TAIWANESE JOURNAL OF MATHEMATICS
Vol. 15, No. 1, pp. 201-210, February 2011
This paper is available online at http://www.tjm.nsysu.edu.tw/

EXISTENCE OF THREE SOLUTIONS FOR A DOUBLY EIGENVALUE FOURTH-ORDER BOUNDARY VALUE PROBLEM

G. A. Afrouzi, S. Heidarkhani and Donal O'Regan

Abstract. In this paper we consider the existence of at least three solutions for the Dirichlet problem

$$\begin{cases} u^{iv} + \alpha u'' + \beta u = \lambda f(x,u) + \mu g(x,u), & x \in (0,1) \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0 \end{cases}$$

where $\alpha,\ \beta$ are real constants, $f, g : [0,1] \times R \to R$ are L^2-Carathéodory functions and $\lambda, \mu > 0$. The approach is based on variational methods and critical points.

1. Introduction

This paper considers the existence of at least three solutions (weak) of the fourth-order boundary value problem

$$(1) \qquad \begin{cases} u^{iv} + \alpha u'' + \beta u = \lambda f(x,u) + \mu g(x,u), & x \in (0,1) \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0 \end{cases}$$

where $\alpha,\ \beta$ are real constants, $f, g : [0,1] \times R \to R$ are Carathéodory functions and $\lambda, \mu > 0$.

Several results are known concerning the existence of multiple solutions for fourth-order boundary value problems, and we refer the reader to [2, 5, 6, 8, 9, 11] and the references cited therein.

The aim of this paper is to establish the existence of a non-empty open set interval $A \subseteq I$ and a positive real number q with the following property: for each $\lambda \in A$ and for each Carathéodory function $g : [0,1] \times R \to R$ such that $\sup_{|\xi| \le s} |g(.,\xi)| \in$

Received January 13, 2009, accepted July 9, 2009.
Communicated by Biagio Ricceri.
2000 *Mathematics Subject Classification*: 34B15.
Key words and phrases: Fourth-order equations, Three solutions, Critical point, Multiplicity results.

$L^2(a,b)$ for all $s>0$, there is $\delta>0$ such that, for each $\mu\in[0,\delta]$, problem (1) admits at least three weak solutions in $W_0^{1,2}([0,1])\cap W^{2,2}([0,1])$ whose norms are less than q. Our analysis is based on the following three critical points theorem (see also [15] for an earlier version as well as [4, 13] for related results).

Theorem A. [Ricceri, 14]. *Let X be a reflexive real Banach space, $I\subseteq R$ an interval, $\Phi : X \longrightarrow R$ a sequentially weakly lower semicontinuous C^1 functional bounded on each bounded subset of X whose derivative admits a continuous inverse on X^* and $J : X \longrightarrow R$ a C^1 functional with compact derivative.*

Assume that

$$\lim_{||x||\to+\infty}(\Phi(x)+\lambda J(x))=+\infty$$

for all $\lambda\in I$, and that there exists $\rho\in R$ such that

$$\sup_{\lambda\in I}\inf_{x\in X}(\Phi(x)+\lambda(J(x)+\rho))<\inf_{x\in X}\sup_{\lambda\in I}(\Phi(x)+\lambda(J(x)+\rho)).$$

Then, there exist a non-empty open set interval $A\subseteq I$ and a positive real number q with the following property: for every $\lambda\in A$ and every C^1 functional $\Psi : X \longrightarrow R$ with compact derivative, there exists $\tau>0$ such that, for each $\mu\in[0,\tau]$, the equation

$$\Phi'(u)+\lambda J'(u)+\mu\Psi'(u)=0$$

has at least three solutions in X whose norms are less than q.

In the proof of our main result we also use the next result to verify the minimax inequality in Theorem A.

Proposition B. [Bonanno, 4]. *Let X be a non-empty set and $\Phi,\ J$ two real functions on X. Assume that $\Phi(x)\geq 0$ for every $x\in X$ and there exists $u_0\in X$ such that $\Phi(u_0)=J(u_0)=0$. Further, assume that exist $u_1\in X$, $r>0$ such that*

(κ_1) $\Phi(u_1)>r$

(κ_2) $\sup_{\Phi(x)<r}(-J(x))<r\frac{-J(u_1)}{\Phi(u_1)}$.

Then, for every $\nu>1$ and for every $\rho\in R$ satisfying

$$\sup_{\Phi(x)<r}(-J(x))+\frac{r\frac{-J(u_1)}{\Phi(u_1)}-\sup_{\Phi(x)<r}(-J(x))}{\nu}<\rho<r\frac{-J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\in R}\inf_{x\in X}(\Phi(x)+\lambda(J(x)+\rho))<\inf_{x\in X}\sup_{\lambda\in[0,\sigma]}(\Phi(x)+\lambda(J(x)+\rho))$$

where

$$\sigma = \frac{\nu r}{r\frac{-J(u_1)}{\Phi(u_1)} - \sup_{\Phi(x)<r}(-J(x))}.$$

For other basic notations and definitions, we refer the reader to [1, 3, 7, 10, 12, 16]. We note that some of the ideas used here were motivated by corresponding ones in [5].

2. Main Results

Let α and β be two real constants such that

$$\max\{\frac{\alpha}{\pi^2},\ -\frac{\beta}{\pi^4},\ \frac{\alpha}{\pi^2} - \frac{\beta}{\pi^4}\} < 1. \tag{1}$$

Put

$$\sigma := \max\{\frac{\alpha}{\pi^2},\ -\frac{\beta}{\pi^4},\ \frac{\alpha}{\pi^2} - \frac{\beta}{\pi^4},\ 0\}$$

and

$$\delta := \sqrt{1-\sigma}.$$

Let $f : [0,1] \times R \to R$ be a L^2-Carathéodory function, namely, $x \to f(x,t)$ is measurable for every $t \in R$, $t \to f(x,t)$ is continuous for almost every $x \in [0,1]$, and for every $s > 0$ there exists a function $l_s \in L^2([0,1])$ such that

$$\sup_{|t|\le s} |f(x,t)| \le l_s(x)$$

for almost every $x \in [0,1]$. Let F be the function defined by putting

$$F(x,t) = \int_0^t f(x,\xi)d\xi$$

for each $(x,t) \in [0,1] \times R$.

A function $u : [0,1] \to R$ is a generalized solution to problem (1) if $u \in C^3([0,1])$, $u''' \in AC([0,1])$, $u(0) = u(1) = 0$, $u''(0) = u''(1) = 0$, and $u^{iv} + \alpha u'' + \beta u = \lambda f(x,u) + \mu g(x,u)$ for almost every $x \in [0,1]$, and it is a weak solution to problem (1) if $u \in W_0^{1,2}([0,1]) \cap W^{2,2}([0,1])$ and

$$\int_0^1 [u''(x)v''(x) - \alpha u'(x)v'(x) + \beta u(x)v(x)]dx - \int_0^1 [\lambda f(x,u(x)) + \mu g(x,u(x))]v(x)dx = 0$$

for every $v \in W_0^{1,2}([0,1]) \cap W^{2,2}([0,1])$.

Standard methods (see [5, Proposition 2.2]) show that a weak solution to (1) is a generalized one when f, g are L^2-Carathéodory functions.

Let $X = W_0^{1,2}([0,1]) \cap W^{2,2}([0,1])$ endowed with the norm

$$|| u ||= \left(\int_0^1 (|u''(x)|^2 - \alpha|u'(x)|^2 + \beta|u(x)|^2)dx \right)^{1/2}.$$

Our main result is the following theorem

Theorem 1. *Assume that there exist a positive constant* r *and a function* $w \in W_0^{1,2}([0,1]) \cap W^{2,2}([0,1])$ *such that:*

(i) $\int_0^1 (|w''(x)|^2 - \alpha|w'(x)|^2 + \beta|w(x)|^2)dx > 2r$,

(ii) $\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}, \frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx < 2r \frac{\int_0^1 F(x,w(x))dx}{\int_0^1 (|w''(x)|^2 - \alpha|w'(x)|^2 + \beta|w(x)|^2)dx}$,

(iii) $\frac{2}{\delta^2\pi^4} \limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} < \frac{1}{\theta}$ *for almost every* $x \in [0,1]$ *and for all* $t \in R$, *and for some* θ *satisfying*

$$\theta > \frac{1}{2r\frac{\int_0^1 F(x,w(x))dx}{\int_0^1 (|w''(x)|^2 - \alpha|w'(x)|^2 + \beta|w(x)|^2)dx} - \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}, \frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx}.$$

Then, there exist a non-empty open interval $A \subseteq]0, r\theta]$ *and a number* $q > 0$ *with the following property: for each* $\lambda \in A$ *and for an arbitrary* L^2*-Carathéodory function* $g : [0,1] \times R \to R$, *there is* $\tau > 0$ *such that, whenever* $\mu \in [0,\tau]$, *problem* (1) *admits at least three generalized solutions whose norm in* X *are less than* q.

Put

$$k = 2\delta^2\pi^2 \left(\frac{2048}{27} - \frac{32}{9}\alpha + \frac{13}{40}\beta \right)^{-1},$$

and it is easy to see (see [5 pg 1169]) that $k > 0$.

Let us first present a consequence of Theorem 1 for a fixed test function w.

Corollary 1. *Assume that there exist two positive constants* c *and* d *with* $c < \frac{d}{\sqrt{k}}$ *such that:*

(j) $F(x,t) \geq 0$ *for each* $(x,t) \in ([0, \frac{3}{8}] \cup [\frac{5}{8}, 1]) \times [0,d]$,

(jj) $\int_0^1 \sup_{t\in[-c,c]} F(x,t)dx < k(\frac{c}{d})^2 \int_{\frac{3}{8}}^{\frac{5}{8}} F(x,d)dx$,

(jjj) $\frac{2}{\delta^2\pi^4} \limsup_{|t|\to+\infty} \frac{F(x,t)}{t^2} < \frac{1}{\theta}$ *for almost every* $x \in [0,1]$ *and for all* $t \in R$, *and for some* θ *satisfying*

$$\theta > \frac{1}{k(\frac{c}{d})^2 \int_{\frac{3}{8}}^{\frac{5}{8}} F(x,d)dx - \int_0^1 \sup_{t\in[-c,c]} F(x,t)dx}.$$

Then, there exist a non-empty open interval $A \subseteq]0, r\theta]$ and a number $q > 0$ with the following property: for each $\lambda \in A$ and for an arbitrary L^2-Carathéodory function $g : [0,1] \times R \to R$, there is $\tau > 0$ such that, whenever $\mu \in [0, \tau]$, problem (1) admits at least three generalized solutions whose norm in X are less than q.

The proof of Corollary 1 is based on the following lemma motivated from a result in [5].

Lemma 1. *Assume that there exist two positive constants c and d with $c < \frac{d}{\sqrt{k}}$. Under Assumptions (j) and (jj) of Corollary 1, there exist $r > 0$ and $w \in X$ such that $||w||^2 > 2r$ and*

$$\int_0^1 \sup_{t \in [-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}, \frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx < 2r \frac{\int_0^1 F(x, w(x))dx}{||w||^2}.$$

Proof. We put

$$w(x) = \begin{cases} -\frac{64d}{9}(x^2 - \frac{3}{4}x) & \text{if } 0 \le x \le \frac{3}{8}, \\ d & \text{if } \frac{3}{8} \le x < \frac{5}{8}, \\ -\frac{64d}{9}(x^2 - \frac{5}{4}x + \frac{1}{4}) & \text{if } \frac{5}{8} \le x < 1 \end{cases} \tag{2}$$

and $r = 2(\delta\pi c)^2$. It is easy to see that $w \in X$ and, in particular, one has

$$||w||^2 = \frac{4\delta^2\pi^2}{k} d^2.$$

Hence, taking into account that $c < \frac{d}{\sqrt{k}}$, one has

$$2r < ||w||^2.$$

Since $0 \le w(x) \le d$ for each $x \in [0,1]$, condition (j) ensures that

$$\int_0^{\frac{3}{8}} F(x, w(x))dx + \int_{\frac{5}{8}}^1 F(x, w(x))dx \ge 0.$$

Moreover from (jj), $r = 2(\delta\pi c)^2$ and the above inequality we have that

$$\int_0^1 \sup_{t \in [-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}, \frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx < k(\frac{c}{d})^2 \int_{\frac{3}{8}}^{\frac{5}{8}} F(x,d)dx \le 2r \frac{\int_0^1 F(x, w(x))dx}{||w||^2}.$$

Thus

$$\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{\tau}{2}},\frac{1}{\delta\pi}\sqrt{\frac{\tau}{2}}]} F(x,t)dx < 2r\frac{\int_0^1 F(x,w(x))dx}{||w||^2},$$

so the proof is complete. ■

Proof of Corollary 1. From Lemma 1 we see that Assumptions (i) and (ii) of Theorem 1 are fulfilled for w given in (3). Also, from (jjj), one has that (iii) is satisfied. Hence, the conclusion follows directly from Theorem 1. ■

Let us present an application of Corollary 1.

Example 1. Consider the problem

$$(3)\quad \begin{cases} u^{iv} + \alpha u'' + \beta u = \lambda(e^{-u}u^7(8-u)+1) + \mu g(x,u), & x \in (0,1) \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0 \end{cases}$$

where $g : [0,1] \times R \to R$ is a fixed L^2-Carathéodory function and $\alpha,\ \beta$ satisfying (2) are such that $k > \frac{4(e^4+e^5)}{5^6}$. Let $f(x,t) = f(t) = e^{-t}t^7(8-t)+1$ and note $F(x,t) = F(t) = e^{-t}t^8 + t$ for all $x \in [0,1]$ and $t \in R$. It is clear that (j) is satisfied. Also, by choosing $c = 1$ and $d = 5$ and taking into account that $0 < k < \frac{1}{2}$ (for more details, see [5 pg 1169]), one has $c < \frac{d}{\sqrt{k}}$, and since $k > \frac{4(e^4+e^5)}{5^6}$ we have $e^{-1}+1 < k\left(\frac{1}{20}\right)e^{-5}5^7 < K\left(\frac{1}{20}\right)(e^{-5}5^7+1)$, and as a result (jj) is satisfied. Also since $\limsup_{|t|\to+\infty}\frac{F(x,t)}{t^2} = 0$ then (jjj) holds. As a result we can apply Corollary 1 for every

$$\theta > \frac{1}{k(\frac{1}{20})(e^{-5}5^7+1) - (e^{-1}+1)}.$$

3. Proof of Theorem 1

We apply Theorem A, taking $X = W_0^{1,2}([0,1]) \cap W^{2,2}([0,1])$ endowed with the norm

$$|| u ||= \left(\int_0^1 (|u''(x)|^2 - \alpha|u'(x)|^2 + \beta|u(x)|^2)dx\right)^{1/2}.$$

Let

$$(4)\qquad \Phi(u) = \frac{||u||^2}{2}$$

and

$$J(u) = -\int_0^1 F(x, u(x))dx \tag{5}$$

for each $u \in X$. First of all, by classical results, the functional Φ is well defined, bounded on each bounded subset of X as well as continuously $G\widehat{a}teaux$ differentiable and sequentially weakly lower semi continuous functional and whose $G\widehat{a}teaux$ derivative admits a continuous inverse on X^*, and the functional J is well defined and a continuously $G\widehat{a}teaux$ differentiable functional whose $G\widehat{a}teaux$ derivative is compact. In particular, for each $u, v \in X$ one has

$$\Phi'(u)v = \int_0^1 [u''(x)v''(x) - \alpha u'(x)v'(x) + \beta u(x)v(x)]dx$$

and

$$J'(u)v = -\int_0^1 f(x, u(x))v(x)dx.$$

Furthermore from (iii) there exist two constants $\gamma,\ \tau \in R$ with $0 < \gamma < \frac{1}{\theta}$ such that

$$\frac{2}{\delta^2\pi^4}F(x,t) \leq \gamma t^2 + \tau \text{ for a.e. } x \in (0,1) \text{ and all } t \in R.$$

Fix $u \in X$. Then

$$F(x, u(x)) \leq \frac{\delta^2\pi^4}{2}(\gamma|u(x)|^2 + \tau) \quad \text{for all } x \in (0,1). \tag{6}$$

Then, for any fixed $\lambda \in]0, \theta]$, since $||u||_{L_2([0,1])} \leq \frac{1}{\delta\pi^2}||u||$ (see [5 pg 1168]), from (5), (6) and (7), we have

$$\begin{aligned} \Phi(u) + \lambda J(u) &= \frac{||u||^2}{2} - \lambda\int_0^1 F(x, u(x))dx \\ &\geq \frac{1}{2}||u||^2 - \frac{\theta\delta^2\pi^4}{2}\left(\gamma\int_0^1 |u(x)|^2dx + \tau\right) \\ &\geq \frac{1}{2}(1-\gamma\theta)||u||^2 - \frac{\theta\delta^2\pi^4}{2}\tau, \end{aligned}$$

and so

$$\lim_{||u||\to+\infty}(\Phi(u) + \lambda J(u)) = +\infty.$$

We claim that there exist $r > 0$ and $w \in X$ such that

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)}(-J(u)) < r\frac{-J(w)}{\Phi(w)}.$$

Note

$$\max_{x\in[0,1]}|u(x)| \leq \frac{1}{2\delta\pi}||u||$$

for each $u \in X$ (see Proposition 2.1 of [5]) and so

$$\begin{aligned}\Phi^{-1}(]-\infty, r[) &= \{u \in X; \Phi(u) < r\} \\ &= \left\{u \in X; \|u\| < \sqrt{2r}\right\} \\ &\subseteq \left\{u \in X; |u(x)| \leq \frac{1}{2\delta\pi}\sqrt{2r} \text{ for all } x \in [0,1]\right\} \\ &= \left\{u \in X; |u(x)| \leq \frac{1}{\delta\pi}\sqrt{\frac{r}{2}} \text{ for all } x \in [0,1]\right\},\end{aligned}$$

and it follows that

$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} (-J(u)) \leq \int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}, \frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx.$$

Now from (ii) we have

$$\int_0^1 \sup_{t\in[-\frac{1}{\delta\pi}\sqrt{\frac{r}{2}}, \frac{1}{\delta\pi}\sqrt{\frac{r}{2}}]} F(x,t)dx < 2r\frac{\int_0^1 F(x,w(x))dx}{\|w\|^2},$$

and so

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-J(u)) < r\frac{-J(w)}{\Phi(w)}.$$

Also from (i) we have $\Phi(w) > r$. Next recall from (iii) that

$$\theta > \frac{1}{r\frac{-J(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u))}.$$

Choose

$$\nu = \theta\left(r\frac{-J(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u))\right),$$

and note $\nu > 1$. Also, since

$$\theta > \frac{1}{r\frac{-J(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u))},$$

we have

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-J(u)) + \frac{1}{\theta} < r\frac{-J(w)}{\Phi(w)},$$

and so with our choice of ν we have

$$\sup_{u\in\Phi^{-1}(]-\infty,r])} (-J(u)) + \frac{r\frac{-J(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u))}{\nu} < r\frac{-J(w)}{\Phi(w)}.$$

Now from Proposition B (with $u_0 = 0$ and $u_1 = w$) for every $\rho \in R$ satisfying

$$\sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u)) + \frac{r\frac{-J(w)}{\Phi(w)} - \sup_{u\in\Phi^{-1}(]-\infty,r])}(-J(u))}{\nu} < \rho < r\frac{-J(w)}{\Phi(w)},$$

we have (note $\sigma = r\theta$)

$$\sup_{\lambda\in R}\inf_{u\in X}(\Phi(u) + \lambda J(u) + \rho\lambda) < \inf_{u\in X}\sup_{\lambda\in[0,r\theta]}(\Phi(u) + \lambda J(u) + \rho\lambda).$$

For any fixed L^2-Carathéodory function $g : [0,1] \times R \to R$, set

$$\Psi(u) = -\int_0^1\int_0^{u(x)} g(x,s)dsdx.$$

It is well known that Ψ is a continuously differentiable functional whose differential $\Psi'(u) \in X^*$, at $u \in X$ is given by

$$\Psi'(u)v = -\int_0^1 g(x,u(x))v(x)dx \ \text{ for every } v \in X,$$

and $\Psi' : X \to X^*$ is a compact operator. Now, all the assumptions of Theorem A, are satisfied. Hence, applying Theorem A, taking into account that the critical points of the functional $\Phi + \lambda J + \mu\Psi$ are exactly the weak solutions of the problem (1), we have that problem (1) admits at least three weak solutions in $W_0^{1,2}([0,1]) \cap W^{2,2}([0,1])$ whose norms in X are less than q.

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