

**ABSENCE OF REAL ROOTS OF CHARACTERISTIC FUNCTIONS OF  
FUNCTIONAL DIFFERENTIAL EQUATIONS  
WITH NINE REAL PARAMETERS**

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**Abstract.** We consider the oscillation of a class of first order neutral differential equations with nine real parameters. This relatively difficult problem is completely solved by applying the Cheng-Lin envelope method to find the exact conditions for the absence of real roots of the associated characteristic function. Several specific examples are also included to illustrate these conditions.

1. INTRODUCTION

Functional differential equations in which multiple delays and/or advancements are involved in the unknown functions or their derivatives can be used to model a variety of physical models and therefore their qualitative properties are important. In recent years, such equations have been the subject of numerous investigations (as can be seen by a quick search of the MathSciNet of the American Mathematical Society). Yet for simple linear differential equations with constant coefficients, explicit necessary and sufficient conditions for these equations to have a specific property are rare.

In this paper, we intend to consider the following neutral type functional differential equation

$$(1) \quad au'(t) + bu(t) + (cu'(t + \sigma) + du(t + \sigma)) + xu(t + \delta) + yu(t + \tau) = 0$$

where  $a, b, c, d, x, y, \sigma, \delta, \tau$  are real parameters, and to find the exact region containing these parameters such that all the solutions of (1) oscillate.

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It is well known that every solution of (1) oscillates if, and only if, its characteristic function

$$(2) \quad F(\lambda) = a\lambda + b + (c\lambda + d)e^{\sigma\lambda} + xe^{\delta\lambda} + ye^{\tau\lambda}$$

does not have any real roots (see e.g. [1]). Therefore, our problem stated above is equivalent to finding the exact region containing these parameters such that  $F(\lambda)$  does not have any real roots.

Several special cases of (1) have been studied. Among these are the equations

$$u'(t) + \tilde{p}u'(t - \tilde{\tau}) + q_1u(t) + q_2u(t - \tilde{\sigma}) = 0, \quad \tilde{\tau}, \tilde{\sigma} > 0 \text{ and } \tilde{p}, q_1, q_2 \text{ are real,}$$

$$[u(t) + \bar{c}u(t - \bar{\tau})]' + \bar{r}u(t) + pu(t - \bar{\tau}) + qu(t + \bar{\sigma}) = 0, \quad \bar{\tau}, \bar{\sigma} > 0 \text{ and } \bar{c}, \bar{r}, p, q \text{ are real,}$$

studied in [2,3] and [6,7] respectively, while other special cases containing not more than 4 real parametrs can be found in numreous studies and in the book by Gyori and Ladas [5].

Needless to say, our problem is a relatively difficult one since we have 9 real parameters. Fortunately, an envelope method is developed recently by Cheng and Lin and formalized recently in [4]. We apply this method together with several new ideas and techniques to the function  $F$  and provide a complete answer to our problem.

## 2. PREPARATORY RESULTS

To facilitate discussions, we first recall a few basic concepts and tools explained in [4]. Let  $\mathbf{R}$  and  $\mathbf{C}$  be respectively the sets of real and complex numbers and let  $\Theta_0$  be the null function, that is  $\Theta_0(x) = 0$  for all  $x \in \mathbf{R}$ . Given an interval  $I$  in  $\mathbf{R}$ , the chi-function  $\chi_I : I \rightarrow \mathbf{R}$  is defined by

$$\chi_I(x) = 1, \quad x \in I.$$

The restriction of a real function  $f$  defined over an interval  $J$  (which is not disjoint from  $I$ ) will be written as  $f\chi_I$ , so that  $f\chi_I$  is now defined on  $I \cap J$  and

$$(f\chi_I)(x) = f(x), \quad x \in I \cap J.$$

A function  $G$  of the form

$$G(\lambda) = f_0(\lambda) + e^{-\lambda\tau_1}f_1(\lambda) + \cdots + e^{-\lambda\tau_m}f_m(\lambda), \quad \lambda \in \mathbf{C},$$

is called a  $\nabla(d_0, d_1, \dots, d_m)$ -polynomial if  $\tau_1, \dots, \tau_m$  are mutually distinct nonzero real numbers, and for each  $i \in \{0, 1, \dots, m\}$ ,  $f_i$  is a polynomial of  $(d_i + 2)$ -variables  $\lambda, a_{d_i}^{(i)}, a_{d_i-1}^{(i)}, \dots, a_{d_0}^{(i)}$

$$f_i(\lambda) := f_i \left( \lambda, a_{d_i}^{(i)}, a_{d_i-1}^{(i)}, \dots, a_{d_0}^{(i)} \right) = a_{d_i}^{(i)}\lambda^{d_i} + a_{d_i-1}^{(i)} + \cdots + a_{d_0}^{(i)}$$

where  $\lambda$  varies over a subset of  $\mathbf{C}$ , each  $a_{d_i-k}^{(i)}$ ,  $k \in \{1, 2, \dots, d_i\}$ , varies over a subset of  $\mathbf{R}$  and  $a_{d_i}^{(i)}$  varies over a subset of  $\mathbf{R}$  containing points other than 0. In particular, the function  $F$  in (2) may be considered as a  $\nabla(1, 1, 0, 0)$ -polynomial when  $a$  and  $c$  are not 0.

Let  $S$  be a plane curve and  $L$  be a plane straight line. Let  $d(A, B)$  denote the distance of two points  $A$  and  $B$ , and let  $d(A, L)$  be the distance between the point  $A$  and the straight line  $L$ . Assume  $S$  and  $L$  have a common point  $P$ . According to the theory of contact (due to Lagrange), the straight line  $L$  is called the tangent of the curve  $S$  at the point  $P$  if

$$(3) \quad \lim_{A \rightarrow P, A \in S} \frac{d(A, L)}{d(A, P)} = 0.$$

In case  $S$  is described by a pair of parametric functions, we have the following result.

**Lemma 1.** *Let the plane curve  $S$  be described by the parametric functions  $x(t)$  and  $y(t)$  on an interval  $I$ . Let  $t_0 \in I$  such that  $x(t) \neq x(t_0)$  for all  $t \in I \setminus \{t_0\}$ . For any  $m \in \mathbf{R}$ , let the straight line  $L_m$  be defined by  $L_m(x) = m(x - x(t_0)) + y(t_0)$  for  $x \in \mathbf{R}$ . Suppose the limit*

$$M := \lim_{t \rightarrow t_0} \frac{y(t) - y(t_0)}{x(t) - x(t_0)}$$

*exists. Then the straight line  $L_M(x)$  is the unique tangent of the curve  $S$  at  $(x(t_0), y(t_0))$ .*

*Proof.* Let  $P_t = (x(t), y(t))$ ,  $\alpha(t) = x(t) - x(t_0)$  and  $\beta(t) = y(t) - y(t_0)$  for  $t \in I$ . Then  $\alpha(t) \neq 0$  for all  $t \in I \setminus \{t_0\}$  and  $\lim_{t \rightarrow t_0} \beta(t)/\alpha(t) = M$ . For any  $m \in \mathbf{R}$ , we see that

$$(4) \quad \lim_{t \rightarrow t_0} \frac{d(P_t, L_m)}{d(P_t, P_{t_0})} = \lim_{t \rightarrow t_0} \left| \frac{m\alpha(t) - \beta(t)}{\sqrt{\alpha^2(t) + \beta^2(t)}} \right| = \lim_{t \rightarrow t_0} \left| \frac{m - \frac{\beta(t)}{\alpha(t)}}{\sqrt{1 + \frac{\beta^2(t)}{\alpha^2(t)}}} \right|.$$

By (4),

$$\lim_{t \rightarrow t_0} \frac{d(P_t, L_M)}{d(P_t, P_{t_0})} = 0,$$

so  $L_M(x)$  is the tangent of the curve  $S$  at  $(x(t_0), y(t_0))$ . If there is another tangent  $L_{M'}$  of the curve  $S$  at  $(x(t_0), y(t_0))$ , then by (3) and (4), we see that

$$0 = \left| \frac{M' - M}{\sqrt{1 + M'^2}} \right|.$$

Then  $M = M'$ . The proof is complete.

We remark that this definition is compatible with the concept of ‘tangent lines’ associated with the graph of a real smooth function  $y = f(x)$  of a real variable. Indeed, let  $S$  be the curve which is also described by the graph of a smooth function  $f$  passing through  $P = (x_0, y_0)$ . By Lemma 1, it is easy to see that (3) holds if, and only if, the straight line  $L$  is the tangent of the graph of the function  $f$ .

A point in the plane is said to be a *dual point* of order  $m$  of the plane curve  $S$ , where  $m$  is a nonnegative integer, if there exist exactly  $m$  mutually distinct tangents of  $S$  that also pass through it. The set of all dual points of order  $m$  of  $S$  in the plane is called the dual set of order  $m$  of  $S$ . We remark that  $m = 0$  is allowed. In this case, there are no tangents of  $S$  that pass through the point in consideration.

Let  $\{C_\lambda : \lambda \in I\}$ , where  $I$  is a real interval, be a family of plane curves. With each  $C_\lambda$ , suppose we can associate just one point  $P_\lambda$  in each  $C_\lambda$  such that the totality of these points form a curve  $S$ . Then  $S$  is called an **envelope** of the family  $\{C_\lambda | \lambda \in I\}$  if the curves  $C_\lambda$  and  $S$  share a common tangent line at the common point  $P_\lambda$ . Suppose we have a family of curves in the  $x, y$ -plane implicitly defined by

$$F(x, y, \lambda) = 0, \lambda \in I,$$

where  $I$  is an interval of  $\mathbf{R}$ . Then it is well known that the envelope  $S$  is described by a pair of parametric functions  $(\psi(\lambda), \phi(\lambda))$  that satisfy

$$\begin{cases} F(\psi(\lambda), \phi(\lambda), \lambda) = 0, \\ F'_\lambda(\psi(\lambda), \phi(\lambda), \lambda) = 0, \end{cases}$$

for  $\lambda \in I$ , provided some “good conditions” are satisfied. In particular, let  $f, g, h : I \rightarrow \mathbf{R}$ . Then for each fixed  $\lambda \in I$ , the equation

$$(5) \quad L_\lambda : f(\lambda)x + g(\lambda)y = h(\lambda), (f(\lambda), g(\lambda)) \neq 0,$$

defines a straight line  $L_\lambda$  in the  $x, y$ -plane, and we have a collection  $\{L_\lambda : \lambda \in I\}$  of straight lines. For such a collection, we have the following result.

**Theorem 1.** (see [4, Theorems 2.3 and 2.5]). *Let  $f, g, h$  be real differentiable functions defined on the interval  $I$  such that  $f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda) \neq 0$  and  $g(\lambda) \neq 0$  for  $\lambda \in I$ . Let  $\Phi$  be the family of straight lines of the form (5). Let the curve  $S$  be defined by the functions  $x = \psi(\lambda), y = \phi(\lambda)$ :*

$$(6) \quad \psi(\lambda) = \frac{g'(\lambda)h(\lambda) - g(\lambda)h'(\lambda)}{f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda)}, \quad \phi(\lambda) = \frac{f(\lambda)h'(\lambda) - f'(\lambda)h(\lambda)}{f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda)}, \quad \lambda \in I.$$

*Suppose  $\psi$  and  $\phi$  are smooth functions over  $I$  and one of the following cases holds: (i)  $\psi'(\lambda) \neq 0$  for  $\lambda \in I$ ; (ii)  $\psi'(\lambda) \neq 0$  for  $I \setminus \{d\}$  where  $d \in I$  and  $\lim_{\lambda \rightarrow d^-} \phi'(\lambda)/\psi'(\lambda)$  as well as  $\lim_{\lambda \rightarrow d^+} \phi'(\lambda)/\psi'(\lambda)$  exist and are equal. Then  $S$  is the envelope of the family  $\Phi$ .*

**Theorem 2.** (see [4, Theorem 2.6]). *Let  $\Lambda$  be an interval in  $\mathbf{R}$ , and  $f, g, h$  be real differentiable functions defined on  $\Lambda$  such that  $f(\lambda)g'(\lambda) - f'(\lambda)g(\lambda) \neq 0$  for  $\lambda \in \Lambda$ . Let  $\Phi$  be the family of straight lines of the form (5), where  $\lambda \in \Lambda$ , and let the curve  $S$  be the envelope of the family  $\Phi$ . Then the point  $(\alpha, \beta)$  in the plane is a dual point of order  $m$  of  $S$ , if, and only if, the function  $f(\lambda)\alpha + g(\lambda)\beta - h(\lambda)$ , as a function of  $\lambda$ , has exactly  $m$  mutually distinct roots in  $\Lambda$ .*

The above result states roughly that the roots of the function  $F(\lambda|\alpha, \beta) = f(\lambda)\alpha + g(\lambda)\beta - h(\lambda)$  in the interval  $\Lambda$  ‘match’ the tangents connecting the point  $(\alpha, \beta)$  to the envelope of the family  $\{L_\lambda|\lambda \in \Lambda\}$  of straight lines, where  $L_\lambda$  is the straight line defined by  $F(\lambda|x, y)$  for  $x, y \in R$ . Therefore we only need to count the number of such tangents for different pairs of  $(\alpha, \beta)$ , that is, to classify dual points of envelopes.

Plane curves can take on complicated forms. Fortunately, for some plane curves, their dual points can be described precisely. Indeed, a complete list of distribution maps of dual points of strictly convex and smooth (i.e. continuously differentiable) graphs of real functions of one variable defined on real intervals can be found in [4, Theorems 3.3-3.20.]. Based on such distribution maps, a partial list of distributions of dual points of piecewise convex-concave and smooth graphs is also available (see [4, Appendix A]). In this paper, we will need some of these distribution maps (see Lemmas 3 through 5 below) and will build some new ones (see Lemmas 6 through 9 below) for use in later discussions.

In deriving the complete list of distribution maps in [4], strictly convex and smooth functions are classified by their monotonicity and behaviors near the boundary points of their domains. Some of these classifications are standard. A less familiar one is recalled here as follows. Let  $g$  be a function defined on an interval  $I$  with  $c = \inf I$  and  $d = \sup I$ . Note that  $c$  or  $d$  may be infinite, or may be outside the interval  $I$ , and that  $g(c^+)$ ,  $g(d^-)$ ,  $g'(c^+)$  or  $g'(d^-)$  may not exist. For  $\lambda \in (c, d)$ , let

$$(7) \quad L_{g|\lambda}(x) = g'(\lambda)(x - \lambda) + g(\lambda), \quad x \in R.$$

In case  $d$  is finite and  $g(d^-)$ ,  $g'(d^-)$  exist, we let

$$(8) \quad L_{g|d}(x) = g'(d^-)(x - d) + g(d^-), \quad x \in R,$$

and in case  $c$  is finite and  $g(c^+)$ ,  $g'(c^+)$  exist, we let

$$(9) \quad L_{g|c}(x) = g'(c^+)(x - c) + g(c^+), \quad x \in R.$$

When  $d$  is finite, we say  $g \sim H_{d^-}$  if  $\lim_{\lambda \rightarrow d^-} L_{g|\lambda}(\alpha) = -\infty$  for any  $\alpha < d$ ; and similarly when  $c$  is finite,  $g \sim H_{c^+}$  if  $\lim_{\lambda \rightarrow c^+} L_{g|\lambda}(\alpha) = -\infty$  for any  $\alpha > c$ .

In case  $d$  is infinite, we say  $g \sim H_{+\infty}$  if  $\lim_{\lambda \rightarrow +\infty} L_{g|\lambda}(\alpha) = -\infty$  for any  $\alpha \in \mathbf{R}$ ; and similarly, when  $c$  is infinite, we say  $g \sim H_{-\infty}$  if  $\lim_{\lambda \rightarrow -\infty} L_{g|\lambda}(\alpha) = -\infty$  for any  $\alpha \in \mathbf{R}$ .

There is a convenient criteria for the determination of functions with the above stated properties.

**Lemma 2.** ([4, Lemmas 3.1 and 3.5]). *Let  $g : (c, d) \rightarrow \mathbf{R}$  be a smooth and strictly convex function. (i) Assume  $d < +\infty$ . If  $g'(d^-) = +\infty$ , then  $g \sim H_{d^-}$ . (ii) Assume  $d = +\infty$ . If  $g'(+\infty) = +\infty$ , or,  $g'(+\infty) = 0$  and  $g(+\infty) = -\infty$ , then  $g \sim H_{+\infty}$ .*

The description of the distribution of dual points of a plane curve can be cumbersome. For this reason, it is convenient to introduce several notations. We say that a point  $(a, b)$  in the plane is strictly above (above, strictly below, below) the graph of a function  $g$  if  $a$  belongs to the domain of  $g$  and  $g(a) < b$  (respectively  $g(a) \leq b$ ,  $g(a) > b$  and  $g(a) \geq b$ ). The notation is  $(a, b) \in \vee(g)$  (respectively  $(a, b) \in \bar{\vee}(g)$ ,  $(a, b) \in \wedge(g)$  and  $(a, b) \in \bar{\wedge}(g)$ ). Suppose we now have two real functions  $g_1$  and  $g_2$  defined on real subsets  $I_1$  and  $I_2$  respectively. We say that  $(a, b) \in \vee(g_1) \oplus \vee(g_2)$  if  $a \in I_1 \cap I_2$  and  $b > g_1(a)$  and  $b > g_2(a)$ , or,  $a \in I_1 \setminus I_2$  and  $b > g_1(a)$ , or,  $a \in I_2 \setminus I_1$  and  $b > g_2(a)$ . The notations  $(a, b) \in \bar{\vee}(g_1) \oplus \vee(g_2)$ ,  $(a, b) \in \bar{\vee}(g_1) \oplus \wedge(g_2)$ , etc. are similarly defined. If we now have  $n$  real functions  $g_1, \dots, g_n$  defined on intervals  $I_1, \dots, I_n$  respectively, we write  $(a, b) \in \vee(g_1) \oplus \vee(g_2) \oplus \dots \oplus \vee(g_n)$  if  $a \in I_1 \cup I_2 \cup \dots \cup I_n$ , and if

$$a \in I_{i_1} \cup I_{i_2} \cup \dots \cup I_{i_m} \Rightarrow b > g_{i_1}(a), b > g_{i_2}(a), \dots, b > g_{i_m}(a), i_1, \dots, i_m \in \{1, \dots, n\}.$$

The notations  $(a, b) \in \bar{\vee}(g_1) \oplus \bar{\vee}(g_2) \oplus \dots \oplus \bar{\vee}(g_n)$ , etc. are similarly defined.

Equipped with the functions  $L_{g|\lambda}$  defined by (7)-(9) and the ordering of points and graphs in the plane, we may now state the following distribution results for dual points.

**Lemma 3.** ([4, Theorem 3.20], see Figure 1). *Let  $g : R \rightarrow R$  be a strictly convex and smooth function such that  $g \sim H_{-\infty}$  and  $g \sim H_{+\infty}$ . Then  $(\alpha, \beta)$  is a dual point of order 0, 1 or 2 of  $g$  if, and only if, respectively  $\beta > g(\alpha)$ ,  $\beta = g(\alpha)$  or  $\beta < g(\alpha)$ .*

**Lemma 4.** ([4, Theorem A.9], see Figure 2). *Let  $c \in \mathbf{R}$ ,  $g_1 \in C^1(c, +\infty)$  and  $g_2 \in C^1[c, +\infty)$ . Suppose the following hold:*

- (i)  $g_1$  is strictly concave on  $(c, +\infty)$  such that  $-g_1 \sim H_{+\infty}$ ;
- (ii)  $g_2$  is strictly convex on  $[c, +\infty)$  such that  $g_2 \sim H_{+\infty}$ ;
- (iii)  $g_1^{(v)}(c^+) = g_2^{(v)}(c)$  for  $v = 0, 1$ .

*Then the intersection of dual sets of order 0 of  $g_1$  and  $g_2$  is empty.*

**Lemma 5.** ([4, Theorem A.3], see Figure 3). *Let  $c, d \in \mathbf{R}$  such that  $c < d$ ,  $g_1 \in C^1(c, d)$  and  $g_2 \in C^1(-\infty, d]$ . Suppose the following hold:*

- (i)  $g_1$  is strictly concave on  $(c, d)$  such that  $g_1(c^+)$ , and  $g_1'(c^+)$  exist;
- (ii)  $g_2$  is strictly convex on  $(-\infty, d]$  such that  $g_2 \sim H_{-\infty}$ ;
- (iii)  $g_1^{(v)}(d^-) = g_2^{(v)}(d)$  for  $v = 0, 1$ .

Then the intersection of the dual sets of order 0 of  $g_1$  and  $g_2$  is  $\bar{\nabla}(L_{g_1|c}) \oplus \nabla(g_2)$ .

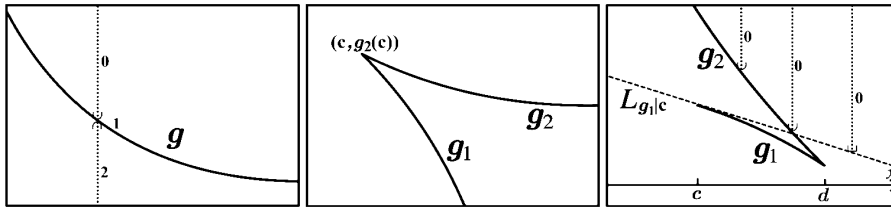


Fig. 1.

Fig. 2.

Fig. 3.

As explained in [4, Appendix A], dual sets of order 0 of plane curves that are made up of several pieces of convex and concave functions can be obtained by intersections. In particular, the following result is easily deduced from Theorems 3.4 and 3.10 in [4].

**Lemma 6.** (See Figure 4). *Let  $c, d \in \mathbf{R}$  such that  $c < d$ ,  $g_1 \in C^1[c, d]$  and  $g_2 \in C^1(-\infty, d)$ . Suppose the following hold:*

- (i)  $g_1$  is strictly convex on  $[c, d]$ ;
- (ii)  $g_2$  is strictly concave on  $(-\infty, d)$  such that  $L_{g_2|-\infty}$  (the asymptote of  $g_2$  at  $-\infty$ ) exists;
- (iii)  $g_1^{(v)}(d) = g_2^{(v)}(d^-)$  for  $v = 0, 1$ .

Then the intersection of the dual sets of order 0 of  $g_1$  and  $g_2$  is  $(\nabla(g_1) \oplus \nabla(L_{g_1|c}) \oplus \bar{\nabla}(L_{g_2|-\infty})) \cup (\wedge(g_2) \oplus \Delta(L_{g_1|c}))$ .

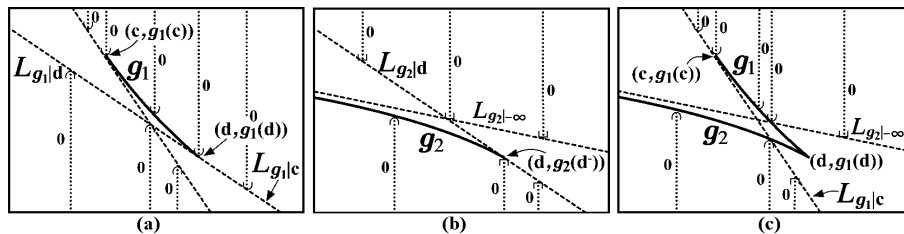


Fig. 4. Intersection of the dual sets of order 0 in (a) and (b) (see [4, Theorems 3.4 and 3.10]) to yield (c).

The following result is easily deduced from Theorems 3.9 and 3.11 in [4]

**Lemma 7.** (See Figure 5). *Let  $a, c \in \mathbf{R}$ ,  $g_1 \in C^1(a, c]$  and  $g_2 \in C^1(-\infty, c)$ . Suppose the following hold:*

- (i)  $g_1$  is strictly convex on  $(a, c]$  such that  $g_1 \sim H_{a+}$ ;
- (ii)  $g_2$  is strictly concave on  $(-\infty, c)$  such that  $-g_2 \sim H_{-\infty}$ ;
- (iii)  $g_1^{(v)}(c) = g_2^{(v)}(c^-)$  for  $v = 0, 1$ .

Then the intersection of the dual sets of order 0 of  $g_1$  and  $g_2$  is  $\wedge(g_2\chi_{(-\infty, a]})$ .

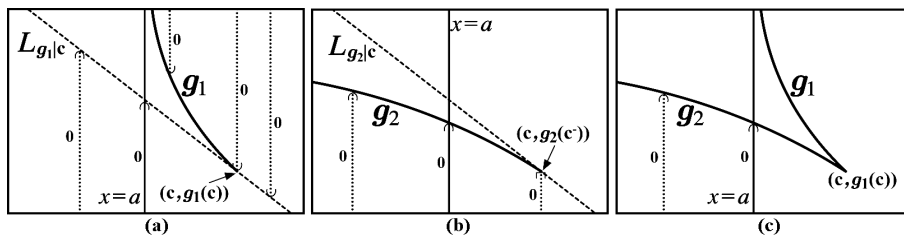


Fig. 5. Intersection of the dual sets of order 0 in (a) and (b) (see [4, Theorems 3.9 and 3.11]) to yield (c).

The following result is easily deduced from Theorems 3.4 and A.3 in [4].

**Lemma 8.** (See Figure 6). *Let  $a, b, c \in \mathbf{R}$ ,  $g_1 \in C^1(a, +\infty)$ ,  $g_2 \in C^1[a, b)$  and  $g_3 \in C^1[c, b]$ . Suppose the following hold:*

- (i)  $g_1$  is strictly convex on  $[a, +\infty)$  such that  $g_1 \sim H_{+\infty}$ ;
- (ii)  $g_2$  is strictly concave on  $(a, b)$ ;
- (iii)  $g_3$  is strictly convex on  $[c, b]$ ;
- (iv)  $g_1^{(v)}(a^+) = g_2^{(v)}(a^+)$  and  $g_2^{(v)}(b^-) = g_3^{(v)}(b^-)$  for  $v = 0, 1$ .

Then the intersection of the dual sets of order 0 of  $g_1, g_2$  and  $g_3$  is  $\vee(g_1) \oplus \vee(g_3) \oplus \vee(L_{g_3|c})$ .

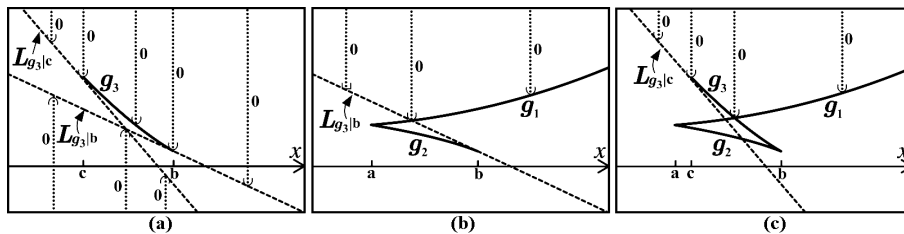


Fig. 6. Intersection of the dual sets of order 0 in (a) and (b) (see [4, Theorems 3.4 and A.3]) to yield (c).



The following result is easily deduced from Theorems 3.8 in [4] and Lemma 5.

**Lemma 9.** (See Figure 7). *Let  $c, d \in \mathbf{R}$ ,  $g_1 \in C^1(0, c]$ ,  $g_2 \in C^1(d, c)$  and  $g_3 \in C^1[d, +\infty)$ . Suppose the following hold:*

- (i)  $g_1$  is strictly convex on  $(0, c]$ ;
- (ii)  $g_2$  is strictly concave on  $(d, c)$ ;
- (iii)  $g_3$  is strictly convex on  $[d, +\infty)$  such that  $g_3 \sim H_{+\infty}$ ;
- (iv)  $g_1^{(v)}(c) = g_2^{(v)}(c^-)$  and  $g_2^{(v)}(d^+) = g_3^{(v)}(d)$  for  $v = 0, 1$ .

Then the intersection of the dual sets of order 0 of  $g_1, g_2$  and  $g_3$  is  $\vee(g_1) \oplus \vee(g_3\chi_{[0, +\infty)}) \oplus \overline{\vee}(\Theta_0\chi_{\{0\}})$ .

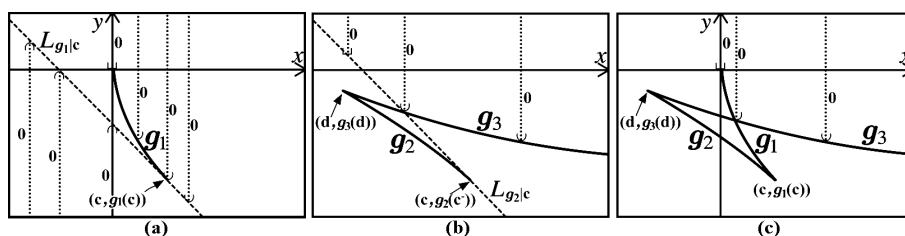


Fig. 7. Intersection of the dual sets of order 0 in (a) and (b) (see [4, Theorems 3.8] and Lemma 5) to yield (c).

Given a pair of parametric functions  $x = \psi(\lambda)$  and  $y = \varphi(\lambda)$  defined on an interval  $I$ . We may sometimes be able to solve for  $\lambda$  from  $x = \psi(\lambda)$  and then substitute it into  $\varphi(\lambda)$  to yield a function  $y = f(x)$ . The following simple result can be used to make sure that smooth graphs can be obtained from parametric curves in this manner.

**Lemma 10.** (see [4, Theorem 2.1]). *Let  $G$  be the curve described by a pair of smooth functions  $\psi(\lambda)$  and  $\phi(\lambda)$  on an interval  $I$  such that  $\psi'(\lambda) > 0$  (or  $\psi'(\lambda) < 0$ ) for  $t \in I$  except at perhaps one point  $r$ . Suppose  $q$  is a continuous function defined on  $I$  such that  $\phi'(\lambda)/\psi'(\lambda) = q(\lambda)$  for  $\lambda \in I \setminus \{r\}$ . Then  $G$  is also the graph of a smooth function  $y = S(x)$  defined on  $\psi(I)$ .*

For the sake of convenience, we will use the same notation to indicate a real function of a real variable and its graph. Therefore, in the above result, we may stay the conclusion in the form “Then the curve  $G$  is the graph of a smooth function  $y = G(x)$  defined over  $\psi(I)$ .”

The graph of a real function  $f$  defined on a set  $J$  of real numbers is the set

$$\{(x, y) \in \mathbf{R}^2 | y = f(x), x \in J\}.$$

For the sake of convenience, we will use the same notation to indicate a (real) function of a real variable and its graph. Therefore, in the sequel, we will meet statements such as 'the set  $S$  is also the graph of a function  $y = S(x)$  defined on the interval  $I \dots$ '.

We now turn to the function  $F(\lambda)$  in (2). In case  $ac = 0$ , or, any two of the numbers  $\tau, \sigma$  and  $\delta$  are the same, or,  $\tau\sigma\delta = 0$ , the resulting function is a simpler one and the corresponding characteristic region has been considered in detail in Chapter 7 of [4].

Therefore we may assume that  $ac \neq 0$  and  $\tau, \sigma, \delta$  are mutually distinct nonzero real numbers. However, if  $\sigma < 0$ , then

$$e^{-\sigma\lambda}F(\lambda) = (c\lambda + d) + (a\lambda + b)e^{-\sigma\lambda} + xe^{(\delta-\sigma)\lambda} + ye^{(\tau-\sigma)\lambda},$$

and the real roots of the functions  $F(\lambda)$  and  $e^{-\sigma\lambda}F(\lambda)$  are the same. So we may assume that  $\sigma > 0$ .

Since  $a \neq 0$ , by dividing  $F(\lambda)$  by  $a$  if necessary, we may further assume without loss of generality that our characteristic equation  $F(\lambda)$  is of the form

$$(10) \quad Q(\lambda|x, y, a, b, c, \sigma, \delta, \tau) = \lambda + a + (b\lambda + c)e^{\sigma\lambda} + xe^{\delta\lambda} + ye^{\tau\lambda}$$

where  $a, c, x, y \in \mathbf{R}$ ,  $b \in \mathbf{R} \setminus \{0\}$  and  $\sigma, \delta, \tau$  are mutually distinct nonzero real numbers with  $\delta < \tau$  and  $\sigma > 0$ . The subset of  $(x, y, a, b, c, \sigma, \delta, \tau) \in \mathbf{R}^8$  such that none of the roots of the corresponding  $\nabla(1, 1, 0, 0)$ -polynomial  $Q$  are real is called its  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region. Before we discuss the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of  $Q$ , we need to first handle the  $\nabla(1, 1)$ -polynomial

$$(11) \quad T(\lambda|\alpha, \beta, w, \sigma) = \lambda + \beta + e^{-\lambda\sigma}(w\lambda + \alpha), \quad \alpha, \beta, w, \sigma \in \mathbf{R}; w \neq 0, \sigma > 0.$$

This quasi-polynomial has been discussed in [4, Section 7.1.4]. However, we need a few more of its properties for later uses.

### 3. PROPERTIES OF $\nabla(1, 1)$ -POLYNOMIAL $T(\lambda|\alpha, \beta, w, \sigma)$

Let  $T$  and the involved parameters be defined by (11). Let the parametric curve  $S$  be defined by

$$(12) \quad \psi(\lambda) = \frac{1}{\sigma}e^{\sigma\lambda} + \frac{w}{\sigma} - w\lambda, \quad \varphi(\lambda) = -\lambda - \frac{1}{\sigma} - \frac{w}{\sigma}e^{-\sigma\lambda}, \quad \lambda \in \mathbf{R}.$$

Observe that  $\psi$  and  $\varphi$  are smooth for  $\lambda \in \mathbf{R}$ , and that

$$(13) \quad \psi'(\lambda) = -(w - e^{\sigma\lambda}), \quad \varphi'(\lambda) = e^{-\sigma\lambda}(w - e^{\sigma\lambda}), \quad \lambda \in \mathbf{R}.$$

Therefore, for each  $\lambda \in \mathbf{R}$  with  $e^{\sigma\lambda} \neq w$ , we have

$$(14) \quad \frac{\varphi'(\lambda)}{\psi'(\lambda)} = -e^{-\sigma\lambda} \quad \text{and} \quad \frac{\frac{d}{d\lambda} \frac{\varphi'(\lambda)}{\psi'(\lambda)}}{\psi'(\lambda)} = \frac{\sigma}{e^{\sigma\lambda}(e^{\sigma\lambda} - w)}.$$

**Lemma 11.** Let  $T(\lambda|\alpha, \beta, w, \sigma)$  be defined by (11) and let the curve  $S$  be the plane curve defined by (12). For any given  $\alpha \in \mathbf{R}$ , let  $y = h(\lambda|\alpha)$  be the function defined by

$$(15) \quad h(\lambda|\alpha) = -e^{-\sigma\lambda}(\alpha - \psi(\lambda)) + \varphi(\lambda) \text{ for } \lambda \in \mathbf{R}.$$

The following results hold:

- (i)  $(\alpha, \beta)$  is a dual point of order  $m$  of  $S$  if, and only if,  $h(\lambda|\alpha) = \beta$  has exactly  $m$  distinct real solutions;
- (ii)  $T(\lambda|\alpha, \beta, w, \sigma)$  has exactly  $m$  distinct real roots if, and only if,  $h(\lambda|\alpha) = \beta$  has exactly  $m$  distinct real solutions;
- (iii)  $T(\lambda|\alpha, \beta, w, \sigma) > 0$  (or  $T(\lambda|\alpha, \beta, w, \sigma) < 0$ ) for some real interval  $I$  if, and only if,  $h(\lambda|\alpha) < \beta$  (respectively  $h(\lambda|\alpha) > \beta$ ) for  $\lambda \in I$ .

*Proof.* From (13), we see that  $\psi'(\lambda)$  has at most one real root, and for each  $\lambda_0 \in \mathbf{R}$ ,  $\psi(\lambda) \neq \psi(\lambda_0)$  for all  $\lambda \in \mathbf{R} \setminus \{\lambda_0\}$  which is sufficiently close to  $\lambda_0$ . Since

$$\lim_{\lambda \rightarrow \lambda_0} \frac{\varphi(\lambda) - \varphi(\lambda_0)}{\psi(\lambda) - \psi(\lambda_0)} = \lim_{\lambda \rightarrow \lambda_0} \frac{\varphi'(\lambda)}{\psi'(\lambda)} = -e^{-\sigma\lambda_0},$$

by Lemma 1, the straight line

$$L_{\lambda_0}(x) = -e^{-\sigma\lambda_0}(x - \psi(\lambda_0)) + \varphi(\lambda_0)$$

is the tangent line of  $S$  at the point  $(\psi(\lambda_0), \varphi(\lambda_0))$ . So by (15),

$$h(\lambda_0|\alpha) = L_{\lambda_0}(\alpha) \text{ for any } \alpha \in \mathbf{R}$$

In other words,  $h(\lambda|\alpha)$  can be interpreted as the  $y$ -coordinate of the point of intersection of the vertical straight line  $x = \alpha$  with the tangent line of  $S$  at the point  $(\psi(\lambda), \varphi(\lambda))$ . Therefore, if there is a tangent line of  $S$  at  $(\psi(\lambda), \varphi(\lambda))$  that passes through the point  $(\alpha, \beta)$ , then  $h(\lambda|\alpha) = \beta$ . Conversely, if  $h(\lambda|\alpha) = \beta$  for some  $\lambda \in \mathbf{R}$ , then there is a tangent line of the graph of  $S$  at  $(\psi(\lambda), \varphi(\lambda))$  that passes through the points  $(\alpha, \beta)$ . The proof of the statement (i) is complete.

Next, by substituting  $\psi(\lambda)$  and  $\varphi(\lambda)$  into (15), we may easily obtain

$$(16) \quad h(\lambda|\alpha) = \beta - T(\lambda|\alpha, \beta, w, \sigma).$$

Clearly, if  $\bar{\lambda}$  is a real root of  $T(\lambda|\alpha, \beta, w, \sigma)$ , then  $\bar{\lambda}$  is a real solution of  $h(\lambda|\alpha) = \beta$ . The converse is also true. The proof of the statement (ii) is complete. By (16),  $T(\lambda|\alpha, \beta, w, \sigma) > 0$  for some real interval  $I$ , then  $h(\lambda|\alpha) < \beta$  on  $I$ . The converse is true. The proof of the statement (iii) is complete.

We remark that, in view of (16), there exist additional relations between the functions  $h(\lambda|\alpha)$  and  $T(\lambda|\alpha, \beta, w, \sigma)$ . For instance, they have the same extremal points, and the properties of the graphs of  $-h(\lambda|\alpha)$  and  $T(\lambda|\alpha, \beta, w, \sigma)$  are similar, etc. So in later discussions, we may investigate the properties of the function  $T(\lambda|\alpha, \beta, w, \sigma)$  by means of the function  $h(\lambda|\alpha)$ . Note also, that by (15),

$$(17) \quad h'_\lambda(\lambda|\alpha) = \sigma e^{-\sigma\lambda}(\alpha - \psi(\lambda)).$$

The statements (i) and (ii) in the above result assert that  $T$  has exactly  $m$  distinct real roots if, and only if,  $(\alpha, \beta)$  is a dual point of order  $m$  of the curve  $S$ . This leads us to investigate the distribution of dual points of  $S$ . We need to consider two cases:  $w > 0$  and  $w < 0$ .

Suppose  $w < 0$ . We see that

$$\lim_{\lambda \rightarrow -\infty} (\psi(\lambda), \varphi(\lambda)) = (-\infty, +\infty) \text{ and } \lim_{\lambda \rightarrow +\infty} (\psi(\lambda), \varphi(\lambda)) = (+\infty, -\infty),$$

and that  $\psi'(\lambda)$  and  $\varphi'(\lambda)$  in (13) have no real roots. By Lemma 10, the curve  $S$  is the graph of a smooth function  $y = S(x)$  over  $\mathbf{R}$ . By the chain rule and other previously obtained information related to  $\psi(\lambda)$  and  $\varphi(\lambda)$ , we may then see that  $S$  is strictly decreasing and strictly convex on  $\mathbf{R}$ , that  $S(+\infty) = -\infty$  and that  $S'(-\infty) = -\infty$  as well as  $S'(+\infty) = 0$ . The latter three properties imply, by Lemma 2, that  $S \sim H_{+\infty}$  and  $S \sim H_{-\infty}$ . We break the plane into three sets (see Figure 8):

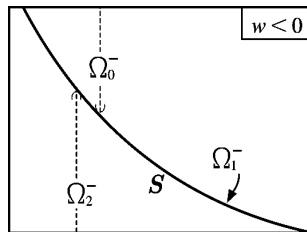


Fig. 8.

$$(18) \quad \begin{aligned} \Omega_0^- &= \{(x, y) : y > S(x)\}, \quad \Omega_1^- = \{(x, y) : y = S(x)\} \\ \text{and } \Omega_2^- &= \{(x, y) : y < S(x)\}. \end{aligned}$$

**Lemma 12.** *Suppose  $w < 0$ . Let  $T(\lambda|\alpha, \beta, w, \sigma)$  be defined by (11) and the sets  $\Omega_0^-$ ,  $\Omega_1^-$  and  $\Omega_2^-$  be defined by (18). Then the following statements hold:*

- (i) *For any  $\alpha, \beta \in \mathbf{R}$ , the function  $T(\lambda|\alpha, \beta, w, \sigma)$  has at most two real roots.*
- (ii) *If  $(\alpha, \beta) \in \Omega_0^- \cup \Omega_1^-$ , then  $T(\lambda|\alpha, \beta, w, \sigma) \geq 0$  for  $\lambda \in \mathbf{R}$ .*

(iii) If  $(\alpha, \beta) \in \Omega_2^-,$  then there are exactly two real roots  $r_1$  and  $r_2,$   $r_1 < r_2,$  of  $T(\lambda|\alpha, \beta, w, \sigma)$  such that  $T(\lambda|\alpha, \beta, w, \sigma) > 0$  on  $\mathbf{R} \setminus [r_1, r_2]$  and  $T(\lambda|\alpha, \beta, w, \sigma) < 0$  on  $(r_1, r_2).$

*Proof.* Under our assumptions, by the properties of the curve  $S$  and Lemma 3,  $\Omega_0^-$  is the dual set of order 0 of  $S,$   $\Omega_1^-$  is the dual set of order 1 of  $S$  and  $\Omega_2^-$  is the dual set of order 2 of  $S.$  Furthermore, there are no dual sets of order  $m,$  where  $m \geq 3,$  of  $S.$  By Lemma 11,  $(\alpha, \beta)$  is a dual point of order  $m$  of  $S$  if, and only if,  $T(\lambda|\alpha, \beta, w, \sigma)$  has exactly  $m$  distinct real roots. So the proof of the statement (i) is complete.

We note that  $T(-\infty|\alpha, \beta, w, \sigma) = T(+\infty|\alpha, \beta, w, \sigma) = +\infty.$  If  $(\alpha, \beta) \in \Omega_0^-,$  then  $T(\lambda|\alpha, \beta, w, \sigma)$  has no real roots and  $T(\lambda|\alpha, \beta, w, \sigma) > 0$  for  $\lambda \in \mathbf{R}.$  If  $(\alpha, \beta) \in \Omega_1^-,$  then  $T(\lambda|\alpha, \beta, w, \sigma)$  has exactly one real root  $r.$  So  $T(\lambda|\alpha, \beta, w, \sigma) > 0$  for  $\lambda \in \mathbf{R} \setminus \{r\}.$  The proof of the statement (ii) is complete. If  $(\alpha, \beta) \in \Omega_2^-,$  then  $T(\lambda|\alpha, \beta, w, \sigma)$  has exactly two real roots  $r_1$  and  $r_2.$  Since  $\psi(\lambda)$  is strictly increasing on  $\mathbf{R}$  by (13),  $\psi(-\infty) = -\infty$  and  $\psi(+\infty) = +\infty,$  we see that  $\psi(\lambda) = \alpha$  has a unique real solution. By (17),  $h(\lambda|\alpha)$  has at most one local extremal point. By (16),  $T(\lambda|\alpha, \beta, w, \sigma)$  has at most one local extremal point as well. Hence, it must be true that  $T(\lambda|\alpha, \beta, w, \sigma) > 0$  on  $\mathbf{R} \setminus [r_1, r_2]$  and  $T(\lambda|\alpha, \beta, w, \sigma) < 0$  on  $(r_1, r_2).$

Next we suppose  $w > 0.$  We see that

$$\lim_{\lambda \rightarrow -\infty} (\psi(\lambda), \varphi(\lambda)) = \lim_{\lambda \rightarrow +\infty} (\psi(\lambda), \varphi(\lambda)) = (+\infty, -\infty),$$

that  $\psi'(\lambda)$  and  $\varphi'(\lambda)$  have exactly common one real root  $\lambda^* = (\ln w)/\sigma$  and

$$r^* := \psi(\lambda^*) = \frac{w(2 - \ln w)}{\sigma}.$$

We may see that the curve  $S$  is composed of two pieces  $S_1$  and  $S_2.$  The first piece  $S_1$  corresponds to the case where  $\lambda \in (-\infty, \lambda^*)$  and the second  $S_2$  corresponds to the case where  $\lambda \in [\lambda^*, +\infty).$  By means of these information together with (12), (13) and (14) and Lemma 10,  $S_1$  is the graph of a function  $y = S_1(x)$  which is strictly decreasing, strictly concave, and smooth over  $(r^*, +\infty);$  and  $S_2$  is the graph of a function  $y = S_2(x)$  which is strictly decreasing, strictly convex, and smooth over  $[r^*, +\infty).$  Since  $S_1'(+\infty) = -\infty,$   $S_2(+\infty) = -\infty$  and  $S_2'(+\infty) = 0,$  according to Lemma 2, we see that  $-S_1 \sim H_{+\infty}$  and  $S_2 \sim H_{+\infty}.$  We break the plane into three sets (see Figure 9):

$$\begin{aligned} \Omega_1^+ &= (\wedge(S_1)) \cup (\vee(S_2\chi_{(r^*, +\infty)})) \cup \{(x, y) : x \leq r^*\}, \\ (19) \quad \Omega_2^+ &= \{(x, y) : x > r^* \text{ and } y = S_1(x)\} \cup \{(x, y) : x > r^* \text{ and } y = S_2(x)\}, \\ \Omega_3^+ &= \{(x, y) : x > r^* \text{ and } S_1(x) < y < S_2(x)\}. \end{aligned}$$

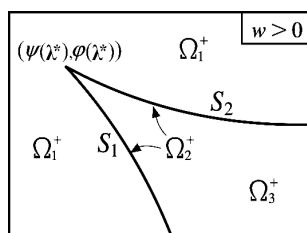


Fig. 9.

**Lemma 13.** Suppose  $w > 0$ . Let  $T(\lambda|\alpha, \beta, w, \sigma)$  be defined by (11) and the sets  $\Omega_1^+$ ,  $\Omega_2^+$  and  $\Omega_3^+$  be defined by (19). Then the following statements hold:

- (1) For any  $\alpha, \beta \in \mathbf{R}$ , the function  $T(\lambda|\alpha, \beta, w, \sigma)$  has at most three real roots.
- (2) If  $(\alpha, \beta) \in \Omega_1^+ \cup \Omega_2^+$ , then there is a real root  $r$  of  $T(\lambda|\alpha, \beta, w, \sigma)$  such that  $T(\lambda|\alpha, \beta, w, \sigma) \leq 0$  on  $(-\infty, r)$  and  $T(\lambda|\alpha, \beta, w, \sigma) \geq 0$  on  $(r, +\infty)$ .
- (3) If  $(\alpha, \beta) \in \Omega_3^+$ , then there are exactly three real roots  $r_1, r_2$  and  $r_3$  of  $T(\lambda|\alpha, \beta, w, \sigma)$  such that  $T(\lambda|\alpha, \beta, w, \sigma) < 0$  on  $(-\infty, r_1) \cup (r_2, r_3)$  and  $T(\lambda|\alpha, \beta, w, \sigma) > 0$  on  $(r_1, r_2) \cup (r_3, +\infty)$ .

*Proof.* By (15),  $h(-\infty|\alpha) = +\infty$  and  $h(+\infty|\alpha) = -\infty$ . From (12) and (13), we have that  $\psi(-\infty) = \psi(+\infty) = +\infty$ , that  $\psi(\lambda)$  is strictly decreasing on  $(-\infty, \lambda^*)$  and  $\psi(\lambda)$  is strictly increasing on  $(\lambda^*, +\infty)$ . So  $\psi(\lambda)$  has the absolute minimal value  $r^*$ .

Assume  $\alpha > r^*$ . The equation  $\psi(\lambda) = \alpha$  has exactly two real solutions  $\lambda_{\min} \in (-\infty, \lambda^*)$  and  $\lambda_{\max} \in (\lambda^*, +\infty)$  so that

$$\psi(\lambda) < \alpha \text{ for } \lambda \in (\lambda_{\min}, \lambda_{\max}) \text{ and } \psi(\lambda) > \alpha \text{ for } \lambda \in \mathbf{R} \setminus [\lambda_{\min}, \lambda_{\max}].$$

By (17), we see that  $h(\lambda|\alpha)$  is strictly increasing on  $(\lambda_{\min}, \lambda_{\max})$  and  $h(\lambda|\alpha)$  is strictly decreasing on  $\mathbf{R} \setminus [\lambda_{\min}, \lambda_{\max}]$ . So  $\lambda_{\min}$  is a local minimal point of  $h$  and  $\lambda_{\max}$  is a local maximal point of  $h$ . Furthermore,

$$h(\lambda_{\min}|\alpha) = \varphi(\lambda_{\min}) = S_1(\psi(\lambda_{\min})) = S_1(\alpha)$$

and

$$h(\lambda_{\max}|\alpha) = \varphi(\lambda_{\max}) = S_2(\psi(\lambda_{\max})) = S_2(\alpha)$$

(see Figure 10(a)). Assume  $\alpha \leq r^*$ . By (17), we see that  $h(\lambda|\alpha)$  is strictly decreasing on  $\mathbf{R}$  (see Figure 10(b)). Therefore, given any  $\alpha, \beta \in \mathbf{R}$ ,  $h(\lambda|\alpha) = \beta$  has at most three real solutions. The proof of the statement (i) is then completed by Lemma 11(i).

If  $(\alpha, \beta_1) \in \Omega_3^+$ , that is  $\alpha > r^*$  and  $S_1(\alpha) < \beta_1 < S_2(\alpha)$ , then  $h(\lambda|\alpha) = \beta_1$  has exactly three real solutions  $r_1, r_2$  and  $r_3$  such that  $h(\lambda|\alpha) > \beta_1$  on  $(-\infty, r_1) \cup (r_2, r_3)$  and  $h(\lambda|\alpha) < \beta_1$  on  $(r_1, r_2) \cup (r_3, +\infty)$ . The proof of the statement (iii) is then completed by Lemma 11(iii) (see Figure 10(a)).

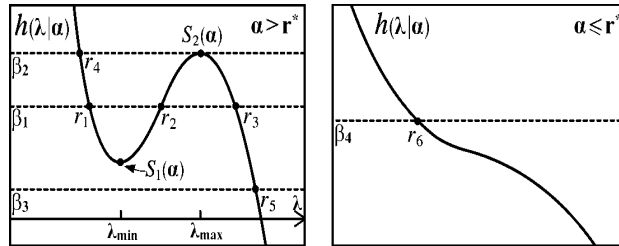


Fig. 10.

To prove the statement (ii), suppose  $(\alpha, \beta_2) \in \Omega_2^+$ . Then,  $\alpha > r^*$  and  $S_2(\alpha) = \beta_2$ , or  $\alpha > r^*$  and  $S_1(\alpha) = \beta_2$ . In the former case,  $h(\lambda|\alpha) = \beta_2$  has exactly two real solutions  $r_4 \in (-\infty, \lambda_{\min})$  and  $r_{\max}$  such that  $h(\lambda|\alpha) > \beta_2$  on  $(-\infty, r_4)$  and  $h(\lambda|\alpha) < \beta_2$  on  $(r_4, r_{\max}) \cup (r_{\max}, +\infty)$ . We take  $r = r_4$ . By Lemma 11(iii),  $T(\lambda|\alpha, \beta, w, \sigma) \leq 0$  on  $(-\infty, r)$  and  $T(\lambda|\alpha, \beta, w, \sigma) \geq 0$  on  $(r, +\infty)$ . The latter case is similarly proved (see Figure 10(a)).

Suppose  $(\alpha, \beta_3) \in \Omega_1^+$ . If  $\alpha > r^*$ , we see that  $S_1(\alpha) > \beta_3$  or  $S_2(\alpha) < \beta_3$ . In the former case,  $h(\lambda|\alpha) = \beta_3$  has exactly one real solution  $r_5$  such that  $h(\lambda|\alpha) > \beta_3$  on  $(-\infty, r_5)$  and  $h(\lambda|\alpha) < \beta_3$  on  $(r_5, +\infty)$ . Take  $r = r_5$ , by Lemma 11(iii),  $T(\lambda|\alpha, \beta, w, \sigma) \leq 0$  on  $(-\infty, r)$  and  $T(\lambda|\alpha, \beta, w, \sigma) \geq 0$  on  $(r, +\infty)$ . The latter case is similarly proved. Suppose  $\alpha \leq r^*$ . For any  $\beta_4 \in \mathbf{R}$ ,  $h(\lambda|\alpha) = \beta_4$  has exactly one real solution  $r_6$  such that  $h(\lambda|\alpha) > \beta_4$  on  $(-\infty, r_6)$  and  $h(\lambda|\alpha) < \beta_4$  on  $(r_6, +\infty)$ . Take  $r = r_6$ , by Lemma 11(iii),  $T(\lambda|\alpha, \beta, w, \sigma) \leq 0$  on  $(-\infty, r)$  and  $T(\lambda|\alpha, \beta, w, \sigma) \geq 0$  on  $(r, +\infty)$  (see Figures 10(a) and 10(b)). The proof of the statement (iii) is complete.

#### 4. C\R-CHARACTERISTIC REGION OF $\nabla(1, 1, 0, 0)$ -POLYNOMIAL

Consider the  $\nabla(1, 1, 0, 0)$ -polynomial  $Q(\lambda|x, y, a, b, c, \sigma, \delta, \tau)$  defined by (10). For each  $\lambda \in \mathbf{R}$ , let  $L_\lambda$  be the straight line in the plane defined by

$$(20) \quad L_\lambda : e^{\delta\lambda}x + e^{\tau\lambda}y = - \left\{ \lambda + a + (b\lambda + c)e^{\sigma\lambda} \right\}.$$

Note that  $L_\lambda$  defined by (20) is of the form (5) and  $f'(\lambda)g(\lambda) - f(\lambda)g'(\lambda) = (\tau - \delta)e^{(\tau+\delta)\lambda} \neq 0$ . From (6), we let  $G$  be the curve defined by the parametric functions

$$(21) \quad x(\lambda) = \Gamma(\lambda; \delta, \tau) \text{ and } y(\lambda) = \Gamma(\lambda; \tau, \delta)$$

for  $\lambda \in \mathbf{R}$ , where

$$\Gamma(\lambda; u, v) = \frac{e^{-\lambda u}}{u - v} \left\{ v(\lambda + a) - 1 + e^{\lambda \sigma} [(v - \sigma)(b\lambda + c) - b] \right\}.$$

Then

$$(22) \quad x'(\lambda) = \frac{e^{(\sigma-\delta)\lambda}}{\delta - \tau} b(\sigma - \delta)(\tau - \sigma)T(\lambda), \quad y'(\lambda) = -\frac{e^{(\sigma-\tau)\lambda}}{\delta - \tau} b(\sigma - \delta)(\tau - \sigma)T(\lambda),$$

where  $T(\lambda) = T(\lambda|A, B, C, \sigma)$  is given by (11),

$$(23) \quad A = \frac{\delta + \tau - \delta\tau a}{(\sigma - \delta)(\tau - \sigma)b},$$

$$(24) \quad B = \frac{c(\sigma - \delta)(\tau - \sigma) + b(\delta + \tau - 2\sigma)}{(\sigma - \delta)(\tau - \sigma)b},$$

and

$$(25) \quad C = \frac{-\delta\tau}{(\sigma - \delta)(\tau - \sigma)b}.$$

Note that our assumptions on  $b, \sigma, \delta, \tau$  in (10) implies  $C \neq 0$ . Let  $\Sigma(A, B, C, \sigma) = \{\lambda \in \mathbf{R} : T(\lambda|A, B, C, \sigma) = 0\}$ . According to Lemma 12(i) and Lemma 13(i), the real roots of  $T$  are finite in number and isolated, hence  $\Sigma(A, B, C, \sigma)$  is a finite set. Furthermore, by (22),  $x'(\lambda) = 0$  if, and only if,  $\lambda \in \Sigma(A, B, C, \sigma)$ . We see that

$$(26) \quad \frac{y'(\lambda)}{x'(\lambda)} = -e^{(\delta-\tau)\lambda} < 0 \text{ for } \lambda \in \mathbf{R} \setminus \Sigma(A, B, C, \sigma)$$

and

$$\lim_{\lambda \rightarrow d^-} \frac{y'(\lambda)}{x'(\lambda)} = \lim_{\lambda \rightarrow d^+} \frac{y'(\lambda)}{x'(\lambda)} = -e^{(\delta-\tau)d} < 0 \text{ for } d \in \Sigma(A, B, C, \sigma).$$

By Theorem 1,  $G$  is the envelope of the family  $\{L_\lambda : \lambda > 0\}$  where  $L_\lambda$  is defined by (20). We have

$$(27) \quad \frac{\frac{d}{d\lambda} \left( \frac{y'(\lambda)}{x'(\lambda)} \right)}{x'(\lambda)} = \frac{-(\delta - \tau)^2 e^{\tau\lambda}}{(\sigma - \delta)(\tau - \sigma)bT(\lambda)} \text{ for } \lambda \in \mathbf{R} \setminus \Sigma(A, B, C, \sigma).$$

In view of Theorem 2, to find the  $C \setminus \mathbf{R}$ -characteristic region of (10) is to find the dual set of order 0 of the envelope  $G$  described by (21). By (22), we see that the



properties of the curve  $G$  are dependent on the function  $T(\lambda|A, B, C, \sigma)$ . Hence, in view of Lemmas 12 and 13, we have to consider the two cases  $C < 0$  and  $C > 0$ .

**4.1. The case where  $C < 0$**

We have assumed that  $b \in \mathbf{R} \setminus \{0\}$  and  $\tau, \delta$  and  $\sigma$  are pairwise distinct real numbers such that  $\tau > \delta$  and  $\sigma > 0$ . When  $C < 0$ , the real numbers  $\sigma, \tau, \delta$  and  $b$  can be broken exactly into the following six cases:

- (b1)  $\sigma > \tau > \delta > 0$  and  $b < 0$ ;
- (b2)  $\tau > \sigma > \delta > 0$  and  $b > 0$ ;
- (b3)  $\tau > \delta > \sigma > 0$  and  $b < 0$ ;
- (b4)  $\sigma > \tau > 0 > \delta$  and  $b > 0$ ;
- (b5)  $\tau > \sigma > 0 > \delta$  and  $b < 0$ ;
- (b6)  $\sigma > 0 > \tau > \delta$  and  $b < 0$ .

The parametric functions  $x(\lambda)$  and  $y(\lambda)$  defined by (21) have several elementary properties. First,

$$(28) \quad \lim_{\lambda \rightarrow -\infty} (x(\lambda), y(\lambda)) = \begin{cases} (+\infty, -\infty) & \text{if (b1), (b2) or (b3) holds} \\ (0, +\infty) & \text{if (b4) holds} \\ (0, +\infty) & \text{if (b5) holds} \\ (0, 0) & \text{if (b6) holds} \end{cases},$$

and

$$(29) \quad \lim_{\lambda \rightarrow +\infty} (x(\lambda), y(\lambda)) = \begin{cases} (-\infty, +\infty) & \text{if (b1) or (b6) holds} \\ (-\infty, 0) & \text{if (b2) holds} \\ (0, 0) & \text{if (b3) holds} \\ (+\infty, -\infty) & \text{if (b4) holds} \\ (+\infty, 0) & \text{if (b5) holds} \end{cases}.$$

**Theorem 3.** *Suppose condition (b1) or (b2) holds (that is,  $\sigma > \tau > \delta > 0$  and  $b < 0$ , or  $\tau > \sigma > \delta > 0$  and  $b > 0$ ). Let the parametric curve  $G$  be defined by  $x(\lambda) = \Gamma(\lambda; \delta, \tau)$  and  $y(\lambda) = \Gamma(\lambda; \tau, \delta)$  for  $\lambda \in \mathbf{R}$  as in (21). Let  $\Omega_0^-, \Omega_1^-$  and  $\Omega_2^-$  be defined by (18), and the function  $T(\lambda|A, B, C, \sigma)$  be defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively.*

- (i) *If  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G)$ .*

(ii) If  $(A, B) \in \Omega_2^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G_1) \oplus \wedge(G_3)$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r_1]$  and  $G_3$  is the part of the parametric curve  $G$  restricted to the interval  $[r_2, +\infty)$  and  $r_1$  as well as  $r_2$  are the (only) real roots of  $T(\lambda|A, B, C, \sigma)$  arranged in the order  $r_1 < r_2$ .

*Proof.* If  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ , then the function  $T(\lambda|A, B, C, \sigma) \geq 0$  for  $\lambda \in \mathbf{R}$  by Lemma 12. So  $x'(\lambda) < 0$  for  $\lambda \in \mathbf{R} \setminus \Sigma(A, B, C, \sigma)$  by (22). Since  $\Sigma(A, B, C, \sigma)$  is finite, by (28) and (29), the curve  $G$  is the graph of the smooth function  $y = G(x)$  over  $\mathbf{R}$  by Lemma 10. By (22)-(27), we may then see that  $G$  is strictly decreasing and strictly concave. Furthermore,  $-G \sim H_{+\infty}$  by  $G'(+\infty) = -\infty$  and Lemma 2. Assume (b1) holds, we have  $G(-\infty) = +\infty$  and  $G'(-\infty) = 0$ . By Lemma 2,  $-G \sim H_{-\infty}$ . By Lemma 3, the dual set of order 0 of  $G$  is  $\wedge(G)$  (see Figure 11(a)). Assume (b2) holds, we have  $G(-\infty) = 0$  and  $G'(-\infty) = 0$ . Then the asymptote of  $G$  at  $-\infty$  exists. By Theorem 3.19 in [4], the dual set of order 0 of  $G$  is  $\wedge(G)$  (see Figure 12(a)).

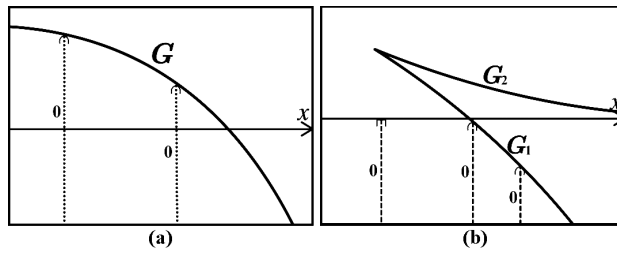


Fig. 11.  $\sigma > \tau > \delta > 0$  and  $b < 0$ .

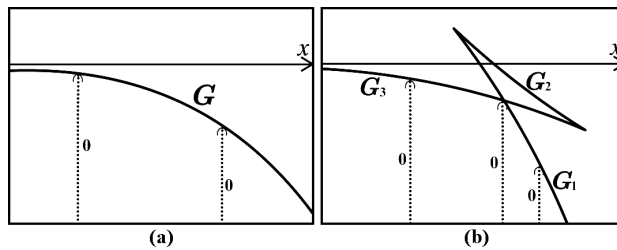


Fig. 12.  $\sigma > \tau > \delta > 0$  and  $b < 0$ .

If  $(A, B) \in \Omega_2^-$ , then there are exactly two real roots  $r_1$  and  $r_2$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) > 0$  on  $\mathbf{R} \setminus [r_1, r_2]$  and  $T(\lambda|A, B, C, \sigma) < 0$  on  $(r_1, r_2)$  by Lemma 12. So  $x'(\lambda) < 0$  on  $\mathbf{R} \setminus [r_1, r_2]$  and  $x'(\lambda) > 0$  on  $(r_1, r_2)$  by (22). The curve  $G$  is composed of three pieces  $G_1$ ,  $G_2$  and  $G_3$  restricted respectively

to  $(-\infty, r_1]$ ,  $(r_1, r_2)$  and  $[r_2, +\infty)$ . By (22)-(27) and Lemmas 2 and 10,  $G_1$  is the graph of a function  $y = G_1(x)$  which is strictly decreasing, strictly concave, and smooth over  $[x(r_1), +\infty)$  such that  $-G_1 \sim H_{+\infty}$ ;  $G_2$  is the graph of a function  $y = G_2(x)$  which is strictly decreasing, strictly convex, and smooth over  $(x(r_1), x(r_2))$ ; and  $G_3$  is the graph of a function  $y = G_3(x)$  which is strictly decreasing, strictly concave, and smooth over  $(-\infty, x(r_2)]$ . Furthermore,

$$G_1^{(v)}(x(r_1)) = G_2^{(v)}(x(r_1)^+) \text{ and } G_2^{(v)}(x(r_2)^-) = G_3^{(v)}(x(r_2)), \quad v = 0, 1.$$

Assume (b1) holds, we have  $-G_3 \sim H_{-\infty}$  by Lemma 2,  $G_3(-\infty) = +\infty$  and  $G_3'(-\infty) = 0$ . By Theorem A.14 in [4], the intersection of the dual sets of order 0 of  $G_1, G_2$  and  $G_3$  is  $\wedge(G_1) \oplus \wedge(G_3)$  (see Figure 11(b)). Assume (b2) holds. Then the asymptote of  $G_3$  at  $-\infty$  exists since  $G_3(-\infty) = 0$  and  $G_3'(-\infty) = 0$ . By Theorem A.16 in [4], the intersection of the dual sets of order 0 of  $G_1, G_2$  and  $G_3$  is  $\wedge(G_1) \oplus \wedge(G_3)$  (see Figure 12(b)). Hence, the dual set of order 0 of  $G$  is  $\wedge(G_1) \oplus \wedge(G_3)$ . The proof is complete.

**Theorem 4.** *Suppose the condition (b3) holds (that is,  $\tau > \delta > \sigma > 0$  and  $b < 0$ ). Let the parametric curve  $G$  be defined by  $x(\lambda) = \Gamma(\lambda; \delta, \tau)$  and  $y(\lambda) = \Gamma(\lambda; \tau, \delta)$  for  $\lambda \in \mathbf{R}$  as in (21). Let  $\Omega_0^-, \Omega_1^-$  and  $\Omega_2^-$  be defined by (18), and the function  $T(\lambda|A, B, C, \sigma)$  be defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively.*

- (i) *If  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G) \oplus \Delta(\Theta_0)$ .*
- (ii) *If  $(A, B) \in \Omega_2^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G_1) \oplus \wedge(G_3) \oplus \Delta(\Theta_0)$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r_1]$  and  $G_3$  is the part of the parametric curve  $G$  restricted to the interval  $[r_2, +\infty)$  and  $r_1$  as well as  $r_2$  are the (only) real roots of  $T(\lambda|A, B, C, \sigma)$  arranged in the order  $r_1 < r_2$ .*

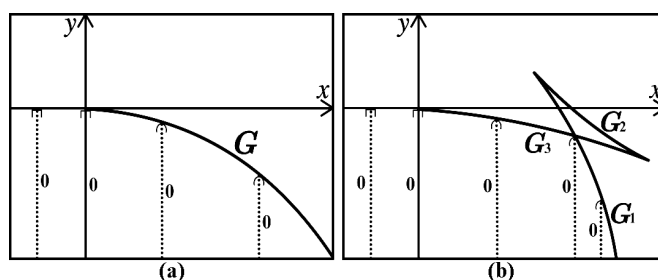


Fig. 13.

*Proof.* If  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ , then the function  $T(\lambda|A, B, C, \sigma) \geq 0$  for  $\lambda \in \mathbf{R}$  by Lemma 12. As in the proof of Theorem 3, we may then see that the curve  $G$  is the graph of the function  $y = G(x)$  which is strictly decreasing, strictly concave and smooth over  $(0, +\infty)$  such that  $-G \sim H_{+\infty}$ . Since  $G'(0^+) = 0$  and  $G(0^+) = 0$ , we see that  $L_{G|0} = \Theta_0$ , where for the definition of  $L_{G|0}$  see (9). By Theorem 3.11 in [4], the dual set of order 0 of  $G$  is  $\wedge(G) \oplus \Delta(\Theta_0)$  (see Figure 13(a)).

If  $(A, B) \in \Omega_2^-$ , then there are exactly two real roots  $r_1$  and  $r_2$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) > 0$  on  $\mathbf{R} \setminus [r_1, r_2]$  and  $T(\lambda|A, B, C, \sigma) < 0$  on  $(r_1, r_2)$  by Lemma 12. The curve  $G$  is composed of three pieces  $G_1$ ,  $G_2$  and  $G_3$  restricted respectively to  $(-\infty, r_1]$ ,  $(r_1, r_2)$  and  $[r_2, +\infty)$ . Similarly,  $G_1$  is the graph of a function  $y = G_1(x)$  which is strictly decreasing, strictly concave, and smooth over  $[x(r_1), +\infty)$  such that  $-G_1 \sim H_{+\infty}$ ;  $G_2$  is the graph of a function  $y = G_2(x)$  which is strictly decreasing, strictly convex, and smooth over  $(x(r_1), x(r_2))$ ; and  $G_3$  is the graph of a function  $y = G_3(x)$  which is strictly decreasing, strictly concave, and smooth over  $(0, x(r_2)]$  such that  $G_3'(0^+) = 0$  and  $G_3(0^+) = 0$ . Furthermore,  $L_{G_3|0} = \Theta_0$ ,

$$G_1^{(v)}(x(r_1)) = G_2^{(v)}(x(r_1)^+) \text{ and } G_2^{(v)}(x(r_2)^-) = G_3^{(v)}(x(r_2)), \quad v = 0, 1.$$

By Theorem A.13 in [4], the intersection of the dual sets of order 0 of  $G_1$ ,  $G_2$  and  $G_3$  is  $\wedge(G_1) \oplus \wedge(G_3) \oplus \Delta(\Theta_0)$ . So the dual set of order 0 of  $G$  is  $\wedge(G_1) \oplus \wedge(G_3) \oplus \Delta(\Theta_0)$  (see Figure 13(b)). The proof is complete.

**Theorem 5.** *Suppose the condition (b4) holds (that is,  $\sigma > \tau > 0 > \delta$  and  $b > 0$ ). Let the parametric curve  $G$  be defined by  $x(\lambda) = \Gamma(\lambda; \delta, \tau)$  and  $y(\lambda) = \Gamma(\lambda; \tau, \delta)$  for  $\lambda \in \mathbf{R}$  as in (21). Let  $\Omega_0^-$ ,  $\Omega_1^-$  and  $\Omega_2^-$  be defined by (18), and the function  $T(\lambda|A, B, C, \sigma)$  be defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively.*

- (i) *If  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \vee(G)$ .*
- (ii) *If  $(A, B) \in \Omega_2^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \vee(G_1) \oplus \vee(G_3 \chi_{(0, +\infty)})$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r_1]$  and  $G_3$  is the part of the parametric curve  $G$  restricted to the interval  $[r_2, +\infty)$  and  $r_1$  as well as  $r_2$  are the (only) real roots of  $T(\lambda|A, B, C, \sigma)$  arranged in the order  $r_1 < r_2$ .*

*Proof.* Suppose  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ . As in the proof of Theorem 3, we may then see that the curve  $G$  is the graph of the function  $y = G(x)$  which is strictly

decreasing, strictly convex and smooth over  $(0, +\infty)$  such that  $G(0^+) = +\infty$  and  $G \sim H_{+\infty}$ . Since  $G'(0^+) = -\infty$  and Lemma 2, we see that  $G \sim H_{0^+}$ . By Theorem 3.15 in [4], the dual set of order 0 of  $G$  is  $\vee(G)$  (see Figure 14(a)).

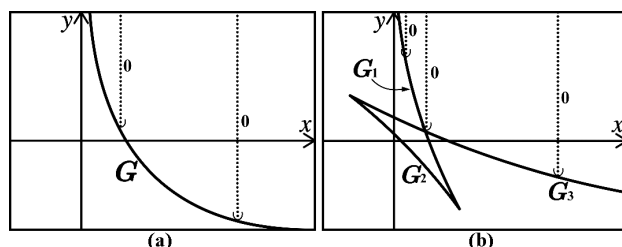


Fig. 14.

If  $(A, B) \in \Omega_2^-$ , then there are exactly two real roots  $r_1$  and  $r_2$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) > 0$  on  $\mathbf{R} \setminus [r_1, r_2]$  and  $T(\lambda|A, B, C, \sigma) < 0$  on  $(r_1, r_2)$  by Lemma 12. Then the curve  $G$  is composed of three pieces  $G_1, G_2$  and  $G_3$  restricted respectively to  $(-\infty, r_1], (r_1, r_2)$  and  $[r_2, +\infty)$ . As in the proof of Theorem 3, we may then see that the curve  $G_1$  is the graph of a function  $y = G_1(x)$  which is strictly decreasing, strictly convex, and smooth over  $[x(r_1), +\infty)$  such that  $G_1(0^+) = +\infty$  and  $G_1 \sim H_{0^+}$ ;  $G_2$  is the graph of a function  $y = G_2(x)$  which is strictly decreasing, strictly concave, and smooth over  $(x(r_2), x(r_1))$ ;  $G_3$  is the graph of a function  $y = G_3(x)$  which is strictly decreasing, strictly concave, and smooth over  $[x(r_2), +\infty)$  such that  $G_3 \sim H_{+\infty}$ . Furthermore,

$$G_1^{(v)}(x(r_1)) = G_2^{(v)}(x(r_1)^-) \text{ and } G_2^{(v)}(x(r_2)^+) = G_3^{(v)}(x(r_2)), \quad v = 0, 1.$$

By Theorem A.15 in [4], the intersection of the dual sets of order 0 of  $G_1, G_2$  and  $G_3$  is  $\vee(G_1) \oplus \vee(G_3 \chi_{(0, +\infty)})$ . So the dual set of order 0 of  $G$  is  $\vee(G_1) \oplus \vee(G_3 \chi_{(0, +\infty)})$  (see Figure 14(b)). The proof is complete.

**Theorem 6.** *Suppose the condition (b5) holds (that is,  $\tau > \sigma > 0 > \delta$  and  $b < 0$ ). Let the parametric curve  $G$  be defined by  $x(\lambda) = \Gamma(\lambda; \delta, \tau)$  and  $y(\lambda) = \Gamma(\lambda; \tau, \delta)$  for  $\lambda \in \mathbf{R}$  as in (21). Let  $\Omega_0^-, \Omega_1^-$  and  $\Omega_2^-$  be defined by (18), and the function  $T(\lambda|A, B, C, \sigma)$  be defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively.*

- (i) *If  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \vee(G) \oplus \Delta(\Theta_0 \chi_{(-\infty, 0]})$ .*
- (ii) *If  $(A, B) \in \Omega_2^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$  characteristic region of (10) if, and only if,  $(x, y) \in \vee(G_1) \oplus \vee(G_3 \chi_{(0, +\infty)})$ , or  $(x, y) \in \wedge(G_2 \chi_{(-\infty, 0]}) \oplus \Delta(\Theta_0 \chi_{(-\infty, 0]})$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r_1]$ ,  $G_2$  is the part of the parametric curve  $G$  restricted to*

the interval  $(r_1, r_2)$  and  $G_3$  is the part of the parametric curve  $G$  restricted to the interval  $[r_2, +\infty)$  and  $r_1$  as well as  $r_2$  are the (only) real roots of  $T(\lambda|A, B, C, \sigma)$  arranged in the order  $r_1 < r_2$ .

*Proof.* Suppose  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ . As in the proof of Theorem 3, we may then see then the curve  $G$  is the graph of the function  $y = G(x)$  which is strictly decreasing, strictly convex and smooth over  $(0, +\infty)$  such that  $G(0^+) = +\infty$ ,  $G(+\infty) = 0$ ,  $G \sim H_{0+}$  and  $G'(+\infty) = 0$ . So the asymptote of  $G$  at  $+\infty$  exists and is just  $\Theta_0$ . By Theorem 3.13 in [4], the dual set of order 0 of  $G$  is  $\vee(G) \oplus \Delta(\Theta_0\chi_{(-\infty, 0]})$  (see Figure 15(a)).

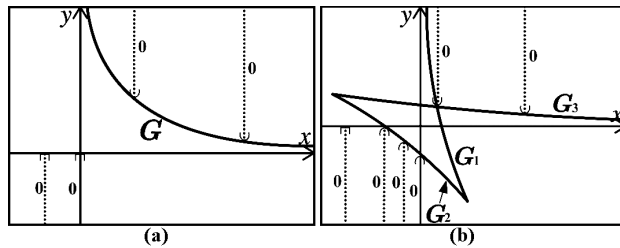


Fig. 15.

If  $(A, B) \in \Omega_2^-$ , then there are exactly two real roots  $r_1$  and  $r_2$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) > 0$  on  $\mathbf{R} \setminus [r_1, r_2]$  and  $T(\lambda|A, B, C, \sigma) < 0$  on  $(r_1, r_2)$  by Lemma 12. Then the curve  $G$  is composed of three pieces  $G_1$ ,  $G_2$  and  $G_3$  restricted respectively to  $(-\infty, r_1]$ ,  $(r_1, r_2)$  and  $[r_2, +\infty)$ . We may further see that the  $G_1$  is the graph of a function  $y = G_1(x)$  which is strictly decreasing, strictly convex, and smooth over  $(0, x(r_1)]$  such that  $G_1(0^+) = +\infty$  and  $G_1 \sim H_{0+}$ ;  $G_2$  is the graph of a function  $y = G_2(x)$  which is strictly decreasing, strictly concave, and smooth over  $(x(r_2), x(r_1))$ ;  $G_3$  is the graph of a function  $y = G_3(x)$  which is strictly decreasing, strictly convex, and smooth over  $[x(r_2), +\infty)$  such that  $G_3'(+\infty) = 0$  and  $G_3(+\infty) = 0$ . Furthermore, the the asymptote of  $G_3$  at  $+\infty$  exists and is just  $\Theta_0$ , and

$$G_1^{(v)}(x(r_1)) = G_2^{(v)}(x(r_1)^-) \text{ and } G_2^{(v)}(x(r_2)^+) = G_3^{(v)}(x(r_2)), \quad v = 0, 1.$$

By Theorem A.17 in [4], the intersection of the dual sets of order 0 of  $G_1, G_2$  and  $G_3$  is

$$(\vee(G_1) \oplus \vee(G_3\chi_{(0, +\infty)})) \cup (\wedge(G_2\chi_{(-\infty, 0]}) \oplus \Delta(\Theta_0\chi_{(-\infty, 0]})).$$

So the dual set of order 0 of  $G$  is the set above (see Figure 15(b)). The proof is complete.

**Theorem 7.** *Suppose the condition (b6) holds (that is,  $\sigma > 0 > \tau > \delta$  and  $b < 0$ ). Let the parametric curve  $G$  be defined by  $x(\lambda) = \Gamma(\lambda; \delta, \tau)$  and  $y(\lambda) = \Gamma(\lambda; \tau, \delta)$  for  $\lambda \in \mathbf{R}$  as in (21). Let  $\Omega_0^-, \Omega_1^-$  and  $\Omega_2^-$  be defined by (18), and the function  $T(\lambda|A, B, C, \sigma)$  be defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively.*

- (i) *If  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G) \oplus \Delta(\Theta_0 \chi_{\{0\}})$ .*
- (ii) *If  $(A, B) \in \Omega_2^-$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G_1) \oplus \wedge(G_3 \chi_{(-\infty, 0]}) \oplus \Delta(\Theta_0 \chi_{\{0\}})$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r_1]$  and  $G_3$  is the part of the parametric curve  $G$  restricted to the interval  $[r_2, +\infty)$  and  $r_1$  as well as  $r_2$  are the (only) real roots of  $T(\lambda|A, B, C, \sigma)$  arranged in the order  $r_1 < r_2$ .*

*Proof.* Suppose  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ . As in the proof of Theorem 3, the curve  $G$  is the graph of the function  $y = G(x)$  which is strictly decreasing, strictly concave and smooth function over  $(-\infty, 0)$  such that  $-G \sim H_{-\infty}$  and  $-G \sim H_{0-}$ . By Theorem 3.14 in [4], the dual set of order 0 of  $G$  is  $\wedge(G) \oplus \Delta(\Theta_0 \chi_{\{0\}})$  (see Figure 16(a)).

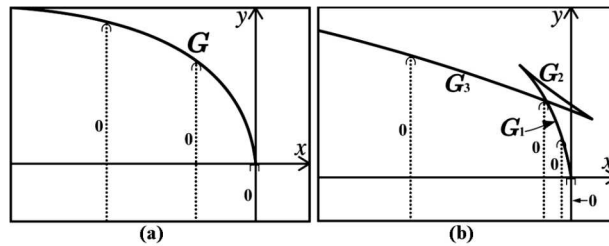


Fig. 16.

If  $(A, B) \in \Omega_2^-$ , then there are exactly two real roots  $r_1$  and  $r_2$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) > 0$  on  $\mathbf{R} \setminus [r_1, r_2]$  and  $T(\lambda|A, B, C, \sigma) < 0$  on  $(r_1, r_2)$  by Lemma 12. The curve  $G$  is composed of three pieces  $G_1, G_2$  and  $G_3$  restricted respectively to  $(-\infty, r_1], (r_1, r_2)$  and  $[r_2, +\infty)$ . We may further see that  $G_1$  is the graph of a function  $y = G_1(x)$  which is strictly decreasing, strictly concave, and smooth over  $[x(r_1), 0)$  such that  $-G_1 \sim H_{0-}$ ;  $G_2$  is the graph of a function  $y = G_2(x)$  which is strictly decreasing, strictly convex, and smooth over  $(x(r_1), x(r_2))$ ;  $G_3$  is the graph of a function  $y = G_3(x)$  which is strictly decreasing, strictly concave, and smooth over  $(-\infty, x(r_2)]$  such that  $-G_3 \sim H_{-\infty}$ . Furthermore,

$$G_1^{(v)}(x(r_1)) = G_2^{(v)}(x(r_1)^+) \text{ and } G_2^{(v)}(x(r_2)^-) = G_3^{(v)}(x(r_2)), \quad v = 0, 1.$$

By Lemma 9, the intersection of the dual sets of order 0 of  $G_1, G_2$  and  $G_3$  is  $\wedge(G_1) \oplus \wedge(G_3\chi_{(-\infty,0]}) \oplus \Delta(\Theta_0\chi_{\{0\}})$ . So the dual set of order 0 of  $G$  is  $\wedge(G_1) \oplus \wedge(G_3\chi_{(-\infty,0]}) \oplus \Delta(\Theta_0\chi_{\{0\}})$  (see Figure 16(b)). The proof is complete.

**4.2. The case where  $C > 0$**

We have assumed that  $b \in \mathbf{R} \setminus \{0\}$  and  $\tau, \delta$  and  $\sigma$  are pairwise distinct real numbers such that  $\tau > \delta$  and  $\sigma > 0$ . Since  $C > 0$ , the real numbers  $\sigma, \tau, \delta$  and  $b$  can be broken exactly into the following six cases:

- (a1)  $\sigma > \tau > \delta > 0$  and  $b > 0$ ;
- (a2)  $\tau > \sigma > \delta > 0$  and  $b < 0$ ;
- (a3)  $\tau > \delta > \sigma > 0$  and  $b > 0$ ;
- (a4)  $\sigma > 0 > \tau > \delta$  and  $b > 0$ ;
- (a5)  $\sigma > \tau > 0 > \delta$  and  $b < 0$ ;
- (a6)  $\tau > \sigma > 0 > \delta$  and  $b > 0$ .

The parametric functions  $x(\lambda)$  and  $y(\lambda)$  defined by (21) have several elementary properties. First,

$$(30) \quad \lim_{\lambda \rightarrow -\infty} (x(\lambda), y(\lambda)) = \begin{cases} (+\infty, -\infty) & \text{if one of (a1), (a2) and (a3) holds} \\ (0, 0) & \text{if (a4) holds} \\ (0, +\infty) & \text{if one of (a5) and (a6) holds} \end{cases},$$

and

$$(31) \quad \lim_{\lambda \rightarrow +\infty} (x(\lambda), y(\lambda)) = \begin{cases} (+\infty, -\infty) & \text{if one of (a1) and (a4) holds} \\ (+\infty, 0) & \text{if (a2) holds} \\ (0, 0) & \text{if (a3) holds} \\ (-\infty, +\infty) & \text{if (a5) holds} \\ (-\infty, 0) & \text{if (a6) holds} \end{cases}.$$

**Theorem 8.** *Suppose the condition (a1) holds (that is,  $\sigma > \tau > \delta > 0$  and  $b > 0$ ). Then the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) is empty.*

*Proof.* Recall that the sets  $\Omega_1^+, \Omega_2^+$  and  $\Omega_3^+$  are defined by (19), and the function  $T(\lambda|A, B, C, \sigma)$  defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively. If  $(A, B) \in \Omega_1^+ \cup \Omega_2^+$ , then by Lemma 13, there



is a real root  $r$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) \leq 0$  on  $(-\infty, r)$  and  $T(\lambda|A, B, C, \sigma) \geq 0$  on  $(r, +\infty)$ . So  $x'(\lambda) < 0$  on  $(-\infty, r) \setminus \Sigma(A, B, C, \sigma)$  and  $x'(\lambda) > 0$  on  $(r, +\infty) \setminus \Sigma(A, B, C, \sigma)$  by (22). The curve  $G$  is composed of two pieces  $G_1$  and  $G_2$  restricted respectively to  $(-\infty, r)$  and  $[r, +\infty)$ . Since  $\Sigma(A, B, C, \sigma)$  is finite, by (30), (31) and Lemma 10, the curve  $G_1$  is the graph of the smooth function  $y = G_1(x)$  over  $(x(r), +\infty)$  and the curve  $G_2$  is the graph of the smooth function  $y = G_2(x)$  over  $[x(r), +\infty)$ . By (22)-(27) and Lemma 2,  $G_1$  is strictly decreasing and strictly concave such that  $-G_1 \sim H_{+\infty}$  and  $G_2$  is strictly decreasing and strictly convex such that  $G_2 \sim H_{+\infty}$ . Furthermore,

$$G_1^{(v)}(x(r)^+) = G_2^{(v)}(x(r)), \quad v = 0, 1.$$

By Lemma 4, the intersection of dual sets of order 0 of  $G_1$  and  $G_2$  is empty. So the dual set of order 0 of  $G$  is empty (see Figure 17).

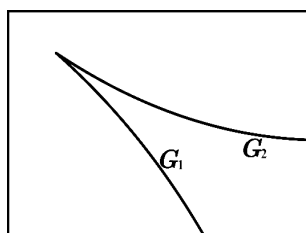


Fig. 17.

If  $(A, B) \in \Omega_3^+$ , then by Lemma 13, there are exactly three real roots  $r_1, r_2$  and  $r_3$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) < 0$  on  $(-\infty, r_1) \cup (r_2, r_3)$  and  $T(\lambda|A, B, C, \sigma) > 0$  on  $(r_1, r_2) \cup (r_3, +\infty)$ . So  $x'(\lambda) < 0$  on  $(-\infty, r_1) \cup (r_2, r_3)$  and  $x'(\lambda) > 0$  on  $(r_1, r_2) \cup (r_3, +\infty)$  by (22). The curve  $G$  is composed of four pieces  $G_1, G_2, G_3$  and  $G_4$  restricted respectively to  $(-\infty, r_1), [r_1, r_2), [r_2, r_3]$  and  $(r_3, +\infty)$ . By (22)-(27), Lemma 2 and Lemma 10, the curve  $G_1$  is the graph of the function  $y = G_1(x)$  which is strictly decreasing, strictly concave and smooth over  $(x(r_1), +\infty)$  such that  $-G_1 \sim H_{+\infty}$ ; the curve  $G_2$  is the graph of the function  $y = G_2(x)$  which is strictly decreasing, strictly convex and smooth over  $[x(r_1), x(r_2))$ ; the curve  $G_3$  is the graph of the function  $y = G_3(x)$  which is strictly decreasing, strictly concave and smooth over  $[x(r_3), x(r_2)]$  and the curve  $G_4$  is the graph of the function  $y = G_4(x)$  which is strictly decreasing, strictly convex and smooth over  $(x(r_3), +\infty)$  such that  $G_4 \sim H_{+\infty}$ . Furthermore,

$$\begin{aligned} G_1^{(v)}(x(r_1)^+) &= G_2^{(v)}(x(r_1)), \quad G_2^{(v)}(x(r_2)^-) \\ &= G_3^{(v)}(x(r_2)) \quad \text{and} \quad G_3^{(v)}(x(r_3)) = G_4^{(v)}(x(r_3)^+), \quad v = 0, 1. \end{aligned}$$

So  $L_{G_3|x(r_3)}(x) = L_{G_4|x(r_3)}(x)$  for  $x \in \mathbf{R}$  where for definitions of  $L_{G_3|x(r_3)}$  and  $L_{G_4|x(r_3)}$  see (8) and (9). By Lemma 8, the intersection of the dual sets of order 0 of  $G_1, G_2$  and  $G_3$  is  $\wedge(G_1) \oplus \wedge(G_3) \oplus \wedge(L_{G_3|x(r_3)})$  (see Figure 18(a)). By Theorem 3.11 in [4], the dual set of order 0 of  $G_4$  is  $\vee(G_4) \oplus \overline{\vee}(L_{G_4|x(r_3)})$  (see Figure 18(b)). The intersection of the two dual sets above, in view of the distribution maps in Figure 18, is empty. Hence, the intersection of dual sets of order 0 of  $G_1, G_2, G_3, G_4$  is empty. The proof is complete.

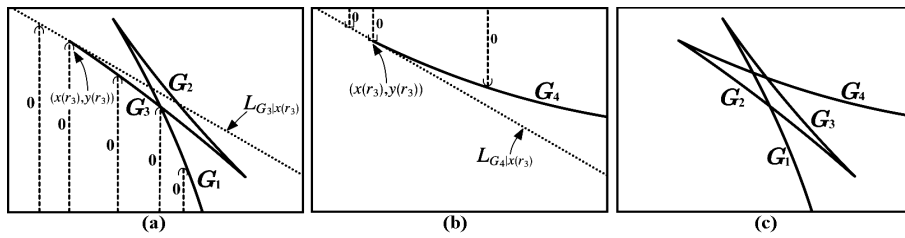


Fig. 18. Intersection of the dual sets of order 0 in (a) and (b) to yield (c).

**Theorem 9.** *Suppose the condition (a2) or (a3) holds (that is,  $\tau > \sigma > \delta > 0$  and  $b < 0$ , or  $\tau > \delta > \sigma > 0$  and  $b > 0$ ). Let the parametric curve  $G$  be defined by  $x(\lambda) = \Gamma(\lambda; \delta, \tau)$  and  $y(\lambda) = \Gamma(\lambda; \tau, \delta)$  for  $\lambda \in \mathbf{R}$  as in (21). Let  $\Omega_1^+, \Omega_2^+$  and  $\Omega_3^+$  be defined by (19), and the function  $T(\lambda|A, B, C, \sigma)$  be defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively.*

- (i) *If  $(A, B) \in \Omega_1^+ \cup \Omega_2^+$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G_1) \oplus \underline{\Delta}(\Theta_0)$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r]$  and  $r$  is the real root of  $T(\lambda|A, B, C, \sigma)$  which is not its extremal point.*
- (ii) *If  $(A, B) \in \Omega_3^+$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G_1) \oplus \wedge(G_3) \oplus \underline{\Delta}(\Theta_0)$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r_1)$  and  $G_3$  is the part of the parametric curve  $G$  restricted to the interval  $[r_2, r_3]$  and  $r_1, r_2$  and  $r_3$  are the (only) real roots of  $T(\lambda|A, C, B, \sigma)$  arranged in the order  $r_1 < r_2 < r_3$ .*

*Proof.* Let

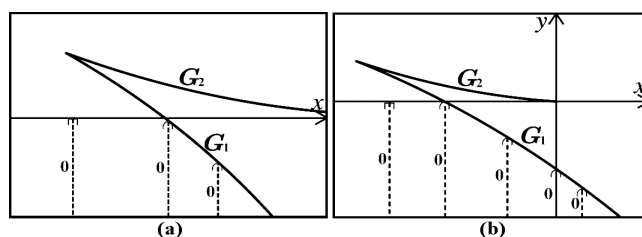
$$\eta = \begin{cases} +\infty & \text{if } \tau > \sigma > \delta > 0 \text{ and } b < 0 \\ 0 & \text{if } \tau > \delta > \sigma > 0 \text{ and } b > 0 \end{cases}.$$

If  $(A, B) \in \Omega_1^+ \cup \Omega_2^+$ , then by Lemma 13, there is a real root  $r$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) \leq 0$  on  $(-\infty, r)$  and  $T(\lambda|A, B, C, \sigma) \geq 0$  on  $(r, +\infty)$ .

So by (22)  $x'(\lambda) < 0$  on  $(-\infty, r) \setminus \Sigma(A, B, C, \sigma)$  and  $x'(\lambda) > 0$  on  $(r, +\infty) \setminus \Sigma(A, B, C, \sigma)$ . The curve  $G$  is composed of two pieces  $G_1$  and  $G_2$  restricted respectively to  $(-\infty, r]$  and  $(r, +\infty)$ . As in the proof of Theorem 8, we may then see that the curve  $G_1$  is the graph of the function  $y = G_1(x)$  which is strictly decreasing, strictly concave and smooth over  $[x(r), +\infty)$  such that  $-G_1 \sim H_{+\infty}$  and the curve  $G_2$  is the graph of the function  $y = G_2(x)$  which is strictly decreasing, strictly convex and smooth over  $(x(r), \eta)$  such that  $G_2'(\eta) = 0$ . Furthermore,

$$G_1^{(v)}(x(r)) = G_2^{(v)}(x(r)^+), \quad v = 0, 1.$$

Assume (a2) holds, we see that  $\Theta_0$  is the asymptote of  $G_2$  at  $+\infty$ . By Theorem A.8 in [4], the intersection of the dual sets of order 0 of  $G_1$  and  $G_2$  is  $\wedge(G_1) \oplus \Delta(\Theta_0)$  (see Figure 19(a)). Assume (a3) holds, we see that  $L_{G_2|_0} = \Theta_0$ . By Lemma 5, the intersection of the dual sets of order 0 of  $G_1$  and  $G_2$  is  $\wedge(G_1) \oplus \Delta(\Theta_0)$  (see Figure 19(b)). Hence, the dual set of order 0 of  $G$  is  $\wedge(G_1) \oplus \Delta(\Theta_0)$



$\tau > \sigma > \delta > 0$  and  $b < 0$        $\tau > \delta > \sigma > 0$  and  $b > 0$

Fig. 19.

If  $(A, B) \in \Omega_3^+$ , by Lemma 13, then there are exactly three real roots  $r_1, r_2$  and  $r_3$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) < 0$  on  $(-\infty, r_1) \cup (r_2, r_3)$  and  $T(\lambda|A, B, C, \sigma) > 0$  on  $(r_1, r_2) \cup (r_3, +\infty)$ . So  $x'(\lambda) < 0$  on  $(-\infty, r_1) \cup (r_2, r_3)$  and  $x'(\lambda) > 0$  on  $(r_1, r_2) \cup (r_3, +\infty)$  by (22). The curve  $G$  is composed of four pieces  $G_1, G_2, G_3$  and  $G_4$  restricted respectively to  $(-\infty, r_1), [r_1, r_2), [r_2, r_3]$  and  $(r_3, +\infty)$ . As in the proof of Theorem 8, we may then see that the curve  $G_1$  is the graph of the function  $y = G_1(x)$  which is strictly decreasing, strictly concave and smooth over  $(x(r_1), +\infty)$  such that  $-G_1 \sim H_{+\infty}$ ; the curve  $G_2$  is the graph of the function  $y = G_2(x)$  which is strictly decreasing, strictly convex and smooth over  $[x(r_1), x(r_2))$ ; the curve  $G_3$  is the graph of the function  $y = G_3(x)$  which is strictly decreasing, strictly concave and smooth over  $[x(r_3), x(r_2)]$  and the curve  $G_4$  is the graph of the function  $y = G_4(x)$  which is strictly decreasing, strictly convex and smooth over  $(x(r_3), \eta)$  such that  $G_4'(\eta) = 0$ . Furthermore,

$$\begin{aligned} G_1^{(v)}(x(r_1)^+) &= G_2^{(v)}(x(r_1)), \quad G_2^{(v)}(x(r_2)^-) \\ &= G_3^{(v)}(x(r_2)) \quad \text{and} \quad G_3^{(v)}(x(r_3)) = G_4^{(v)}(x(r_3)^+), \quad v = 0, 1. \end{aligned}$$

So  $L_{G_3|x(r_3)}(x) = L_{G_4|x(r_3)}(x)$  for  $x \in \mathbf{R}$ . By Lemma 8, the intersection of the dual sets of order 0 of  $G_1, G_2$  and  $G_3$  is  $\wedge(G_1) \oplus \wedge(G_3) \oplus \wedge(L_{G_3|x(r_3)})$  (see Figures 20(a) and 21(a)). Assume (a2) holds, we see that  $\Theta_0$  is the asymptote of  $G_4$  at  $+\infty$ . By Theorem 3.10 in [4], the dual set of order 0 of  $G_4$  is

$$(32) \quad (\vee(G_4) \oplus \bar{\vee}(L_{G_4|x(r_3)})) \cup (\triangle(L_{G_4|x(r_3)}) \oplus \triangle(\Theta_0)).$$

(see Figure 20(b)) The intersection of the two sets  $\wedge(G_1) \oplus \wedge(G_3) \oplus \wedge(L_{G_3|x(r_3)})$  and (32), in view of the distribution maps in Figure 20, is  $\wedge(G_1) \oplus \wedge(G_3) \oplus \triangle(\Theta_0)$ . Assume (a3) holds, we see that  $L_{G_4|0} = \Theta_0$ . By Theorem 3.1 in [4], the dual set of order 0 of  $G_4$  is

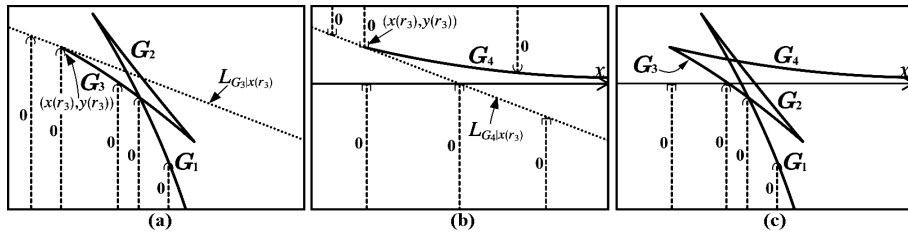


Fig. 20. In case  $\tau > \sigma > \delta > 0$  and  $b < 0$ , intersection of the dual sets of order 0 in (a) and (b) to yield (c).

$$(33) \quad (\vee(G_4) \oplus \bar{\vee}(L_{G_4|x(r_3)}) \oplus \bar{\vee}(\Theta_0)) \cup (\triangle(L_{G_4|x(r_3)}) \oplus \triangle(\Theta_0)).$$

(see Figure 21(b)). The intersection of the two sets  $\wedge(G_1) \oplus \wedge(G_3) \oplus \wedge(L_{G_3|x(r_3)})$  and (33), in view of the distribution maps in Figure 21, is  $\wedge(G_1) \oplus \wedge(G_3) \oplus \triangle(\Theta_0)$ . Hence, the intersection of dual sets of order 0 of  $G_1, G_2, G_3, G_4$  is  $\wedge(G_1) \oplus \wedge(G_3) \oplus \triangle(\Theta_0)$ . The proof is complete.

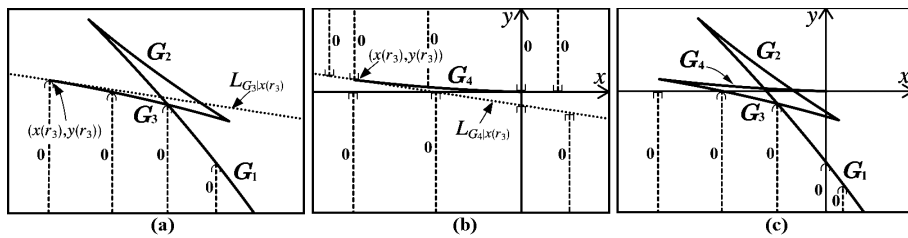


Fig. 21. In case  $\tau > \delta > \sigma > 0$  and  $b > 0$ , intersection of the dual sets of order 0 in (a) and (b) to yield (c).

**Theorem 10.** Suppose the condition (a4) holds (that is,  $\sigma > 0 > \tau > \delta$  and  $b > 0$ ). Let the parametric curve  $G$  be defined by  $x(\lambda) = \Gamma(\lambda; \delta, \tau)$  and

$y(\lambda) = \Gamma(\lambda; \tau, \delta)$  for  $\lambda \in \mathbf{R}$  as in (21). Let  $\Omega_1^+$ ,  $\Omega_2^+$  and  $\Omega_3^+$  be defined by (19), and the function  $T(\lambda|A, B, C, \sigma)$  be defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively.

- (i) If  $(A, B) \in \Omega_1^+ \cup \Omega_2^+$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \vee(G_2\chi_{[0,+\infty)})$ , where  $G_2$  is the part of the parametric curve  $G$  restricted to the interval  $(r, +\infty)$  and  $r$  is the real root of  $T(\lambda|A, B, C, \sigma)$  which is not its extremal point.
- (ii) If  $(A, B) \in \Omega_3^+$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \vee(G_2\chi_{[0,+\infty)}) \oplus \vee(G_4\chi_{[0,+\infty)})$ , where  $G_2$  is the part of the parametric curve  $G$  restricted to the interval  $[r_1, r_2]$  and  $G_4$  is the part of the parametric curve  $G$  restricted to the interval  $(r_3, +\infty)$  and  $r_1, r_2$  and  $r_3$  are the (only) real roots of  $T(\lambda|A, B, C, \sigma)$  arranged in the order  $r_1 < r_2 < r_3$ .

*Proof.* If  $(A, B) \in \Omega_1^+ \cup \Omega_2^+$ , then by Lemma 13, there is a real root  $r$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) \leq 0$  on  $(-\infty, r)$  and  $T(\lambda|A, B, C, \sigma) \geq 0$  on  $(r, +\infty)$ . The curve  $G$  is composed of two pieces  $G_1$  and  $G_2$  restricted respectively to  $(-\infty, r]$  and  $(r, +\infty)$ . As in the proof of Theorem 8, we may then see that the curve  $G_1$  is the graph of the function  $y = G_1(x)$  which is strictly decreasing, strictly concave and smooth function over  $[x(r), 0)$  and the curve  $G_2$  is the graph of the function  $y = G_2(x)$  which is strictly decreasing, strictly convex and smooth function over  $(x(r), +\infty)$  such that  $G_2 \sim H_{+\infty}$ . Furthermore,

$$G_1^{(v)}(x(r)) = G_2^{(v)}(x(r)^+), \quad v = 0, 1.$$

Since  $G_1'(0^-) = -\infty$ , by Lemma 2, we see that  $-G_1 \sim H_{0^-}$ . By Theorem A.1 in [4], the intersection of the dual sets of order 0 of  $G_1$  and  $G_2$  is  $\vee(G_2\chi_{[0,+\infty)})$ . So the dual set of order 0 of  $G$  is  $\vee(G_2\chi_{[0,+\infty)})$  (see Figure 22).

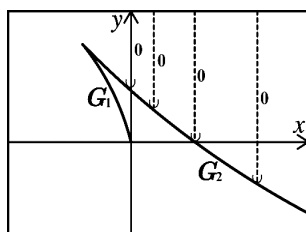


Fig. 22.

If  $(A, B) \in \Omega_3^+$ , then by Lemma 13, there are exactly three real roots  $r_1, r_2$  and  $r_3$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) < 0$  on  $(-\infty, r_1) \cup (r_2, r_3)$

and  $T(\lambda|A, B, C, \sigma) > 0$  on  $(r_1, r_2) \cup (r_3, +\infty)$ . The curve  $G$  is composed of four pieces  $G_1, G_2, G_3$  and  $G_4$  restricted respectively to  $(-\infty, r_1), [r_1, r_2], (r_2, r_3]$  and  $(r_3, +\infty)$ . Similarly, we may then see that the curve  $G_1$  is the graph of the function  $y = G_1(x)$  which is strictly decreasing, strictly concave and smooth function over  $(x(r_1), 0)$  such that  $-G_1 \sim H_{0-}$ ; the curve  $G_2$  is the graph of the function  $y = G_2(x)$  which is strictly decreasing, strictly convex and smooth function over  $[x(r_1), x(r_2)]$ ; the curve  $G_3$  is the graph of the function  $y = G_3(x)$  which is strictly decreasing, strictly concave and smooth function over  $[x(r_3), x(r_2))$  and the curve  $G_4$  is the graph of the function  $y = G_4(x)$  which is strictly decreasing, strictly convex and smooth function over  $(x(r_3), +\infty)$  such that  $G_4 \sim H_{+\infty}$ . Furthermore,

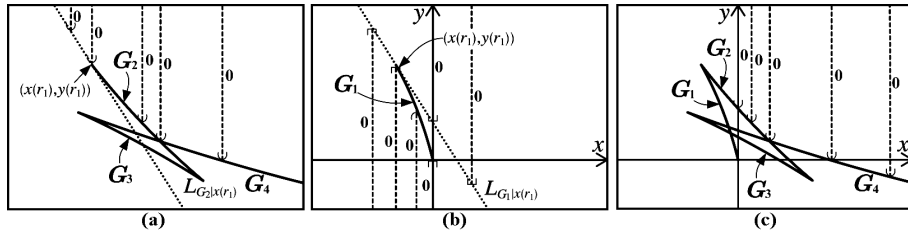


Fig. 23. Intersection of the dual sets of order 0 in (a) and (b) to yield (c).

$$\begin{aligned}
 G_1^{(v)}(x(r_1)^+) &= G_2^{(v)}(x(r_1)), \quad G_2^{(v)}(x(r_2)) \\
 &= G_3^{(v)}(x(r_2)^-) \text{ and } G_3^{(v)}(x(r_3)) = G_4^{(v)}(x(r_3)^+), \quad v = 0, 1.
 \end{aligned}$$

So  $L_{G_1|x(r_1)}(x) = L_{G_2|x(r_1)}(x)$  for  $x \in \mathbf{R}$ . By Lemma 8, then the intersection of the dual sets of order 0 of  $G_2, G_3$  and  $G_4$  is  $\vee(G_2) \oplus \vee(G_4) \oplus \vee(L_{G_2|x(r_1)})$  (see Figure 25(a)). By Theorem 3.5 in [4], then the dual set of order 0 of  $G_1$  is

$$(\wedge(G_1) \oplus \wedge(L_{G_1|x(r_1)}) \oplus \wedge(\Theta_0\chi_{\{0\}})) \cup (\nabla(L_{G_1|x(r_1)}\chi_{[0,+\infty)})) .$$

(see Figure 25(b)). The intersection of the two resultant sets, in veiw of the distribution maps in Figure 25, is  $\vee(G_2\chi_{[0,+\infty)}) \oplus \vee(G_4\chi_{[0,+\infty)})$ . Hence, the intersection of dual sets of order 0 of  $G_1, G_2, G_3, G_4$  is  $\vee(G_2\chi_{[0,+\infty)}) \oplus \vee(G_4\chi_{[0,+\infty)})$ . The proof is complete.

**Theorem 11.** *Suppose the condition (a5) holds (that is,  $\sigma > \tau > 0 > \delta$  and  $b < 0$ ). Let the parametric curve  $G$  be defined by  $x(\lambda) = \Gamma(\lambda; \delta, \tau)$  and  $y(\lambda) = \Gamma(\lambda; \tau, \delta)$  for  $\lambda \in \mathbf{R}$  as in (21). Let  $\Omega_1^+, \Omega_2^+$  and  $\Omega_3^+$  be defined by (19), and the function  $T(\lambda|A, B, C, \sigma)$  be defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively.*

- (i) *If  $(A, B) \in \Omega_1^+ \cup \Omega_2^+$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G_2\chi_{(-\infty, 0]})$ , where  $G_2$  is the part of the*

parametric curve  $G$  restricted to the interval  $(r, +\infty)$  and  $r$  is the real root of  $T(\lambda|A, B, C, \sigma)$  which is not its extremal point.

- (ii) If  $(A, B) \in \Omega_3^+$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \wedge(G_2\chi_{(-\infty, 0]}) \oplus \wedge(G_4\chi_{(-\infty, 0]})$ , where  $G_2$  is the part of the parametric curve  $G$  restricted to the interval  $[r_1, r_2]$  and  $G_4$  is the part of the parametric curve  $G$  restricted to the interval  $(r_3, +\infty)$  and  $r_1, r_2$  and  $r_3$  are the (only) real roots of  $T(\lambda|A, B, C, \sigma)$  arranged in the order  $r_1 < r_2 < r_3$ .

*Proof.* If  $(A, B) \in \Omega_1^+ \cup \Omega_2^+$ , then by Lemma 13, there is a real root  $r$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) \leq 0$  on  $(-\infty, r)$  and  $T(\lambda|A, B, C, \sigma) \geq 0$  on  $(r, +\infty)$ . The curve  $G$  is composed of two pieces  $G_1$  and  $G_2$  restricted respectively to  $(-\infty, r]$  and  $(r, +\infty)$ . As in the proof of Theorem 8, we may then see that the curve  $G_1$  is the graph of the function  $y = G_1(x)$  which is strictly decreasing, strictly convex and smooth over  $(0, x(r)]$  and the curve  $G_2$  is the graph of the function  $y = G_2(x)$  which is strictly decreasing, strictly concave and smooth over  $(-\infty, x(r))$  such that  $-G_2 \sim H_{-\infty}$ . Furthermore,

$$G_1^{(v)}(x(r)) = G_2^{(v)}(x(r)^-), \quad v = 0, 1.$$

Since  $G_1'(0^-) = -\infty$ , by Lemma 2, we see that  $G_1 \sim H_{0^-}$ . By Lemma 7, the intersection of the dual sets of order 0 of  $G_1$  and  $G_2$  is  $\wedge(G_2\chi_{(-\infty, 0]})$ . So the dual set of order 0 of  $G$  is  $\wedge(G_2\chi_{(-\infty, 0]})$  (see Figure 24).

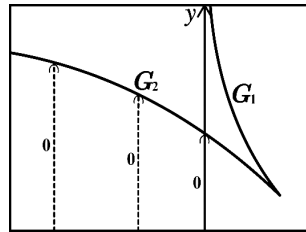


Fig. 24.

If  $(A, B) \in \Omega_3^+$ , then by Lemma 13, there are exactly three real roots  $r_1, r_2$  and  $r_3$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) < 0$  on  $(-\infty, r_1) \cup (r_2, r_3)$  and  $T(\lambda|A, B, C, \sigma) > 0$  on  $(r_1, r_2) \cup (r_3, +\infty)$ . The curve  $G$  is composed of four pieces  $G_1, G_2, G_3$  and  $G_4$  restricted respectively to  $(-\infty, r_1), [r_1, r_2], (r_2, r_3]$  and  $(r_3, +\infty)$ . We may further see that the curve  $G_1$  is the graph of the function  $y = G_1(x)$  which is strictly decreasing, strictly convex and smooth over  $(0, x(r_1))$  such that  $G_1 \sim H_{0^-}$ ; the curve  $G_2$  is the graph of the function  $y = G_2(x)$  which is strictly decreasing, strictly concave and smooth over  $[x(r_2), x(r_1)]$ ; the curve  $G_3$  is

the graph of the function  $y = G_3(x)$  which is strictly decreasing, strictly convex and smooth over  $(x(r_2), x(r_3)]$  and the curve  $G_4$  is the graph of the function  $y = G_4(x)$  which is strictly decreasing, strictly concave and smooth over  $(-\infty, x(r_3))$  such that  $-G_4 \sim H_{-\infty}$ . Furthermore,

$$\begin{aligned} G_1^{(v)}(x(r_1)^-) &= G_2^{(v)}(x(r_1)), G_2^{(v)}(x(r_2)) \\ &= G_3^{(v)}(x(r_2)^+) \text{ and } G_3^{(v)}(x(r_3)) = G_4^{(v)}(x(r_3)^-), v = 0, 1. \end{aligned}$$

So  $L_{G_1|x(r_1)}(x) = L_{G_2|x(r_1)}(x)$  for  $x \in \mathbf{R}$ . By Lemma 8, the intersection of the dual sets of order 0 of  $G_2, G_3$  and  $G_4$  is  $\wedge(G_2) \oplus \wedge(G_4) \oplus \wedge(L_{G_2|x(r_1)})$  (see Figure 25(a)). By Theorem 3.6 in [4], the dual set of order 0 of  $G_1$  is

$$(\vee(G_1) \oplus \nabla(L_{G_1|x(r_1)}\chi_{(0,+\infty)})) \cup (\Delta(L_{G_1|x(r_1)}\chi_{(-\infty,0]})).$$

(see Figure 25(b)). The intersection of the two resultant sets, in view of the distribution maps in Figure 25, is  $\wedge(G_2\chi_{(-\infty,0]}) \oplus \wedge(G_4\chi_{(-\infty,0]})$ . Hence, the intersection of dual sets of order 0 of  $G_1, G_2, G_3, G_4$  is  $\wedge(G_2\chi_{(-\infty,0]}) \oplus \wedge(G_4\chi_{(-\infty,0]})$ . The proof is complete.

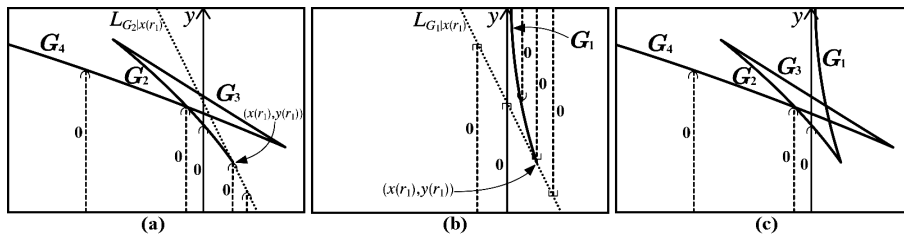


Fig. 25. Intersection of the dual sets of order 0 in (a) and (b) to yield (c).

**Theorem 12.** Suppose the condition (a6) holds (that is,  $\tau > \sigma > 0 > \delta$  and  $b > 0$ ). Let the parametric curve  $G$  be defined by  $x(\lambda) = \Gamma(\lambda; \delta, \tau)$  and  $y(\lambda) = \Gamma(\lambda; \tau, \delta)$  for  $\lambda \in \mathbf{R}$  as in (21). Let  $\Omega_1^+, \Omega_2^+$  and  $\Omega_3^+$  be defined by (19), and the function  $T(\lambda|A, B, C, \sigma)$  be defined by (11) where  $A, B$  and  $C$  are defined by (23), (24) and (25) respectively.

- (i) If  $(A, B) \in \Omega_1^+ \cup \Omega_2^+$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \vee(G_1) \oplus \nabla(\Theta_0\chi_{(0,+\infty)})$ , or  $(x, y) \in \wedge(G_2\chi_{(-\infty,0]})$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r]$  and  $G_2$  is the part of the parametric curve  $G$  restricted to the interval  $(r, +\infty)$  and  $r$  is the real root of  $T(\lambda|A, B, C, \sigma)$  which is not its extremal point.
- (ii) If  $(A, B) \in \Omega_3^+$ , then  $(x, y)$  is a point of the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of (10) if, and only if,  $(x, y) \in \vee(G_1) \oplus \vee(G_3\chi_{(0,+\infty)}) \oplus \nabla(\Theta_0\chi_{(0,+\infty)})$ ,



or  $(x, y) \in \wedge(G_2\chi_{(-\infty,0]}) \oplus \wedge(G_4\chi_{(-\infty,0]})$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r_1]$  and  $G_2$  is the part of the parametric curve  $G$  restricted to the interval  $(r_1, r_2)$  and  $G_3$  is the part of the parametric curve  $G$  restricted to the interval  $[r_2, r_3)$  and  $G_4$  is the part of the parametric curve  $G$  restricted to the interval  $[r_3, +\infty)$  and  $r_1, r_2$  and  $r_3$  are the (only) real roots of  $T(\lambda|A, B, C, \sigma)$  arranged in the order  $r_1 < r_2 < r_3$ .

*Proof.* If  $(A, B) \in \Omega_1^+ \cup \Omega_2^+$ , then by Lemma 13, there is a real root  $r$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) \leq 0$  on  $(-\infty, r)$  and  $T(\lambda|A, B, C, \sigma) \geq 0$  on  $(r, +\infty)$ . The curve  $G$  is composed of two pieces  $G_1$  and  $G_2$  restricted respectively to  $(-\infty, r]$  and  $(r, +\infty)$ . As in the proof of Theorem 8, we may then see that the curve  $G_1$  is the graph of the function  $y = G_1(x)$  which is strictly decreasing, strictly convex and smooth over  $(0, x(r)]$  such that  $G_1 \sim H_{0-}$  and the curve  $G_2$  is the graph of the function  $y = G_2(x)$  which is strictly decreasing, strictly concave and smooth over  $(-\infty, x(r))$  such that  $G_2'(-\infty) = 0$ . and  $G_2(-\infty) = 0$ . Furthermore,

$$G_1^{(v)}(x(r)) = G_2^{(v)}(x(r)^-), \quad v = 0, 1.$$

By Theorem A2 in [4], the intersection of the dual sets of order 0 of  $G_1$  and  $G_2$  is

$$(\vee(G_1) \oplus \bar{\vee}(\Theta_0\chi_{(0,+\infty)})) \cup (\wedge(G_2\chi_{(-\infty,0]})$$

which is the dual set of order 0 of  $G$  (see Figure 26).

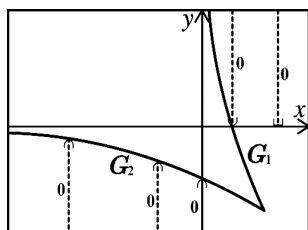


Fig. 26.

If  $(A, B) \in \Omega_3^+$ , then by Lemma 13, there are exactly three real roots  $r_1, r_2$  and  $r_3$  of  $T(\lambda|A, B, C, \sigma)$  such that  $T(\lambda|A, B, C, \sigma) < 0$  on  $(-\infty, r_1) \cup (r_2, r_3)$  and  $T(\lambda|A, B, C, \sigma) > 0$  on  $(r_1, r_2) \cup (r_3, +\infty)$ . The curve  $G$  is composed of four pieces  $G_1, G_2, G_3$  and  $G_4$  restricted respectively to  $(-\infty, r_1), [r_1, r_2), [r_2, r_3]$  and  $(r_3, +\infty)$ . We may further see that the curve  $G_1$  is the graph of the function  $y = G_1(x)$  which is strictly decreasing, strictly convex and smooth over  $(0, x(r_1))$  such that  $G_1 \sim H_{0-}$ ; the curve  $G_2$  is the graph of the function  $y = G_2(x)$  which is

strictly decreasing, strictly concave and smooth over  $(x(r_2), x(r_1))$ ; the curve  $G_3$  is the graph of the function  $y = G_3(x)$  which is strictly decreasing, strictly convex and smooth over  $[x(r_2), x(r_3)]$  and the curve  $G_4$  is the graph of the function  $y = G_4(x)$  which is strictly decreasing, strictly concave and smooth over  $(-\infty, x(r_3))$  such that  $G_4'(-\infty) = 0$ . and  $G_4(-\infty) = 0$ . Furthermore,

$$\begin{aligned} G_1^{(v)}(x(r_1)^-) &= G_2^{(v)}(x(r_1)), G_2^{(v)}(x(r_2)^+) \\ &= G_3^{(v)}(x(r_2)) \text{ and } G_3^{(v)}(x(r_3)) = G_4^{(v)}(x(r_3)^-), v = 0, 1. \end{aligned}$$

So  $L_{G_1|x(r_1)}(x) = L_{G_2|x(r_1)}(x)$  for  $x \in \mathbf{R}$ . By Theorem A.5 in [4], the intersection of the dual sets of order 0 of  $G_1$  and  $G_2$  is

$$(\vee(G_1) \oplus \nabla(L_{G_2|x(r_2)})\chi_{(0,+\infty)}) \cup (\wedge(G_2\chi_{(-\infty,0]}) \oplus \Delta(L_{G_2|x(r_2)}\chi_{(-\infty,0]})) .$$

(see Figure 27(a)). By Lemma 6, the dual set of order 0 of  $G_3$  and  $G_4$  is

$$(\vee(G_3) \oplus \vee(L_{G_3|x(r_2)}) \oplus \nabla(\Theta_0)) \cup (\wedge(G_4) \oplus \wedge(L_{G_3|x(r_2)})) .$$

(see Figure 27(b)). The intersection of the two resultant sets, in view of the distribution maps in Figure 27, is

$$(\vee(G_1) \oplus \vee(G_3\chi_{(0,+\infty)}) \oplus \nabla(\Theta_0\chi_{(0,+\infty)})) \cup (\wedge(G_2\chi_{(-\infty,0]}) \oplus \wedge(G_4\chi_{(-\infty,0]})) .$$

which is the desired dual set of order 0. The proof is complete.

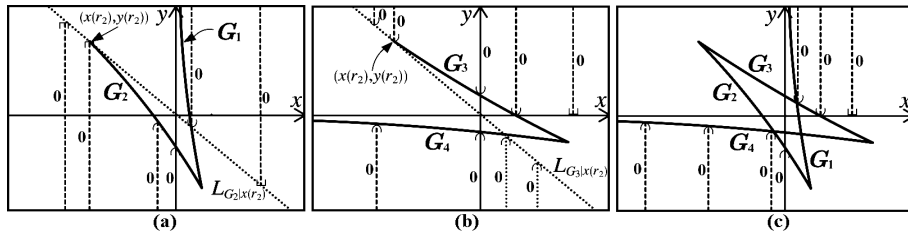


Fig. 27. Intersection of the dual sets of order 0 in (a) and (b) to yield (c).

### 5. EXAMPLES

Although we have given the exact conditions for the absence of real roots of the function  $Q$  defined by (10), these conditions may require further specifications since parametric curves may be involved. We illustrate how these details can be taken care of in several different cases.

**Example 1.** Assume  $1/3 \leq b \leq 7/5$ ,  $p < (1 - 7b)/3$  and  $q < -(1 + 2b)/3$ . Then every solution of the differential equation

$$(34) \quad u'(t) + b(u'(t + 2) + 2u(t + 2) + u(t)) + pu(t + 1) + qu(t + 4) = 0$$

is oscillatory.

*Proof.* The characteristic equation of our differential equation is

$$Q(\lambda) = \lambda + b + (b\lambda + 2b)e^{2\lambda} + pe^\lambda + qe^{4\lambda}, \text{ for } \lambda \in \mathbf{R}.$$

We have  $A = 5/(2b) - 2$ ,  $B = 5/2$  and  $C = -2/b$ . Let the curve  $S$  be defined by

$$\psi(\lambda) = 0.5e^{2\lambda} - \frac{1}{b} + \frac{\lambda}{2} \text{ and } \varphi(\lambda) = -\lambda - 0.5 + \frac{1}{b}e^{-2\lambda},$$

for  $\lambda \in \mathbf{R}$ . Then  $\psi(0) = 0.5 - 1/b$  and  $\varphi(0) = -0.5 + 1/b$ . By the assumption  $1/3 < b < 7/5$ , we see that  $\psi(0) \leq A$  and  $\varphi(0) \leq B$  (see Figure 28(a)). Since  $S$  is strictly decreasing,  $(A, B) \in \Omega_0^- \cup \Omega_1^-$ . By Theorem 3,  $Q(\lambda)$  has no real roots if, and only if,  $(p, q) \in \wedge(G)$  where  $G$  is the curve defined by

$$\begin{aligned} x(\lambda) &= \frac{-1}{3} \left\{ e^{-\lambda}(4\lambda + 4b - 1) + e^\lambda b(2\lambda + 3) \right\} \text{ and } y(\lambda) \\ &= \frac{1}{3} \left\{ e^{-4\lambda}(\lambda + b - 1) - e^{-2\lambda}b(\lambda + 3) \right\}, \end{aligned}$$

for  $\lambda \in \mathbf{R}$ . Note that  $p < (1 - 7b)/3 = x(0)$  and  $q < -(1 + 2b)/3 = y(0)$  (see Figure 28(b)). Since  $G$  is strictly decreasing,  $(p, q) \in \wedge(G)$ . The proof is complete.

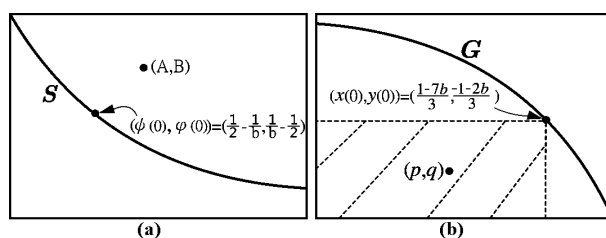


Fig. 28.

**Example 2.** Suppose  $p < -2$ . Then every solution of the differential equation

$$(35) \quad u'(t) + u(t) + (-u'(t+2) + u(t+2)) + pu(t+1) + qu(t+4) = 0$$

is oscillatory if, and only if,  $q \leq 0$ .

*Proof.* The characteristic equation of our differential equation is

$$Q(\lambda) = \lambda + 1 + (1 - \lambda)e^{2\lambda} + pe^\lambda + qe^{4\lambda}, \text{ for } \lambda \in \mathbf{R}.$$

We have  $A = B = -0.5$  and  $C = 2$ . Let the curve  $S$  be defined by

$$\psi(\lambda) = \frac{1}{2}e^{\sigma\lambda} + 1 - 2\lambda, \varphi(\lambda) = -\lambda - \frac{1}{2} - e^{-2\lambda}, \text{ for } \lambda \in \mathbf{R},$$

and  $\lambda^* = 0.5 \ln 2$ . Then  $(\psi(\lambda^*), \varphi(\lambda^*)) = (2 - \ln 2, -1 - 0.5 \ln 2)$ . The curve  $S$  is composed of two pieces  $S_1$  and  $S_2$ .  $S_1$  and  $S_2$  are the graphs of the functions  $y = S_1(x)$  over  $[x(\lambda^*), +\infty)$  and  $y = S_2(x)$  over  $(x(\lambda^*), +\infty)$ . Since  $-0.5 < 0 < 2 - \ln 2$ , the point  $(-0.5, -0.5)$  lies in the set  $\Omega_1^+$  (see Figure 29(a)). Let  $G$  be the parametric curve defined by

$$x(\lambda) = \frac{-1}{3} \left\{ e^{-\lambda}(4\lambda + 3) + e^\lambda(3 - 2\lambda) \right\} \text{ and } y(\lambda) = \frac{1}{3} e^{-4\lambda}(1 + e^{2\lambda})\lambda$$

for  $\lambda \in \mathbf{R}$ . By Theorem 9,  $Q(\lambda)$  has no real roots if, and only if,  $(p, q) \in \Delta(\Theta_0) \oplus \wedge(G_1)$ , where  $G_1$  is the part of the parametric curve  $G$  restricted to the interval  $(-\infty, r]$  and  $r$  is the real roots of  $T(\lambda|A, B, C, \sigma) = \lambda - 0.5 + (2\lambda - 0.5)e^{-2\lambda}$  which is not its extremal point. It is easy to see that the point of intersection of  $G_1$  and  $\Theta_0$  is  $(-2, 0)$ . Since  $G_1$  is strictly decreasing and  $p < -2$ , we see that

$$\begin{aligned} & \{ \Delta(\Theta_0) \oplus \wedge(G_1) \} \cap \{ (x, y) \in \mathbf{R}^2 : x < -2 \text{ and } y \in \mathbf{R} \} \\ &= \{ (x, y) \in \mathbf{R}^2 : x < -2 \text{ and } y \leq 0 \}. \end{aligned}$$

Hence, every solution of the equation (35) is oscillatory if, and only if,  $q \leq 0$  (see Figure 29(b)).

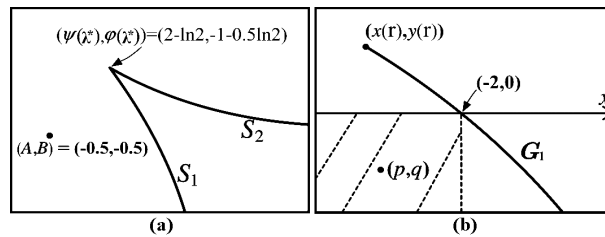


Fig. 29.

**Example 3.** Assume  $p < 0$  and  $q < 0$ . Assume further that one of the following conditions holds:

- (1)  $\sigma > \delta > \tau > 0$  and  $(\delta + \tau)/3 \leq \sigma/2 \leq \delta + \tau$ ;
- (2)  $\tau > \delta > \sigma > 0$ ;
- (3)  $\tau > \sigma > 0 > \delta$ ;
- (4)  $\sigma > 0 > \tau > \delta$ .

Then every solution of

$$(36) \quad u'(t) - u'(t + \sigma) + pu(t + \delta) + qu(t + \tau) = 0$$

is oscillatory.

*Proof.* The characteristic equation of our differential equation is

$$Q(\lambda) = \lambda - \lambda e^{\sigma\lambda} + p e^{\delta\lambda} + q e^{\tau\lambda}, \text{ for } \lambda \in \mathbf{R}.$$

By assumption,  $b = -1 < 0$ ,

$$A = \frac{\delta + \tau}{(\sigma - \delta)(\sigma - \tau)}, \quad B = \frac{(\delta + \tau - 2\sigma)}{(\sigma - \delta)(\tau - \sigma)} \text{ and } C = \frac{-\delta\tau}{(\sigma - \delta)(\sigma - \tau)} < 0.$$

Let the curve  $S$  be defined by

$$\psi(\lambda) = \frac{1}{\sigma} e^{\sigma\lambda} + \frac{C}{\sigma} - C\lambda, \quad \varphi(\lambda) = -\lambda - \frac{1}{\sigma} - \frac{C}{\sigma} e^{-\sigma\lambda}$$

for  $\lambda \in \mathbf{R}$ , and let the curve  $G$  be defined by

$$\begin{aligned} x(\lambda) &= \frac{e^{-\lambda\delta}}{\delta - \tau} \left\{ \tau\lambda - 1 - e^{\lambda\sigma} [(\tau - \sigma)\lambda - 1] \right\} \text{ and } y(\lambda) \\ &= \frac{e^{-\lambda\tau}}{\tau - \delta} \left\{ \delta\lambda - 1 - e^{\lambda\sigma} [(\delta - \sigma)\lambda - 1] \right\}, \end{aligned}$$

for  $\lambda \in \mathbf{R}$ . We have that  $x(0) = y(0) = 0$ , and that  $\psi(0) = (C + 1)/\sigma$  and  $\varphi(0) = -(C + 1)/\sigma$ .

Assume  $\sigma > \delta > \tau > 0$  and  $(\delta + \tau)/3 \leq \sigma/2 \leq \delta + \tau$  hold. Then  $\psi(0) < A$  and  $\varphi(0) < B$  (see Figure 30(a)). Since  $S$  is strictly decreasing, we see that  $(A, B) \in \Omega_0^-$ . By Theorem 3, the  $\mathbf{C} \setminus \mathbf{R}$ -characteristic region of  $Q$  is  $\wedge(G)$ . Since  $G$  is strictly decreasing and passes through the origin  $(0, 0)$ ,  $(p, q) \in \wedge(G)$ . The proof of (i) is complete (see Figure 30(b)).

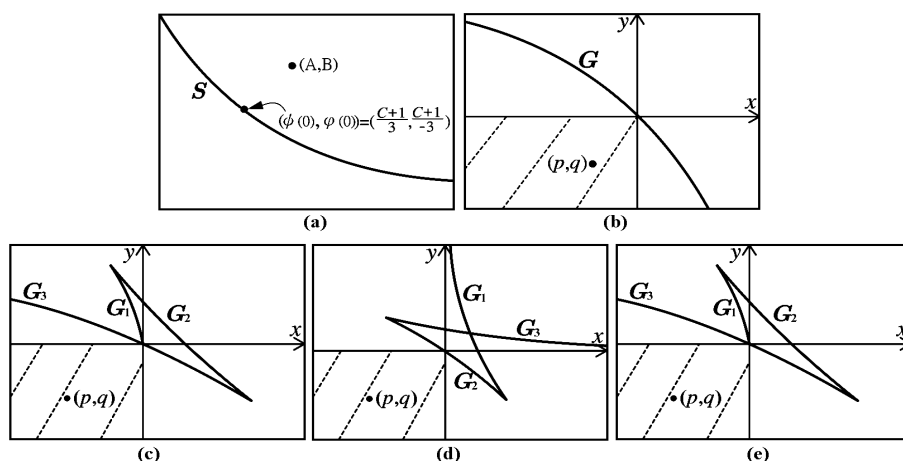


Fig. 30.

The conditions (ii), (iii) and (iv) satisfy (b3), (b5) and (b6) respectively. We observe that the curve  $G$  passes through the origin. Thus  $(A, B) \in \Omega_2^-$ . Otherwise, the curve  $G$  does not pass through the origin by Theorems 4, 6 and 7. By the same Theorems again, in view of Figures 30(c), 30(d) and 30(e), the point  $(p, q)$  lies in the  $\mathbb{C} \setminus \mathbb{R}$ -characteristic region of  $Q$ . Hence, every solution of (36) is oscillatory.

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