\Omega_i})(x,y) > \frac{1}{10}\}.$$

Note that

$$|\widetilde{\Omega}_i| \le C\int_{R^n\times R^m} 10|\chi_{\Omega_i}|\left(1+log^+(10|\chi_{\Omega_i}|)\right)^{n-1}dxdy \le C|\Omega_i|$$

By Hölder's inequality,

$$\left\| \sum_{(j,k,I,J)\in\mathcal{B}_i} |J||I|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J) \right\|_p^p$$

$$\leq |\widetilde{\Omega}_i|^{1-\frac{p}{2}} \| \sum_{(j,k,I,J)\in\mathcal{B}_i} |J||I|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)\|_2^p.$$

For all $g \in L^2$ with $\|g\|_2 \leq 1$,

$$\left| < \sum_{(j,k,I,J)\in\mathcal{B}_i} |J||I|\widetilde{\phi}_{j,k}(x-x_I,y-y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J), g > \right|$$

$$= \left| \sum_{(j,k,I,J)\in\mathcal{B}_i} |J||I|\widetilde{\phi}_{j,k}*g(x_I,y_J)\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J) \right|$$

$$\leq C \left(\sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J||\phi_{j,k}*\left(T_N^{-1}(f)\right)(x_I,y_J)|^2 \right)^{\frac{1}{2}}$$

$$\cdot \left(\sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J||\widetilde{\phi}_{j,k}*g(x_I,y_J)|^2 \right)^{\frac{1}{2}}.$$

While,

$$\left(\sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J||\widetilde{\phi}_{j,k}*g(x_I,y_J)|^2 \right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{(j,k,I,J)\in\mathcal{B}_i} |I||J| \left(M_s\left(\widetilde{\phi}_{j,k}*g\right)(x,y)\chi_I(x)\chi_J(y)\right)^2 \right)^{\frac{1}{2}}$$

$$\leq C \left(\sum_{j,k} \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} \left(\widetilde{\phi}_{j,k}*g\right)^2(x,y)dxdy \right)^{\frac{1}{2}}$$

$$= C \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} \left(\sum_{j,k} |\widehat{\widetilde{\phi}_{j,k}}|^2 \right) \cdot |\widehat{g}(\xi,\eta)|^2 d\xi d\eta \right)^{\frac{1}{2}}$$

$$\leq C \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} \widehat{g}(\xi,\eta)|^2 d\xi d\eta \right)^{\frac{1}{2}} = C\|g\|_2$$

where $\widehat{\tilde{\phi}_{j,k}}$ and $\hat{g}$ are Fourier transforms of $\tilde{\phi}_{j,k}$ and g respectively.

In addition,

$$\begin{aligned} C2^{2i}|\Omega_i| &\geq \int_{\widetilde{\Omega_i}\backslash\Omega_{i+1}} \widetilde{S}^2(f)(x,y)dxdy \\ &\geq \sum_{(j,k,I,J)\in\mathcal{B}_i} |\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I,y_J)|^2 |(I\times J)\cap\widetilde{\Omega}_i\backslash\Omega_{i+1}| \\ &\geq \frac{1}{2}\sum_{(j,k,I,J)\in\mathcal{B}_i} |I|\,|J|\,|\phi_{j,k} * \left(T_N^{-1}(f)\right)(x_I,y_J)|^2, \end{aligned}$$

where in the last inequality we use the fact that $|(I\times J)\cap\widetilde{\Omega_i}\backslash\Omega_{i+1}| > \frac{1}{2}|I\times J|$ when $(j,k,I,J)\in\mathcal{B}_i$. By the duality argument, we get Lemma 4.5. ■

As a consequence of Lemma 4.5, we have the following result

Corollary 4.6. $H_F^1(\mathbb{R}^n\times\mathbb{R}^m)$ *is a subspace of* $L^1(\mathbb{R}^n\times\mathbb{R}^m)$.

Proof. Suppose $f\in H_F^1(\mathbb{R}^{n+m})$, since $L^2(\mathbb{R}^{n+m})$ is a dense subspace of $H_R^1(\mathbb{R}^{n+m})$, there exists a sequence $\{f_n\}\subset L^2(\mathbb{R}^{n+m})\cap H_F^1(\mathbb{R}^{n+m})$ such that f_n converges to f in the norm of $H_F^1(\mathbb{R}^{n+m})$. By Lemma 4.5, $\{f_n\}$ is a Cauchy sequence in $L^1(\mathbb{R}^{n+m})$, thus there exists some $g\in L^1(\mathbb{R}^{n+m})$ such that f_n converges to g in $L^1(\mathbb{R}^{n+m})$. By taking subsequences which converge almost everywhere, we get that $f=g$ and then $f\in L^1(\mathbb{R}^{n+m})$. ■

Proof of Theorem 1.5. Since T is bounded on L^2 and on H_F^p, it follows that $T(f)\in H_F^p\cap L^2$ for $f\in H_F^p\cap L^2$, so by Lemma 4.5, $\|T(f)\|_{L^p}\leq C\|T(f)\|_{H_F^p}$ for $f\in H_F^p\cap L^2$. Then we have from Theorem 1.4, $\|T(f)\|_{L^p}\leq C\|f\|_{H_F^p}$ for $f\in H_F^p\cap L^2$. Since $H_F^p\cap L^2$ is dense in H_F^p, Theorem 1.5 is obtained. ■

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