TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 1, pp. 381-394, February 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

INTEGRATION WITH ONE-DIMENSIONAL SPACE OF GAUGES

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Abstract. We offer an alternative definition for the H_1-integral of Garces, Lee, and Zhao.

1. Introduction

1.1. H_1-integral

The H_1-integral has been introduced by Garces, Lee, and Zhao [3], in an attempt to define an integral with nearly the Kurzweil–Henstock integral power, but in terms of Moore–Smith limits. Later advances in the theory of H_1-integration have shown that this challenge was not successful in some sense. From the most important results, except a Harnack extension theorem, we saw that H_1-integral is rather far from than close to the Kurzweil–Henstock integral. For example, it integrates neither all derivatives nor all almost everywhere null functions. Moreover, all standard limit passages under the integral sign are not permitted (i.e., the limit function can fail to be integrable), if not under a rather artificial equi-integrability (Vitali-type) assumption (not easy to handle). This makes H_1-integration less convenient in the sense of any potential applications.

However, as it turned out, H_1-integrability of a Kurzweil–Henstock integrable (H-integrable, for short) function depends on some continuity-type property; this fact has been formulated in a characterization theorem, similar to the classical Lebesgue theorem for Riemann integrability. In this connection, one can find the H_1-integral interesting not as being an alternative tool for integration, with some advantages over other integration processes, but as producing the class of H_1-integrable functions with its relations to other classes of functions. A few important results have been received in the latter direction, but, it seems, some investigation opportunities remain, especially those related to characterizations of classes generated by some

Received June 30, 2007, accepted July 25, 2009.
Communicated by Yuh-Jia Lee.
2000 *Mathematics Subject Classification*: 26A39.
Key words and phrases: Kurzweil-Henstock integral, H_1-integral, Stieltjes integral, Variational measure.

arithmetic operations with H_1-integrable functions as one of the arguments (for a sample problem see [12, Question 2.3]).

Since H_1-integration is in fact a gauge integration with respect to some special net of gauges, properties of H_1-integrable functions help to understand better the role of gauges in Riemann-type integration. Actually, the concept of new integration that gave rise to the present work, gives a similar tribute to the theory.

The references contain the whole bibliography related to H_1-integration that the author is aware of [1-4, 7-14, 16].

1.2. Motivation

In our considerations on H_1-integration in its Stieltjes version [13], we sought for the widest possible (i.e., concerning also some integrators of unbounded variation) characterization theorem for integrable functions. Nevertheless, the class of integrators covered by the characterization we gave in [13, Theorem 3.17], is lesser than it was expected at first, as its members have to obey some constraints at discontinuity points. In present paper, instead of modifying the characterizing condition from [13], we make an attempt to provide another Stieltjes integration, for which that condition becomes accurate for the class of integrators originally expected for H_1-integration, namely for all *VBG*$_*$-functions. Reduced to the *non-Stieltjes case* (i.e., with the integrator being id), it results in an integration equivalent to the H_1 one, so giving a nice (since simpler than the original) definition of that integration process.

We shall refer to this new integration concept as just to *new integration* or *our new integration*, thus suggesting that any name is rather auxiliary here, and this new concept shall be referred to as some modification of H_1-integral.

The work is structured as follows. In Section 2 we provide all necessary notions and notation to be used. In Section 3 we prove a Riemann–Lebesgue type theorem: a characterization of newly integrable functions; it is the main result of this paper. Section 4 offers a direct proof that H_1-integrability implies new integrability (in the non-Stieltjes case).

2. Preliminaries

Let $E \subset \mathbb{R}$; by $\operatorname{int} E$, $\operatorname{cl} E$, $\operatorname{Fr} E$, $|E|$ we denote the interior, closure, boundary, and Lebesgue outer measure of E, respectively. If $F\colon E \to \mathbb{R}$ and $A \subset E$ is nonvoid, then we write $\omega_F(A) = \sup F(A) - \inf F(A)$; i.e., $\omega_F(A)$ is the oscillation of F on A. If $c, d \in E$, $I = [c, d]$, we write $\Delta F(I)$ for $F(d) - F(c)$. We say that F is *Baire*1* if for every set $A \subset E$, closed in E, there is a *portion* $I \cap A \neq \varnothing$, I an interval, of A such that $F \restriction (I \cap A)$ is continuous. Recall that for $E = \operatorname{cl} E$, F is Baire*1 iff there exists a sequence $(E_n)_{n=1}^{\infty}$ of closed sets, such that $\bigcup_{n=1}^{\infty} E_n = E$ and for each n, $F \restriction E_n$ is continuous, see [6, Theorem 2.3].

By a *tagged interval* we mean a pair (I, x), where $I = [c, d] \subset \mathbb{R}$, $x \in \mathbb{R}$. Let $\mathcal{P}$ be a finite collection of tagged intervals. We write $U_{\mathcal{P}}$ for $\bigcup_{(I,x)\in\mathcal{P}} I$. Having a function $\delta\colon \mathbb{R} \to (0, \infty)$, called a *gauge*, we say that $\mathcal{P}$ is *δ-fine* if for each $(I, x) \in \mathcal{P}$ we have $I \subset (x - \delta(x), x + \delta(x))$. Let $f, G\colon [a, b] \to \mathbb{R}$. Let $I \subset [a, b]$, $x \in [a, b]$ for each $(I, x) \in \mathcal{P}$. We write

$$\sigma_G(\mathcal{P}, f) = \sum_{(I,x)\in\mathcal{P}} f(x) \cdot \Delta G(I), \quad |\sigma_G|(\mathcal{P}, f) = \sum_{(I,x)\in\mathcal{P}} |f(x) \cdot \Delta G(I)|.$$

Also, $\sigma(\mathcal{P}, f) = \sigma_{\text{id}}(\mathcal{P}, f)$, where $\text{id}\,(x) = x$, $\Delta G(\mathcal{P}) = \sigma_G(\mathcal{P}, 1)$, $|\Delta|G(\mathcal{P}) = |\sigma_G|(\mathcal{P}, 1)$.

By a *division in* a set E we mean a finite collection of tagged intervals (I, x), where $x \in I \subset E$ and intervals I are pairwise nonoverlapping. If for all $(I, x) \in \mathcal{P}$ we have $x \in E$, then we say that a division $\mathcal{P}$ is *anchored* in E. A division $\mathcal{P}$ in $[a, b]$ is called a *partition of* $[a, b]$ if $U_{\mathcal{P}} = [a, b]$. For two divisions $\mathcal{P}_1$ and $\mathcal{P}_2$ we will write $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ if for every $(I, x) \in \mathcal{P}_1$ there is a $(J, y) \in \mathcal{P}_2$ with $I \subset J$.

By $|E|_G$ we mean the *variational measure of* $E \subset [a, b]$ induced by $G\colon [a, b] \to \mathbb{R}$, see [15, Chapter 3]; i.e.,

$$|E|_G = \inf_{\delta} \sup_{\mathcal{P}} |\Delta|G(\mathcal{P}),$$

where $\sup$ is taken over all δ-fine divisions $\mathcal{P}$ in $[a, b]$, anchored in E, while $\inf$ ranges over all gauges δ. The family $\mathcal{I}_G$ of subsets of $[a, b]$ is defined as follows

$$E \in \mathcal{I}_G \quad \text{if} \quad \text{there exists an } A \in \mathcal{F}_\sigma,\ |A|_G = 0,\ A \supset E;$$

$\mathcal{I} = \mathcal{I}_{\text{id}}$. We will write that a condition holds *G-almost everywhere* if the exceptional set E has $|E|_G = 0$.

For classical notions of *AC*$_*$-, *VB*$_*$-, and *VBG*$_*$-functions, with their basic properties, let us refer the reader to [5, Chapter 6].

Definition 1. We call a function $f\colon [a, b] \to \mathbb{R}$, *H-integrable* with respect to (w.r.t., for short) $G\colon [a, b] \to \mathbb{R}$, with the integral $\mathbf{I} \in \mathbb{R}$, if for each $\varepsilon > 0$ there exists a gauge δ, such that for every δ-fine partition π of $[a, b]$,

$$|\sigma_G(\pi, f) - \mathbf{I}| < \varepsilon. \tag{1}$$

Definition 2. We call a function $f\colon [a, b] \to \mathbb{R}$, *$H_1$-integrable* w.r.t. $G\colon [a, b] \to \mathbb{R}$, with the integral $\mathbf{I} \in \mathbb{R}$, if there exists a gauge δ such that for each $\varepsilon > 0$ there is a partition π_1 of $[a, b]$, such that for every δ-fine partition $\pi \sqsupseteq \pi_1$ of $[a, b]$ one has (1).

Definition 3. We call a function $f\colon [a, b] \to \mathbb{R}$ *newly integrable* w.r.t. $G\colon [a, b] \to \mathbb{R}$, with the integral $\mathbf{I} \in \mathbb{R}$, if there exists a gauge δ such that for each $\varepsilon > 0$ there is a constant $c > 0$, such that for every $c\delta$-fine partition π of $[a, b]$ one has (1).

For both Definitions 2 and 3 we will say f is integrable *using* δ. The number **I** will be denoted with $\int_a^b f\,\mathrm{d}G$ or $\int_a^b f$. Evidently,

Corollary 1. *If an* $f\colon [a,b]\to\mathbb{R}$ *is* H_1*- or newly integrable w.r.t.* G*, then it is H-integrable w.r.t.* G.

Theorem 1. [13, Theorem 3.17]. *Let* $f, G\colon [a,b]\to\mathbb{R}$*. Consider the following two assertions:* (i) f *is H-integrable w.r.t.* G *and*

(2) *for some* $B\in\mathfrak{I}_G$ *the restriction* $f\restriction([a,b]\setminus B)$ *is Baire*1 in its domain.*

(ii) f *is* H_1*-integrable w.r.t.* G.
If $G\in VBG_*$*, one has* (i)⇒(ii)*. The converse holds if* G *is normalized.*

Recall that $G\colon [a,b]\to\mathbb{R}$ is said to be *normalized* if $2G(x)=G(x+)+G(x-)$ at each $x\in(a,b)$.

3. Riemann-Lebesgue Theorem for Our New Integral

The general line of reasoning in this section, actually follows the pattern of [13]. From now on, $f, G\colon [a,b]\to\mathbb{R}$ unless otherwise stated.

Lemma 1. (Cauchy extension). *Suppose that* f *is H-integrable w.r.t.* G *on* $[a,b]$*, and newly integrable w.r.t.* G *on every* $[c,d]\subset(a,b)$*. Then, it is newly integrable w.r.t.* G *on* $[a,b]$.

Proof. Routine argument. ∎

The next lemma is a corollary from [15, Theorem 43.1].

Lemma 2. *For each* $E\subset[a,b]$ *one has* $|G(E)|\leqslant|E|_G$.

Lemma 3. [13, Lemma 3.6]. *Let a set* $D\subset[a,b]$ *be closed. Suppose that* G *is* AC_* *on* D *and* $|D|_G=0$*. Then, for each* $\varepsilon>0$ *there are closed intervals* $I_1,\dots,I_n$, $\bigcup_{i=1}^n I_i\supset D$*, such that for every division* $\mathcal{P}$ *in* $\bigcup_{i=1}^n I_i$*, anchored in* D*, one has* $|\Delta|G(\mathcal{P})<\varepsilon$.

Lemma 4. (Harnack extension). *Suppose that a set* $D\subset[a,b]$ *is perfect and*
(i) f *is H-integrable w.r.t.* G *on* $[a,b]$,
(ii) f *is newly integrable w.r.t.* G *on every* $[c,d]\subset[a,b]\setminus D$,
(iii) G *is* VB_* *on* D,
(iv) F *is* VB_* *on* D*, where* F *is the indefinite integral of* f*; i.e.,*

$$F(x)=\int_a^x f\,\mathrm{d}G,\quad x\in[a,b],$$

(v) *$f \restriction D$ is bounded and G-almost everywhere continuous on D.*

Then, f is newly integrable w.r.t. G on $[a, b]$.

Proof. Let $I_1, I_2, \ldots$ be closed intervals contiguous to D in $[a, b]$. Define a gauge δ on $[a, b]$ so that $(x - \delta(x), x + \delta(x)) \subset I_i$ if $x \in \operatorname{int} I_i$, and so that f is newly integrable on each I_i using δ (Lemma 1). We may assume that $\delta < 1$ and that for every δ-fine division $\mathcal{P}$ in I_i the inequality

$$|\sigma_G(\mathcal{P}, f) - \Delta F(\mathcal{P})| < \frac{1}{2^i}. \tag{3}$$

holds.

Take an arbitrary $\varepsilon > 0$. Consider the set

$$E_\varepsilon = \{x \in D : \omega(x) \geqslant \varepsilon\};$$

$\omega(x)$ is the oscillation of $f \restriction D$ at x; i.e., $\omega(x) = \inf_{r>0} \omega_f\big((x - r, x + r) \cap D\big)$. The set $E_\varepsilon \subset D$ is closed. Since by (v), $|E_\varepsilon|_G = 0$, the integrator G is continuous at each point of E_ε, and it satisfies the condition $\mathcal{N}$ on E_ε (Lemma 2). By (iii), from Banach–Zarecki lemma [5, Theorem 6.16] we get G is AC_* on E_ε. In virtue of Lemma 3 there are closed intervals J_j, $j = 1, \ldots, m$, $\bigcup_{j=1}^m J_j \supset E_\varepsilon$, such that for each division $\mathcal{P}$ in $\bigcup_{j=1}^m J_j$, anchored in E_ε, we have $|\Delta|G(\mathcal{P}) < \varepsilon$. Since G is VB_* on D, we can remove from each J_j finitely many open intervals, missing E_ε but not necessarily D, so that (after labelling the so obtained closed intervals again as J_j, $j = 1, \ldots, m$) $|\Delta|G(\mathcal{P}) < 2\varepsilon$ will hold for each division $\mathcal{P}$ in $\bigcup_{j=1}^m J_j$, which is anchored in D. It is not hard to understand that we can assume D to miss the boundary of $\bigcup_{j=1}^m J_j$; i.e., that $\tilde{D} = D \cap \bigcup_{j=1}^m J_j \subset \operatorname{int} \bigcup_{j=1}^m J_j = O$. Put $c_1 = \operatorname{dist}(\tilde{D}, [a, b] \setminus O) > 0$.

Since $\omega(x) < \varepsilon$ at each $x \in D \setminus O$, there exists a number $c_2 > 0$ such that

$$\omega_f\big((x - c_2, x + c_2) \cap D\big) < 2\varepsilon$$

for all $x \in D \setminus O$.

As both F and G are VB_* on D, there is an N with

$$\sum_{i=N+1}^{\infty} \left(\omega_G(I_i) + \omega_F(I_i) + \frac{1}{2^i}\right) < \varepsilon. \tag{4}$$

For each $i \leqslant N$ choose c_i' so that for all $c_i'\delta$-fine divisions $\mathcal{P}$ in I_i,

$$|\sigma_G(\mathcal{P}, f) - \Delta F(\mathcal{P})| < \frac{\varepsilon}{N} \tag{5}$$

(Saks–Henstock lemma for our new integral). Put $c_3 = \min\{c_1', \ldots, c_N'\}$.

Denote $I_i = [a_i, b_i]$. There is an $[a_i', b_i'] \subset (a_i, b_i)$ such that

$$\begin{aligned} \Big|\Delta F(I_i) - (F(\beta) - F(\alpha)) \\ - f(a_i)(G(\alpha) - G(a_i)) - f(b_i)(G(b_i) - G(\beta))\Big| < \frac{\varepsilon}{N}, \end{aligned} \tag{6}$$

whenever $a_i < \alpha \leqslant a_i'$, $b_i' \leqslant \beta < b_i$. Put

$$c_4 = \min_{1 \leqslant i \leqslant N} \left\{a_i' - a_i, b_i - b_i'\right\}.$$

By (v), for each $z \in D$ either $f \restriction D$ or G are continuous at z. Hence, since $f \restriction D$ is bounded and G is *VB*$_*$ on D, there is a $c_5 > 0$ such that

$$\omega_f\big((z - c_5, z + c_5) \cap D\big) \cdot \omega_G\big((z - c_5, z + c_5)\big) < \frac{\varepsilon}{N}$$

for each $z \in \{a_1, b_1, \ldots, a_N, b_N\}$.

Now, consider two arbitrary $c\delta$-fine partitions π_1, π_2 of $[a, b]$, where

$$c = \min\{c_1, c_2, c_3, c_4, c_5, 1\}.$$

We assume that all tags in π_1, π_2 are endpoints of the intervals attached.

Let $\mathcal{P}_s^i = \{(I, x) \in \pi_s : \ I \subset I_i\}$, $s = 1, 2$, $i = 1, 2, \ldots$. Each $\mathcal{P}_s^i \neq \varnothing$ is a partition of a subinterval of I_i. Since π_s is c_4-fine,

$$\mathcal{P}_s^i \neq \varnothing \quad \text{if} \quad i \leqslant N. \tag{7}$$

Now, divisions $\mathcal{Q}_s = \pi_s \setminus \bigcup_{i=1}^N \mathcal{P}_s^i$, $s = 1, 2$, are to be replaced by some collections $\mathcal{R}_s$ and $\tilde{\mathcal{R}}_s$ according to the recipe that follows. Denote $\tilde{\mathcal{Q}}_s = \{(I, x) \in \mathcal{Q}_s : \ D \cap \operatorname{int} I \neq \varnothing\}$. Include each pair $(I, x) \in \mathcal{Q}_s \setminus \tilde{\mathcal{Q}}_s$ into $\mathcal{R}_s$. Let $(I, x) \in \tilde{\mathcal{Q}}_s$. Notice that $x \in D$. If $\operatorname{Fr} I \subset D$; i.e., if both endpoints of I are in D, include (I, x) into an auxiliary division $\mathcal{O}_s$. In the opposite case, one of the endpoints of $I = [a', b']$, say the left one, belongs to some (a_i, b_i). Then, if $i > N$ include the pair $([a', b_i], b_i)$ into $\mathcal{R}_s$, while if $i \leqslant N$ put it into $\tilde{\mathcal{R}}_s$; let the pair $([b_i, b' = x], b' = x)$ go to $\mathcal{O}_s$. Similarly for the right endpoint situation. For all $(I, x) \in \mathcal{O}_s$ we have $\operatorname{Fr} I \subset D$. Include the collections

$$\big\{(I \cap J, x) : (I, x) \in \mathcal{O}_1,\ (J, y) \in \mathcal{O}_2\big\} \ \text{and} \ \big\{(I \cap J, y) : (I, x) \in \mathcal{O}_1,\ (J, y) \in \mathcal{O}_2\big\},$$

where only nondegenerate intersections $I \cap J$ are considered, into $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. Since D is perfect, $U_{\mathcal{O}_s} \supset D$, $s = 1, 2$. Hence, for the closure J of any compound interval of the set

$$U_{\mathcal{O}_s} \setminus U_{\mathcal{O}_{3-s}},$$

$\operatorname{int} J$ must miss D. Hence $J = I_l = [a_l, b_l]$ for some l. By (7), $l > N$. Choose any $p \in (a_l, b_l)$ and include into $\mathcal{R}_s$ tagged intervals $([a_l, p], a_l)$ and $([p, b_l], b_l)$. We accomplished the construction of $\mathcal{R}_s$ and $\tilde{\mathcal{R}}_s$. Clearly, $\mathcal{R}_s$ need not be a division. It is seen that

$$U_{\mathcal{Q}_s} = U_{\mathcal{R}_s} \cup U_{\tilde{\mathcal{R}}_s}$$

and

$$\{I:\ (I,x) \in \mathcal{S}_1\} = \{I:\ (I,x) \in \mathcal{S}_2\}, \tag{8}$$

where $\mathcal{S}_s = \{(I,x) \in \mathcal{R}_s :\ \operatorname{Fr} I \subset D\}$. Moreover, if $(I,x) \in \mathcal{R}_s \setminus \mathcal{S}_s$ then

$$D \cap \operatorname{int} I = \varnothing. \tag{9}$$

Recall that in the passage from $\mathcal{Q}_s$ to $\mathcal{R}_s$&$\tilde{\mathcal{R}}_s$, each I, $(I,x) \in \mathcal{Q}_s$, has been split into a number of segments. Some segments of the kind $[a_i, z]$ or $[z, b_i]$, $a_i < z < b_i$, $i = 1, 2, \ldots$, in place of $x \in D$ have been given new tags: a_i or b_i. Let us estimate how these changes contribute to $\sigma_G(\mathcal{Q}_s, f)$. Tags' change for all (at most two) segments within I_i costs at most $4M\omega_G(I_i)$, where M is an upper bound of $|f| \restriction D$. However, for $i \leqslant N$, since $\mathcal{Q}_s$ is c_5-fine, this cost does not exceed $2N \cdot \frac{\varepsilon}{N} = 2\varepsilon$ in total ($i = 1, \ldots, N$). Summarizing,

$$\sigma_G(\mathcal{Q}_s, f) = r_s + \sigma_G(\tilde{\mathcal{R}}_s, f) + \sigma_G(\mathcal{R}_s, f), \tag{10}$$

where

$$|r_s| \leqslant 4M \sum_{i=N+1}^{\infty} \omega_G(I_i) + 2\varepsilon \overset{(4)}{<} (4M+2)\varepsilon. \tag{11}$$

Now, we are to make use of (8). Denote $\mathcal{T}_s = \{(I,x) \in \mathcal{S}_s :\ I \subset O\}$. For each $(I,x) \in \mathcal{T}_s$ we have $I \cap D \neq \varnothing$, whence by the definition of O, $|\Delta| G(\mathcal{T}_s) < \varepsilon$ and so ($x \in D$)

$$|\sigma_G(\mathcal{T}_s, f)| < M\varepsilon. \tag{12}$$

Observe that if $(I,x) \in \mathcal{S}_s$, x is the tag I 'had' in $\mathcal{Q}_s$. Therefore $\mathcal{S}_s$ is c_1-fine, whence if $(I,x) \in \mathcal{S}_s$ and $I \not\subset O$, then $x \in D \setminus O$. In consequence ($\mathcal{S}_s$ is c_2-fine too), $\omega_f(I \cap D) < 2\varepsilon$. So,

$$|\sigma_G(\mathcal{S}_1 \setminus \mathcal{T}_1, f) - \sigma_G(\mathcal{S}_2 \setminus \mathcal{T}_2, f)| < 2W\varepsilon, \tag{13}$$

where the number W comes from the *VB*$_*$ property of G on D (we use (8): the intervals in $\mathcal{S}_1$ and $\mathcal{S}_2$, thus so in $\mathcal{S}_1 \setminus \mathcal{T}_1$ and $\mathcal{S}_2 \setminus \mathcal{T}_2$ are the same).

Let $\mathcal{U}_s^i = \{(I,x) \in \mathcal{R}_s \setminus \mathcal{S}_s : I \subset I_i,\ x \in D\}$, $\mathcal{V}_s^i = \{(I,x) \in \mathcal{R}_s \setminus \mathcal{S}_s : I \subset I_i,\ x \notin D\}$, $i > N$. By (9),

$$\mathcal{R}_s \setminus \mathcal{S}_s = \bigcup_{i=N+1}^{\infty} (\mathcal{U}_s^i \cup \mathcal{V}_s^i).$$

Notice that $\mathcal{V}_s^i \subset \mathcal{P}_s^i$ and so every $\mathcal{V}_s^i \neq \varnothing$ is a δ-fine partition of some subinterval of I_i. Thus, by (3) and (4),

$$\begin{aligned} &|\sigma_G(\mathcal{R}_s \setminus \mathcal{S}_s, f)| \\ &\quad\leqslant \sum_{i=N+1}^{\infty} \big(|\sigma_G(\mathcal{U}_s^i, f)| + |\sigma_G(\mathcal{V}_s^i, f) - \Delta F(\mathcal{V}_s^i)| + |\Delta F(\mathcal{V}_s^i)|\big) \\ &\quad< \sum_{i=N+1}^{\infty} \left(2M\omega_G(I_i) + \frac{1}{2^i} + \omega_F(I_i)\right) < 2M\varepsilon. \end{aligned} \tag{14}$$

For $i \leqslant N$ let $\tilde{\mathcal{R}}_s^i = \{(I,x) \in \tilde{\mathcal{R}}_s : I \subset I_i\}$. Notice that each $\tilde{\mathcal{R}}_s^i$ has at most two members, $\tilde{\mathcal{R}}_s = \bigcup_{i=1}^N \tilde{\mathcal{R}}_s^i$, and

$$\mathcal{P}_s^i \cup \tilde{\mathcal{R}}_s^i$$

is a partition of I_i. Moreover, $\mathcal{P}_s^i$ is a $c_i'\delta$-fine partition of an $[\alpha, \beta] \supset [a_i', b_i']$. As a_i and b_i are tags in $\mathcal{P}_s^i \cup \tilde{\mathcal{R}}_s^i$, from (5) and (6) we get

$$|\sigma_G(\mathcal{P}_s^i \cup \tilde{\mathcal{R}}_s^i, f) - \Delta F(I_i)| < 2\frac{\varepsilon}{N}. \tag{15}$$

Summing these estimates up, we obtain

$$\begin{aligned} &|\sigma_G(\pi_1, f) - \sigma_G(\pi_2, f)| \\ &\quad= \left|\sigma_G(\mathcal{Q}_1, f) - \sigma_G(\mathcal{Q}_2, f) + \sum_{i=1}^{N}\big(\sigma_G(\mathcal{P}_1^i, f) - \sigma_G(\mathcal{P}_2^i, f)\big)\right| \\ &\quad\overset{(10)}{\leqslant} \sum_{s=1}^{2}\left(|r_s| + \sum_{i=1}^{N}\big(|\sigma_G(\mathcal{P}_s^i \cup \tilde{\mathcal{R}}_s^i, f) - \Delta F(I_i)|\big)\right) + |\sigma_G(\mathcal{R}_1, f) - \sigma_G(\mathcal{R}_2, f)| \\ &\quad\overset{(11),(15)}{\leqslant} (8M+4)\varepsilon + 4\varepsilon + \sum_{s=1}^{2}\big(|\sigma_G(\mathcal{R}_s \setminus \mathcal{S}_s, f)| + |\sigma_G(\mathcal{T}_s, f)|\big) + \\ &\qquad + |\sigma_G(\mathcal{S}_1 \setminus \mathcal{T}_1, f) - \sigma_G(\mathcal{S}_2 \setminus \mathcal{T}_2, f)| \\ &\quad\overset{(14),(12),(13)}{\leqslant} (8M+4)\varepsilon + 4\varepsilon + 4M\varepsilon + 2M\varepsilon + 2W\varepsilon = (14M + 2W + 8)\varepsilon. \end{aligned}$$

Thus, f fulfils the Cauchy Criterion for new integrability. ∎

Lemma 5. [13, Lemma 3.12]. *Let $G \in$ VBG$_*$. Suppose that a function f is H-integrable w.r.t. G. Then, the indefinite integral F of f has the VBG$_*$ property as well.*

Remark 1. Let $E \subset \mathbb{R}$ and assume that $f\colon E \to \mathbb{R}$ is bounded and continuous. Define

$$g(x) = \begin{cases} f(x) & \text{if } x \in E, \\ \liminf\limits_{t \to x,\, t \in E} f(t) & \text{if } x \in \operatorname{cl} E \setminus E. \end{cases}$$

Then $g\colon \operatorname{cl} E \to \mathbb{R}$ is bounded and g is continuous at each $x \in E$.

Lemma 6. *Let G be VBG$_*$ and $E \in \mathfrak{I}_G$. Then, every function f is newly integrable w.r.t. G on E.*

Proof. Let $E = \bigcup_{n=1}^{\infty} E_n$, where G is *VB*$_*$ on E_n and $|\operatorname{cl} E_n|_G = 0$, $n = 1, 2, \ldots$. We may assume that the sets E_n are pairwise disjoint and each restriction $f \restriction E_n$ is bounded. G is continuous at each point of $\operatorname{cl} E_n$ and it satisfies $\mathcal{N}$ on $\operatorname{cl} E_n$ (Lemma 2). So (Banach–Zarecki lemma [5, Theorem 6.16]), G is *AC*$_*$ on $\operatorname{cl} E_n$. Fix n and $\varepsilon > 0$. By Lemma 3, we can find intervals $I_1, \ldots, I_k$ covering $\operatorname{cl} E_n$, such that for all divisions $\mathcal{P}$ in $\bigcup_{i=1}^{k} I_i$, anchored in $\operatorname{cl} E_n$, one has $|\Delta| G(\mathcal{P}) < \varepsilon / M$, where $M > 0$ is an upper bound of $|f| \restriction E_n$. Obviously, we may assume that $O = \operatorname{int} \bigcup_{i=1}^{k} I_i \supset \operatorname{cl} E_n$. Put $c_n(\varepsilon) = \operatorname{dist}(E_n, [a, b] \setminus O)$ and consider any $c_n(\varepsilon)$-fine division $\mathcal{P}$ anchored in E_n. Clearly, $\mathcal{P}$ is a division in O, whence

$$|\sigma_G|(\mathcal{P}, f) \leqslant |\sigma_G|(\mathcal{P}, M) \leqslant M \frac{\varepsilon}{M} = \varepsilon. \tag{16}$$

Now, put $\delta(x) = \min\{c_n(1/2^n), 1\}$ at $x \in E_n$, anything outside of E. There is an N such that $1/2^N < \varepsilon$. Put

$$c = \min\bigl\{c_1(\varepsilon/N), \ldots, c_N(\varepsilon/N), 1\bigr\}.$$

Consider a $c\delta$-fine partition π of $[a, b]$ and denote $\mathcal{P}_n = \{(I, x) \in \pi : x \in E_n\}$, $n \in \mathbb{N}$. For each n, $\mathcal{P}_n$ is $c_n(1/2^n)$-fine, while for $n \leqslant N$ it is also $c_n(\varepsilon/N)$-fine. Hence, by (16),

$$|\sigma_G(\pi, f\chi_E)| \leqslant \sum_{n=1}^{N} |\sigma_G|(\mathcal{P}_n, f) + \sum_{n=N+1}^{\infty} |\sigma_G|(\mathcal{P}_n, f) <$$

$$< N\frac{\varepsilon}{N} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \varepsilon + \frac{1}{2^N} < 2\varepsilon. \qquad \blacksquare$$

Lemma 7. [8, Lemma 3.1]. *Let* $E = \bigcup_{n=1}^{\infty} E_n$ *be a* $\mathcal{G}_\delta$ *set and* $f\colon E \to \mathbb{R}$. *If the sequence* $(E_n)_{n=1}^{\infty}$ *is ascending and the restriction* $f \restriction E_n$ *is continuous for each* n*, then there exists an open interval* J *such that* $E \cap J \neq \varnothing$ *and the restriction* $f \restriction (E \cap J)$ *is continuous.*

Theorem 2. *Assume* $G \in VBG_*$. *For any* f *the following two assertions are equivalent:*

(i) f *is H-integrable w.r.t.* G *and* ***for each nonempty closed set*** $D \subset [a, b]$***, one can find an*** $A \in \mathfrak{I}_G$ ***and an interval*** I***,*** $I \cap D \setminus A \neq \varnothing$***, such that*** $f \restriction (I \cap D \setminus A)$ ***is continuous****;*

(ii) f *is newly integrable w.r.t.* G.

Proof. (i)⇐(ii). Suppose that f does not satisfy the condition in bold. We will show that f cannot be newly integrable w.r.t. G. Consider arbitrary gauge δ on $[a, b]$. Let D be a closed subset of $[a, b]$ such that for each $A \in \mathfrak{I}_G$ the set of discontinuity points of $f \restriction (D \setminus A)$ is dense in $D \setminus A \neq \varnothing$. Of course, $D \notin \mathfrak{I}_G$. Put $D_n = \{x \in D : \delta(x) > 1/n\}$, $n \in \mathbb{N}$. In virtue of Lemma 7, there exists an n such that

$$C = \big\{x \in D_n : f \restriction D_n \text{ is discontinuous at } x\big\} \notin \mathfrak{I}_G.$$

For $x \in C$, denote by $\omega(x)$ the oscillation of $f \restriction D_n$ at x; one has $\omega(x) > 0$. Since $C \notin \mathfrak{I}_G$, for some m the set $C_m = \{x \in C : \omega(x) > 1/m\}$ satisfies $|\mathrm{cl}\, C_m|_G > M > 0$.

Take any $c > 0$. There is a $\frac{c}{2n}$-fine division $\mathcal{P}$ in $[a, b]$, anchored in $\mathrm{cl}\, C_m$, with

$$|\Delta| G(\mathcal{P}) > M.$$

We can assume all intervals in $\mathcal{P}$ have tags at their endpoints. Thus, if $\mathcal{P}' \subset \mathcal{P}$ is the collection of all members of $\mathcal{P}$ that are tagged at their *left* endpoints,

$$|\Delta| G(\mathcal{P}') > \frac{M}{2} \quad \text{or} \quad |\Delta| G(\mathcal{P} \setminus \mathcal{P}') > \frac{M}{2}.$$

We can assume the first case holds. Find a subdivision $\mathcal{P}'' \subset \mathcal{P}'$ with segments pairwise disjoint and such that

$$|\Delta| G(\mathcal{P}'') > \frac{M}{4}. \tag{17}$$

Consider any $(I = [\zeta_I, \xi_I], \zeta_I) \in \mathcal{P}''$. Since $\zeta_I \in \mathrm{cl}\, C_m$, we can pick

$$x_I \in C_m \cap \left(\zeta_I - \frac{c}{2n}, \xi_I\right), \qquad y_I \in D_n \cap \left(\zeta_I - \frac{c}{2n}, \xi_I\right)$$

such that $|f(x_I) - f(y_I)| > 1/m$. There are two cases to be considered: if $\zeta_I \leqslant x_I$, $\zeta_I < y_I$, we put $I' = I$. If $y_I < \zeta_I$, $x_I \leqslant \zeta_I$, we designate as I' one of the intervals

$$I' = [\min(y_I, x_I), \zeta_I] \quad \text{or} \quad I' = [\min(y_I, x_I), \xi_I]$$

for which

$$|\Delta G(I')| \geqslant \frac{1}{2}|\Delta G(I)|. \tag{18}$$

Clearly, x_I's and y_I's can be picked step by step so that the intervals I' chosen for different I's will not overlap; i.e.,

$$\mathcal{R}_1 = \{(I', x_I) : (I, \zeta_I) \in \mathcal{P}''\}, \qquad \mathcal{R}_2 = \{(I', y_I) : (I, \zeta_I) \in \mathcal{P}''\}$$

will form divisions in $[a, b]$. Since $x_I, y_I \in D_n$, $\xi_I - \min(x_I, y_I) < \frac{c}{n}$, both $\mathcal{R}_1$ and $\mathcal{R}_2$ are $c\delta$-fine. By (17) and (18)

$$\sum_{(I,\zeta_I)\in\mathcal{P}''} |f(x_I) - f(y_I)||\Delta G(I')| > \frac{M}{8m}.$$

M and m were found independently of c, whence Saks-Henstock lemma for our new integral does not hold for f using δ. Thus, f is not newly integrable w.r.t. G. (This part of proof follows for any G).

(i)⇒(ii). Suppose f is *H*-integrable, but not newly integrable w.r.t. G. Let $P \neq \varnothing$ be the set of all points $x \in [a, b]$ such that f is integrable on no neighbourhood of x. Lemma 1 implies that P is perfect and that f is integrable on the closure of every interval contiguous to P. Suppose that f satisfies the condition in bold. There exists a portion $I \cap P \neq \varnothing$ of P such that for some $A \in \mathfrak{I}_G$, the restriction $f \restriction (I \cap P \setminus A)$ is continuous and bounded, and both the integrator G and the *H*-Stieltjes indefinite integral of f are *VB*$_*$ on $I \cap P$ (Lemma 5). Extend the restriction $f \restriction (I \cap P \setminus A)$ to a g on $I \cap P$ as it is described in Remark 1. Let $\tilde{f} = g$ on $I \cap P$, $\tilde{f} = f$ on $I \setminus P$. As $f = \tilde{f}$ outside of A, by Lemma 6 and Corollary 1, $\tilde{f}$ is *H*-integrable w.r.t. G on I. By Lemma 4, $\tilde{f}$ is newly integrable w.r.t. G on I, whence by Lemma 6 again, f is newly integrable w.r.t. G on I. This contradicts the definition of P. ■

Lemma 8. [13, Lemma 3.16]. *The condition in bold from Theorem* 2 *is equivalent to the condition* (2)*; i.e., the set $A \in \mathfrak{I}_G$ can be chosen independently of D.*

Putting together Theorem 2 and Lemma 8 we obtain the following Riemann-Lebesgue type theorem for our new integral. It resembles Theorem 1.

Theorem 3. *Let G be VBG$_*$. The condition* (i) *of Theorem* 1 *is equivalent to* (ii') *f is newly integrable w.r.t. G.*

From Theorems 1 and 3 we deduce that for a *VBG*$_*$ integrator, each newly integrable function is H_1-integrable.

Problem 1. Does it stay true if the integrator is not assumed to be *VBG*$_*$?

This problem can be reduced to the question: is it possible for an f to satisfy the condition in bold from Theorem 2, to be *H*-integrable w.r.t. a $G \notin$ *VBG*$_*$, and not to be H_1-integrable w.r.t. G? This question has already been asked [13, Problem 3.19]. From Theorems 1 and 3 we can deduce also that if G is normalized and *VBG*$_*$, then each H_1-integrand is an integrand in our new sense. But again,

Problem 2. Can we, for the latter statement, drop the assumption that G is *VBG*$_*$?

This is not clear for a continuous integrator G, even though the proof in the next section gives some hint how to handle this matter.

4. A Direct Proof

Theorem 4. *In the non-Stieltjes case, H_1-integral and our new integral coincide.*

Proof. It follows from Theorems 1 and 3. ∎

We would have liked to offer the reader a direct proof of the above theorem. Alas, we are not able to show that new integrability implies H_1-integrability without referring to the two aforementioned theorems and the continuity property therein. Perhaps such a direct proof would suggest how to answer in affirmative our Problem 1.

Proof. Assume a function f is H_1-integrable using $\delta < 1$. We will show it is newly integrable using the same δ. Pick an ε and consider a partition $\pi_\varepsilon = \{([a_{i-1}, a_i], x_i)\}_{i=1}^n$ of $[a, b]$ such that for each δ-fine partition $\pi \sqsupseteq \pi_\varepsilon$,

$$\left|\sigma(\pi, f) - \int_a^b f\right| < \varepsilon.$$

Choose a $\zeta_i > 0$, $a_{i-1} < a_i - \zeta_i < a_i < a_i + \zeta_i < a_{i+1}$, such that $\omega_F([a_i - \zeta_i, a_i + \zeta_i]) < \frac{\varepsilon}{n}$, $i = 1, \dots, n-1$, F the indefinite integral of f. Put

$$c_i = \min\{\zeta_i, a_{i+1} - a_i - \zeta_i, 1\}$$

and

$$c = \min_{1 \leqslant i \leqslant n-1} c_i.$$

Consider a $c\delta$-fine partition π of $[a,b]$. For each $i = 1, \dots, n-1$ there is at most one interval $(J_i, \xi_i) \in \pi$ with $a_i \in \operatorname{int} J_i = (\alpha_i, \beta_i)$. By the definition of c, since (J_i, ξ_i) is c_i-fine,

$$\xi_i \in (a_i - \zeta_i, a_i + \zeta_i).$$

Moreover, $M_i = [\alpha_i, a_i] \subset [a_i - \zeta_i, a_i]$, $L_i = [a_i, \beta_i] \subset [a_i, a_{i+1}]$ if $\xi_i \in [a_i, a_i + \zeta_i)$ (1st case), and $L_i = [\alpha_i, a_i] \subset [a_{i-1}, a_i]$, $M_i = [a_i, \beta_i] \subset [a_i, a_i + \zeta_i]$ if $\xi_i \in (a_i - \zeta_i, a_i)$ (2nd case). Let $\mathcal{I}_s$ be the collection of $i \in \{1, \dots, n-1\}$ for which the sth case holds, $s = 1, 2$. For an $i \in \mathcal{I}_1$ take an interval $K_i \subset [a_i, a_i + \zeta_i]$ such that $K_i \ni \xi_i$ and $|K_i| = |M_i|$; it is possible since $M_i \subset [a_i - \zeta_i, a_i]$. Symmetrically, for an $i \in \mathcal{I}_2$ take an interval $K_i \subset [a_i - \zeta_i, a_i]$ with $K_i \ni \xi_i$ and $|K_i| = |M_i|$. Write

$$\sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} f(\xi_i)|J_i| = \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} f(\xi_i)|L_i| + \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} f(\xi_i)|K_i|. \tag{19}$$

$\mathcal{K} = \{(K_i, \xi_i)\}_{i \in \mathcal{I}_1 \cup \mathcal{I}_2}$ is a δ-fine (since $c\delta$-fine) division $\sqsupseteq \pi_\varepsilon$. Denote $\mathcal{P} = \pi \cup \{(L_i, \xi_i)\}_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} \setminus \{(J_i, \xi_i)\}_{i \in \mathcal{I}_1 \cup \mathcal{I}_2}$; $\mathcal{P} \sqsupseteq \pi_\varepsilon$ and $\mathcal{P}$ is also δ-fine. By (19),

$$\sigma(\pi, f) = \sigma(\mathcal{P}, f) + \sigma(\mathcal{K}, f),$$

and so with Saks–Henstock lemma for the H_1-integral,

$$\begin{aligned} &|\sigma(\pi, f) - (F(b) - F(a))| \\ &\quad \leqslant |\sigma(\mathcal{P}, f) - \Delta F(\mathcal{P})| + |\sigma(\mathcal{K}, f) - \Delta F(\mathcal{K})| + |\Delta| F(\mathcal{K}) \\ &\quad\quad + \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} |\Delta F(M_i)| < \varepsilon + \varepsilon + n \cdot \frac{\varepsilon}{n} + n \cdot \frac{\varepsilon}{n} = 4\varepsilon. \end{aligned}$$

It means f is newly integrable. ■

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