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A NOTE ON A CONJECTURED NESBITT TYPE INEQUALITY

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Abstract. In this note, a modified version of a conjectured Nesbitt type inequality is given and a generalization of Nesbitt's inequality involving three parameters is established.

1. Introduction

Nesbitt's inequality, (see [3] or [5]), states that if x, y, and z are positive real numbers, then

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}, \tag{1}$$

with equality if and only if the three variables are equal. The motivation of this note is the following Nesbitt type inequality conjectured in [7].

Conjecture 1. Let $x_1, x_2, \cdots, x_n$ be positive real numbers, $n \geq 2$, $\alpha \leq \frac{1}{2}$. Then,

$$\left(\frac{x_1}{x_1+x_2}\right)^{\alpha}+\left(\frac{x_2}{x_2+x_3}\right)^{\alpha}+\cdots+\left(\frac{x_{n-1}}{x_{n-1}+x_n}\right)^{\alpha}+\left(\frac{x_n}{x_n+x_1}\right)^{\alpha}\leq\frac{n}{2^{\alpha}}. \tag{2}$$

As stated the conjecture is not valid, a counterexample being $n = 4$, $\alpha = \frac{1}{2}$, and x_1, x_2, x_3, and x_4 positive real numbers such that $\frac{x_2}{x_1} = \frac{x_3}{x_2} = \frac{x_4}{x_3} = 0.1$ and $\frac{x_1}{x_4} = 1000$. A calculation shows that

$$\frac{3}{\sqrt{1.1}}+\frac{1}{\sqrt{1001}}=2.8919>\frac{4}{\sqrt{2}}=2.8284,$$

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which proves that the conjecture is false. In this paper we prove that, under the additional hypothesis that $n \leq \frac{\alpha+1}{\alpha}$, inequality (2) becomes valid and we will also establish a new generalization of Nesbitt's inequality (1) by introducing three parameters. It is worth mentioning that inequality (1) is a particular case of Peixoto's inequality, (see [6]). For related cyclic type inequalities the interested reader is reffered to [1] and [2].

2. A Modified Version of the Conjectured Nesbitt Type Inequality

In this section we prove that if n is a positive integer that verifies the additional hypothesis that $n \leq \frac{\alpha+1}{\alpha}$ then the conjecture is true and we will give the proof of it. Our main result is the following theorem.

Theorem 2. *1) Let $0 \leq \alpha \leq 1$ and let $x \geq 0$. The following inequality holds*

$$\frac{1}{(1+x)^\alpha} + \left(\frac{x}{1+x}\right)^\alpha \leq \frac{2}{2^\alpha}.$$

2) Let $x_1, x_2, \cdots, x_n$ be positive real numbers such that $x_1 x_2 \cdots x_n = 1$. Let $3 \leq n \leq \frac{\alpha+1}{\alpha}$ and let $0 < \alpha \leq \frac{1}{2}$. The following inequality holds

$$\frac{1}{(1+x_1)^\alpha} + \frac{1}{(1+x_2)^\alpha} + \cdots + \frac{1}{(1+x_n)^\alpha} \leq \frac{n}{2^\alpha}.$$

Before we give the proof of Theorem 2 we collect some lemmas that we need for proving the main result of the paper. Recall that, the vector $x = (x_1, x_2, \cdots, x_n)$ majorizes the vector $y = (y_1, y_2, \cdots, y_n)$, and we write $x \succ y$, if and only if the following conditions are satisfied

$$\begin{cases} x_1 \geq y_1, \\ x_1 + x_2 \geq y_1 + y_2, \\ \quad \vdots \\ x_1 + x_2 + \cdots + x_{n-1} \geq y_1 + y_2 + \cdots + y_{n-1}, \\ x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n. \end{cases}$$

Lemma 3. (The Majorization Inequality). *If $x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n$ are real numbers from an interval I such that the vector $x = (x_1, x_2, \cdots, x_n)$ majorizes the vector $y = (y_1, y_2, \cdots, y_n)$, and if $f : I \to \mathbb{R}$ is a convex (concave) function then*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \geq (\leq) f(y_1) + f(y_2) + \cdots + f(y_n).$$

For topics related to the theory of majorization and its applications the interested reader may consult [4].

Lemma 4. *Let* $0 < \alpha \leq \frac{1}{2}$ *and let* $n \geq 3$ *be a natural number such that* $n \leq \frac{\alpha+1}{\alpha}$. *The following inequalities hold*

$$\begin{cases} x^{1-n\frac{\alpha}{\alpha+1}} - \dfrac{1+x}{1+x^{n-1}} \leq 0, & 0 \leq x \leq 1 \\ x^{1-n\frac{\alpha}{\alpha+1}} - \dfrac{1+x}{1+x^{n-1}} \geq 0, & x \geq 1. \end{cases}$$

Proof. Let $u : [0, \infty) \to \mathbb{R}$ be the function defined by

$$u(x) = x^{1-n\frac{\alpha}{\alpha+1}} - \frac{1+x}{1+x^{n-1}}.$$

First we consider the case when $0 \leq x \leq 1$. We have, since $n-1 \geq 1$, that $x^{n-1} \leq x$, and this implies that $\frac{1+x}{1+x^{n-1}} \geq 1$. It follows that $u(x) \leq x^{1-n\frac{\alpha}{\alpha+1}} - 1 \leq 0$. When $x \geq 1$ we have that $\frac{1+x}{1+x^{n-1}} \leq 1$ which implies that $u(x) \geq x^{1-n\frac{\alpha}{\alpha+1}} - 1 \geq 0$, and the lemma is proved. ■

Lemma 5. *Let* $0 < \alpha \leq 1$ *and let* $n \leq \frac{\alpha+1}{\alpha}$. *Then for all* $x \geq 0$ *the following inequality holds*

$$\frac{n-1}{(1+x)^{\alpha}} + \left(\frac{x^{n-1}}{1+x^{n-1}}\right)^{\alpha} \leq \frac{n}{2^{\alpha}}.$$

Proof. Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{n-1}{(1+x)^{\alpha}} + \left(\frac{x^{n-1}}{1+x^{n-1}}\right)^{\alpha}.$$

A calculation shows that

$$f'(x) = -\frac{\alpha(n-1)}{(1+x)^{\alpha+1}x^{\alpha+1-n\alpha}} \left(x^{\alpha+1-n\alpha} - \left(\frac{1+x}{1+x^{n-1}}\right)^{\alpha+1}\right).$$

It follows that the sign of f' is given by the sign of

$$x^{1-n\frac{\alpha}{\alpha+1}} - \frac{1+x}{1+x^{n-1}}.$$

An application of Lemma 4 shows that $f'(x) \geq 0$ when $x \in (0, 1]$ and $f'(x) \leq 0$ when $x \geq 1$. It follows that f increases on the interval $[0, 1]$ and it decreases on

$[1,\infty)$, and hence, f attains its maximum at 1. Thus $f(x) \le f(1) = \frac{n}{2^\alpha}$ and the lemma is proved. ■

Now we are ready to give the proof of Theorem 2 which is based on an application of the Majorization Inequality combined with the classical Jensen's Inequality for concave functions.

Proof of Theorem 2.

(1) If $\alpha = 0$ or $\alpha = 1$ there is nothing to prove. So let $\alpha \in (0,1)$. Let $g : [0,\infty) \to \mathbb{R}$ be the function defined by

$$g(x) = \frac{1}{(1+x)^\alpha} + \left(\frac{x}{1+x}\right)^\alpha .$$

A calculation shows that

$$g'(x) = \frac{\alpha(1-x^{1-\alpha})}{x^{1-\alpha}(1+x)^{\alpha+1}}.$$

Thus g increases on $[0,1]$ and it decreases on the interval $[1,\infty)$ attaining its maximum at 1. It follows that $g(x) \le g(1) = \frac{2}{2^\alpha}$ and the first part of the theorem is proved.

(2) Let $n \ge 3$. Let $x_1 = e^{t_1}, x_2 = e^{t_2}, \cdots, x_n = e^{t_n}$. The inequality to prove reads

$$\frac{1}{(1+e^{t_1})^\alpha}+\frac{1}{(1+e^{t_2})^\alpha}+\cdots+\frac{1}{(1+e^{t_n})^\alpha}\le\frac{n}{2^\alpha}, \text{ subject to } t_1+t_2+\cdots+t_n=0.$$

Let $h : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$h(t) = \frac{1}{(1+e^t)^\alpha}.$$

A calculation shows that

$$h'(t) = -\frac{\alpha e^t}{(1+e^t)^{\alpha+1}} \quad \text{and} \quad h''(t) = \frac{\alpha e^t(\alpha e^t - 1)}{(1+e^t)^{\alpha+2}}.$$

Thus h has exactly one inflection point namely $k = \ln\frac{1}{\alpha}$. We distinguish here two cases.

Case 1. $k \ge t_1 \ge t_2 \ge \cdots \ge t_n$. Since h is concave on $(-\infty, k)$, it follows, based on Jensen's inequality for concave functions, that

$$\sum_{j=1}^{n} h(t_j) \le nh\left(\frac{t_1+t_2+\cdots+t_n}{n}\right) = nh(0) = \frac{n}{2^\alpha}.$$

Case 2. There is an integer m such that $t_1 \geq t_2 \geq \cdots \geq t_m \geq k \geq t_{m+1} \geq \cdots \geq t_n$. We note that

$$(t_1 + t_2 + \cdots + t_m - (m-1)k, k, \cdots, k) \succ (t_1, t_2, \cdots, t_m).$$

It follows, based on Lemma 3 combined with the fact that h is convex on (k, ∞), that

$$h(t_1) + h(t_2) + \cdots + h(t_m) \leq (m-1)h(k) + h(t_1 + t_2 + \cdots + t_m - (m-1)k).$$

This implies that

$$\begin{aligned} &\sum_{j=1}^{n} h(t_j) \\ &\leq (m-1)h(k) + h(t_{m+1}) + \cdots + h(t_n) + h(t_1 + t_2 + \cdots + t_m - (m-1)k). \end{aligned} \tag{3}$$

On the other hand, since h is concave on $(-\infty, k)$ we get, based on Jensen's Inequality, that

$$\begin{aligned} &(m-1)h(k) + h(t_{m+1}) + \cdots + h(t_n) \\ &\leq (n-1)h\left(\frac{(m-1)k + t_{m+1} + \cdots + t_n}{n-1}\right). \end{aligned} \tag{4}$$

Combining (3) and (4) we obtain that

$$\begin{aligned} &\sum_{j=1}^{n} h(t_j) \\ &\leq (n-1)h\left(\frac{(m-1)k + t_{m+1} + \cdots + t_n}{n-1}\right) + h(t_1 + t_2 + \cdots + t_m - (m-1)k). \end{aligned} \tag{5}$$

Let

$$t^* = \frac{(m-1)k + t_{m+1} + \cdots + t_n}{n-1},$$

and we note that $(n-1)t^* + t_1 + t_2 + \cdots + t_m - (m-1)k = 0$. Thus, we have

$$\sum_{j=1}^{n} h(t_j) \leq (n-1)h(t^*) + h(-(n-1)t^*) = \frac{n-1}{(1+e^{t^*})^\alpha} + \left(\frac{e^{(n-1)t^*}}{1 + e^{(n-1)t^*}}\right)^\alpha, \quad (t^* < 0).$$

This implies, by letting $e^{t^*} = x$, that it suffices to study the maximum of the function

$$f(x) = \frac{n-1}{(1+x)^\alpha} + \left(\frac{x^{n-1}}{1+x^{n-1}}\right)^\alpha$$

over the interval $(0, 1)$. An application of Lemma 5 shows that the maximum of f equals $\frac{n}{2^\alpha}$ and the theorem is proved.

Next we give the modified version of inequality (2).

Theorem 6. *Let $x_1, x_2, \cdots, x_n$ be positive real numbers, let $3 \le n \le \frac{\alpha+1}{\alpha}$ and let $0 < \alpha \le \frac{1}{2}$. Then,*

$$\left(\frac{x_1}{x_1+x_2}\right)^\alpha + \left(\frac{x_2}{x_2+x_3}\right)^\alpha + \cdots + \left(\frac{x_{n-1}}{x_{n-1}+x_n}\right)^\alpha + \left(\frac{x_n}{x_n+x_1}\right)^\alpha \le \frac{n}{2^\alpha}.$$

Proof. The inequality follows by using the substitutions

$$x_1 \longmapsto \frac{x_2}{x_1},\ x_2 \longmapsto \frac{x_3}{x_2},\ \cdots,\ x_{n-1} \longmapsto \frac{x_n}{x_{n-1}},\ x_n \longmapsto \frac{x_1}{x_n}$$

combined with the second part of Theorem 2. ■

3. A Parametrized Nesbitt's Inequality

Theorem 7. *Let x, y, z, $tx + ky + lz$, $ty + kz + lx$, $tz + kx + ly$ be positive real numbers and let $-k - l < t \le \frac{k+l}{2}$. Then,*

$$\frac{x}{tx+ky+lz} + \frac{y}{ty+kz+lx} + \frac{z}{tz+kx+ly} \ge \frac{3}{t+k+l}, \tag{6}$$

with equality if and only if $t = k = l$ or $x = y = z$.

Proof. We have, based on Cauchy-Schwarz inequality, that

$$\left(\frac{x}{tx+ky+lz} + \frac{y}{ty+kz+lx} + \frac{z}{tz+kx+ly}\right)$$
$$\cdot\,(x(tx+ky+lz) + y(ty+kz+lx) + z(tz+kx+ly))$$

is greater than or equal to $(x+y+z)^2$. It follows that

$$\begin{aligned} &\frac{x}{tx+ky+lz} + \frac{y}{ty+kz+lx} + \frac{z}{tz+kx+ly} \\ &\ge \frac{(x+y+z)^2}{x(tx+ky+lz) + y(ty+kz+lx) + z(tz+kx+ly)}, \end{aligned} \tag{7}$$

whith equality if and only if

$$tx + ky + lz = ty + kz + lx = tz + kx + ly. \tag{8}$$

On the other hand, direct calculations give

$$\begin{aligned}
&\frac{(x+y+z)^2}{x(tx+ky+lz)+y(ty+kz+lx)+z(tz+kx+ly)}\\
&=\frac{3}{t+k+l}+\frac{(k+l-2t)(x^2+y^2+z^2-xy-yz-zx)}{(t+k+l)[x(tx+ky+lz)+y(ty+kz+lx)+z(tz+kx+ly)]}\\
&=\frac{3}{t+k+l}+\frac{(k+l-2t)[(x-y)^2+(y-z)^2+(z-x)^2]}{2(t+k+l)[x(tx+ky+lz)+y(ty+kz+lx)+z(tz+kx+ly)]}\\
&\geq\frac{3}{t+k+l}.
\end{aligned}\tag{9}$$

The equality holds in inequality (9) if and only if

$$k+l=2t \quad \text{or} \quad x=y=z. \tag{10}$$

Combining (7) and (9) we obtain the desired inequality (6). To establish the cases of equality, we have, based on (8) and (10), that equality holds in (6) if and only if $t=k=l$ or $x=y=z$, and this completes the proof of Theorem 7. ■

Corollary 8. *If x, y, z are positive real numbers and $0<t\leq 1$, then*

$$\frac{x}{tx+y+z}+\frac{y}{ty+z+x}+\frac{z}{tz+x+y}\geq\frac{3}{t+2}.$$

Proof. This follows from Theorem 7 when $k=l=1$ and $0<t\leq 1$. ■

Corollary 9. *If x, y, z, k, l are positive real numbers, then*

$$\frac{x}{ky+lz}+\frac{y}{kz+lx}+\frac{z}{kx+ly}\geq\frac{3}{k+l}.$$

Proof. This follows from Theorem 7 when $t=0$. ■

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