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TAUBERIAN THEOREMS FOR THE WEIGHTED MEANS OF MEASURABLE FUNCTIONS OF SEVERAL VARIABLES

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Abstract. Let $f, \omega : \mathbb{R}_+^n \to \mathbb{C}$ and $T_\omega f(x)$ denote the weighted mean of f at x with respect to the weight function ω. We prove that the conditions of slow oscillation and slow decrease are Tauberian conditions for the implications: $f(x) \overset{st}{\to} l \Longrightarrow f(x) \to l$ and $T_\omega f(x) \overset{st}{\to} l \Longrightarrow f(x) \to l$. We also prove that the statistical version of the conditions of deferred means are Tauberian conditions for the implication: $T_\omega f(x) \overset{st}{\to} l \Longrightarrow f(x) \overset{st}{\to} l$. These generalize several well-known results.

1. Introduction

Let $\mathbb{R}_+ = [0, \infty)$ and $f, \omega : \mathbb{R}_+^n \to \mathbb{C}$ be Lebesgue measurable. Suppose $W(x) = \int_{[0,x_1]\times\cdots\times[0,x_n]} \omega(y)dy \neq 0$ for each $x = (x_1, \cdots, x_n) > \mathbf{0} = (0, \cdots, 0)$. Here $x > \mathbf{0}$ means that $x_k > 0$ for all k. The weighted mean $T_\omega f(x)$ of f at x is defined by

$$T_\omega f(x) = W(x)^{-1} \int_{[0,x_1]\times\cdots\times[0,x_n]} f(y)\omega(y)dy.$$

We say that f is $(\overline{N}, \omega)$ summable to l at ∞ and write $f(x) \to l$ $(\overline{N}, \omega)$ if $T_\omega f(x) \to l$ in the sense of Pringsheim, that is, $T_\omega f(x) \to l$ as $x \to \infty$. Here "$x \to \infty$" means "$\min(x_1, \cdots, x_n) \to \infty$". The notion of $(\overline{N}, \omega)$ summability defined here is the integral analogue of the one given in [8, p.57]. Following [13], we say that $f(x)$ is statistically convergent to l at ∞, in symbols, $f(x) \overset{st}{\to} l$ or $st\text{-}\lim_{x\to\infty} f(x) = l$, if the following equality holds for all $\epsilon > 0$:

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$$\lim_{x\to\infty}\frac{1}{x_1\cdots x_n}\Big|\Big\{u:\mathbf{0}\le u\le x, |f(u)-l|\ge\epsilon\Big\}\Big|=0.$$

Here $|E|$ denotes the Lebesgue measure of $E\subset\mathbb{R}^n_+$ and $\mathbf{0}\le u\le x$ means that $0\le u_k\le x_k$ for all k. We write $f(x)\stackrel{st}{\to} l\ (\overline{N},\omega)$ if $T_\omega f(x)\stackrel{st}{\to} l$. The readers can easily prove that the ordinary convergence implies the corresponding statistical convergence. For $n=1$ and $W(x)\to\infty$ as $x\to\infty$, ones can also deduce

$$f(x)\to l \implies f(x)\to l\ (\overline{N},\omega)\implies f(x)\stackrel{st}{\to} l\ (\overline{N},\omega). \tag{1.1}$$

But the converse implications of (1.1) are false, in general.

The purpose of this paper is to investigate the following converse implication:

$$f(x)\stackrel{st}{\to} l\ (\overline{N},\omega) \implies f(x)\to l. \tag{1.2}$$

We try to find conditions under which (1.2) holds. These conditions are known as Tauberian conditions and the corresponding results are called Tauberian results. Such kind of problems have been investigated in the literature for a long time (*cf.* e.g., [5, 6, 8, 12, 14, 15] for $n=1$, and [9] for $n=2$). In particular, in [9], Móricz investigated the variant of (1.2) with $f(x)\to l\ (\overline{N},\omega)$ instead of $f(x)\stackrel{st}{\to} l\ (\overline{N},\omega)$ for the case that $\omega=1$ and $n=2$. He proved that (1.3) is a Tauberian condition for this implication:

$$\inf_{\rho>1}\left\{\limsup_{x\to\infty}\left(\sup_{\substack{x_k<u_k<\rho x_k\\ u_\ell=x_\ell\ for\ \ell\neq k}}\big|f(u)-f(x)\big|\right)\right\}=0 \qquad (k=1,\cdots,n). \tag{1.3}$$

Here the limit superior "$\limsup_{x\to\infty}$" is defined in [2, p.1243] and [4, p.632]. Condition (1.3) is the n-dimensional analogue of the condition of slow oscillation. In [14], Móricz also showed that (1.3) is a Tauberian condition for (1.2), whenever $n=1$ and $\omega=1$. However, it is unknown whether (1.3) is a Tauberian condition of (1.2) for $n\ge 2$ and general ω. This problem for the discrete case was posed by Móricz [10] and solved by the present authors in [2]. In this paper, we shall prove that the following weak form of (1.3) is a Tauberian condition of (1.2):

$$\inf_{\rho>1}\left\{\limsup_{x\to\infty}\left(\sup_{x<u<\rho x}\big|f(u)-f(x)\big|\right)\right\}=0 \tag{1.4}$$

(see Corollary 6.1). For real-valued f, a similar result is also established (see Corollary 6.4). Our results not only extend [9, 14, 15] from $\omega=1$ to general ω, and [5, 6, 12, 14, 15] from 1-dimensional case to n-dimensional case, but also relax the $(\overline{N},\omega)$ summability to its statistical version.

In order to derive Corollaries 6.1 and 6.4, we first deduce in §2 the convergence property of subsequence type from $f(x)\stackrel{st}{\to} l$ and check the convergence of deferred means from $f(x) \xrightarrow{st} l\ (\overline{N}, \omega)$. We indicate that the subsequences involved here must be restricted. Next, we investigate the Tauberian problem of the implication: $f(x) \xrightarrow{st} l \Longrightarrow f(x) \to l$ (see §3). In §4, we present two Tauberian results for the implication: $f(x) \xrightarrow{st} l\ (\overline{N}, \omega) \Longrightarrow f(x) \xrightarrow{st} l$ (see Theorems 4.1 & 4.2). Based on the above results, we present the Tauberian conditions for the implication: $f(x) \xrightarrow{st} l\ (\overline{N}, \omega) \Longrightarrow f(x) \to l$ (see Theorems 5.1 & 5.2). As a consequence, several special cases of the last two theorems are deduced, which include Corollaries 6.1 and 6.4. We refer the readers to §6 for details.

Throughout this paper, $\mathbf{0} = (0, \cdots, 0)$, $\mathbf{1} = (1, \cdots, 1)$, $x, y, u, \mathbf{a}, \mathbf{b}, \alpha, \beta, \gamma, \cdots$ will denote the points in $\mathbb{R}^n_+$, $s, t, \rho, \cdots \in \mathbb{R}$, and $\lambda, \omega, \cdots$ are functions defined on $\mathbb{R}^n_+$. For $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \{0, 1\}^n$, let $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$.

2. Preliminaries

In this paper, we write $\lambda \in \mathcal{S}_I$ if $\lambda : \mathbb{R}^n_+ \mapsto \mathbb{R}^n$ is of the form $\lambda(x) = (\lambda_1(x_1), \cdots, \lambda_n(x_n))$ and each $\lambda_k : [0, \infty) \to [0, \infty)$ is strictly increasing, $\lambda_k(0) = 0$, and $\lim_{t\to\infty} \lambda_k(t) = \infty$. Without loss of generality, we shall further assume that each λ_k is piecewise smooth. More precisely, for each k, there exist countably many ℓ and a disjoint decomposition of subintervals of $[0, \infty)$, say $\cup_\ell [a_\ell, b_\ell)$, so that λ_k is C^1 on $[a_\ell, b_\ell]$ for each ℓ.

The following is an integral analogue of [1, Lemma 2.3]. It examines the convergence problem of subsequence type from the statistical convergence of a given function, and will be used to derive the convergence of deferred means from the $(\overline{N}, \omega)$ summability (see Theorem 2.3).

Theorem 2.1. *Let $f(x) \xrightarrow{st} l$ and $\lambda \in \mathcal{S}_I$. Suppose that (2.1) holds for all k:*

$$\lambda_k(t) \le Mt \quad (t \ge t_0) \quad \text{and} \quad \lambda_k'(t) \ge m \quad (\text{almost all } t > 0), \tag{2.1}$$

where $M > 0$, $t_0 > 0$, and $m > 0$ are constants. Then $f(\lambda_1^{\beta_1}(x_1), \cdots, \lambda_n^{\beta_n}(x_n)) \xrightarrow{st} l$ for all $\mathbf{0} \le \beta \le \mathbf{1}$, where

$$\lambda_k^\ell(t) = \begin{cases} t & \text{for } \ell = 0, \\ \lambda_k(t) & \text{for } \ell = 1. \end{cases} \tag{2.2}$$

Proof. It suffices to prove the case $\beta = \mathbf{1}$. Let $\epsilon > 0$ and $\mathbf{a} > \mathbf{0}$. Set $E^* = \lambda(E)$, where

$$E = \{x : \mathbf{0} \le x \le \mathbf{a}, |f(\lambda_1(x_1), \cdots, \lambda_n(x_n)) - l| \ge \epsilon\}.$$

The Jacobian of the mapping $x \mapsto \lambda(x)$ is $\lambda_1'(x_1)\cdots\lambda_n'(x_n)$, so by the second condition in (2.1) and [17, Theorem 7.26],

$$|E^*| = \int_{\lambda(E)} dy = \int_E |J_\lambda(x)|\, dx \geq m^n \int_E dx = m^n |E|.$$

Putting this with the first condition in (2.1) together yields

$$\begin{aligned}\frac{1}{a_1\cdots a_n}|E| &\leq \left(\frac{M}{m}\right)^n \frac{1}{\lambda_1(a_1)\cdots\lambda_n(a_n)}|E^*| \\ &= \left(\frac{M}{m}\right)^n \frac{1}{\lambda_1(a_1)\cdots\lambda_n(a_n)}|\,\{y : \mathbf{0} \leq y \leq \lambda(\mathbf{a}), |f(y)-l| \geq \epsilon\}\,| \\ &\longrightarrow 0 \text{ as } \min(a_1,\cdots,a_n) \to \infty.\end{aligned}$$

Hence, $f(\lambda_1(x_1),\cdots,\lambda_n(x_n)) \xrightarrow{st} l$. ∎

Theorem 2.1 is false if any of the two conditions in (2.1) is removed. Consider the functions $f : [0,\infty) \to [0,1]$ and $\lambda : [0,\infty) \to [0,\infty)$, defined by the rules:

$$f(x) = \begin{cases} 0 & \text{if } x \in [k+\phi(k), k+\phi(k+1)] \text{ for some } k \in \mathbb{N}\cup\{0\}, \\ 1 & \text{otherwise} \end{cases}$$

and

$$\lambda(t) = \begin{cases} k+\phi(k) & \text{for } t = 2k \text{ with } k \in \mathbb{N}\cup\{0\}, \\ k+\phi(k+1) & \text{for } t = 2k+1 \text{ with } k \in \mathbb{N}\cup\{0\}, \end{cases}$$

where $\phi(k)\uparrow$, $\phi(0) = 0$, and λ is linear on each subinterval. Whenever $k+\phi(k) \leq a < k+1+\phi(k+1)$, we have

$$\sum_{\ell=0}^{k-1}(\phi(\ell+1)-\phi(\ell)) \leq |\{0 \leq x \leq a : f(x) = 0\}| \leq \sum_{\ell=0}^{k}(\phi(\ell+1)-\phi(\ell))$$

and $k \leq |\{0 \leq x \leq a : f(x) = 1\}| \leq k+1$. These imply

$$f(x) \xrightarrow{st} \begin{cases} 0 & \text{if } \phi(k)/k \to \infty, \\ 1 & \text{if } \phi(k)/k \to 0. \end{cases}$$

On the other hand, $st\text{-}\lim_{t\to\infty} f(\lambda(t))$ does not exist. Moreover, $\lambda \in \mathcal{S}_I$. For $\phi(k) = \sum_{\ell=0}^{k} \ell^2$, the first condition in (2.1) fails, but the second one holds. We get another case, if $\phi(0) = 0$ and $\phi(k) = \sum_{\ell=1}^{k} 1/\ell^2$ for $k = 1,2,\cdots$. These two examples indicate that both conditions in (2.1) are required in Theorem 2.1.

Next, we consider the convergence problem of deferred means. For this purpose, we introduce an equality in the following.

Lemma 2.2. *For* $x_\alpha, y_\alpha \in \mathbb{C}$, *where* $\mathbf{0} \le \alpha \le \mathbf{1}$, *we have*

$$\sum_{\mathbf{0}\le\alpha\le\mathbf{1}} (-1)^{|\alpha|} x_\alpha y_\alpha = \sum_{\mathbf{0}\le\alpha\le\mathbf{1}} \left(\sum_{\alpha\le\beta\le\mathbf{1}} (-1)^{|\alpha|+|\beta|} x_\beta \right) \left(\sum_{\mathbf{0}\le\gamma\le\alpha} (-1)^{|\gamma|} y_\gamma \right). \tag{2.3}$$

Proof. We prove (2.3) by the mathematical induction. For $a_0, a_1, b_0, b_1 \in \mathbb{C}$, it is trivial that $a_0 b_0 - a_1 b_1 = (a_0 - a_1) b_0 + a_1 (b_0 - b_1)$. In particular,

$$\sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} (a_0^{\tilde\alpha} b_0^{\tilde\alpha} - a_1^{\tilde\alpha} b_1^{\tilde\alpha}) = \sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} \left[(a_0^{\tilde\alpha} - a_1^{\tilde\alpha}) b_0^{\tilde\alpha} + a_1^{\tilde\alpha} (b_0^{\tilde\alpha} - b_1^{\tilde\alpha}) \right],$$

where

$$a_0^{\tilde\alpha} = \sum_{\tilde\alpha\le\tilde\beta\le\tilde{\mathbf{1}}} (-1)^{|\tilde\alpha|+|\tilde\beta|} x_{(\tilde\beta,0)}, \qquad b_0^{\tilde\alpha} = \sum_{\tilde{\mathbf{0}}\le\tilde\gamma\le\tilde\alpha} (-1)^{|\tilde\gamma|} y_{(\tilde\gamma,0)},$$

$$a_1^{\tilde\alpha} = \sum_{\tilde\alpha\le\tilde\beta\le\tilde{\mathbf{1}}} (-1)^{|\tilde\alpha|+|\tilde\beta|} x_{(\tilde\beta,1)}, \qquad b_1^{\tilde\alpha} = \sum_{\tilde{\mathbf{0}}\le\tilde\gamma\le\tilde\alpha} (-1)^{|\tilde\gamma|} y_{(\tilde\gamma,1)},$$

and $\overline{\xi} \in \mathbb{C}^{n-1}$ is obtained from $\xi \in \mathbb{C}^n$ by deleting the last coordinate. Suppose that (2.3) holds for the case $n-1$. Then

$$\begin{aligned}
\sum_{\mathbf{0}\le\alpha\le\mathbf{1}} (-1)^{|\alpha|} x_\alpha y_\alpha &= \sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} (-1)^{|\tilde\alpha|} x_{(\tilde\alpha,0)} y_{(\tilde\alpha,0)} - \sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} (-1)^{|\tilde\alpha|} x_{(\tilde\alpha,1)} y_{(\tilde\alpha,1)} \\
&= \sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} \left(\sum_{\tilde\alpha\le\tilde\beta\le\tilde{\mathbf{1}}} (-1)^{|\tilde\alpha|+|\tilde\beta|} x_{(\tilde\beta,0)} \right) \left(\sum_{\tilde{\mathbf{0}}\le\tilde\gamma\le\tilde\alpha} (-1)^{|\tilde\gamma|} y_{(\tilde\gamma,0)} \right) \\
&\quad - \sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} \left(\sum_{\tilde\alpha\le\tilde\beta\le\tilde{\mathbf{1}}} (-1)^{|\tilde\alpha|+|\tilde\beta|} x_{(\tilde\beta,1)} \right) \left(\sum_{\tilde{\mathbf{0}}\le\tilde\gamma\le\tilde\alpha} (-1)^{|\tilde\gamma|} y_{(\tilde\gamma,1)} \right) \\
&= \sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} (a_0^{\tilde\alpha} b_0^{\tilde\alpha} - a_1^{\tilde\alpha} b_1^{\tilde\alpha}) = \sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} \left[(a_0^{\tilde\alpha} - a_1^{\tilde\alpha}) b_0^{\tilde\alpha} + a_1^{\tilde\alpha} (b_0^{\tilde\alpha} - b_1^{\tilde\alpha}) \right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} \left[(a_0^{\tilde\alpha} - a_1^{\tilde\alpha}) b_0^{\tilde\alpha} + a_1^{\tilde\alpha} (b_0^{\tilde\alpha} - b_1^{\tilde\alpha}) \right] \\
&= \sum_{\tilde{\mathbf{0}}\le\tilde\alpha\le\tilde{\mathbf{1}}} \left\{ \left[\sum_{\tilde\alpha\le\tilde\beta\le\tilde{\mathbf{1}}} \left((-1)^{|\tilde\alpha|+|\tilde\beta|} x_{(\tilde\beta,0)} - (-1)^{|\tilde\alpha|+|\tilde\beta|} x_{(\tilde\beta,1)} \right) \right] \left[\sum_{\tilde{\mathbf{0}}\le\tilde\gamma\le\tilde\alpha} (-1)^{|\tilde\gamma|} y_{(\tilde\gamma,0)} \right] \right.
\end{aligned}$$

$$+\left[\sum_{\tilde\alpha\le\tilde\beta\le\tilde{\mathbf{1}}}(-1)^{|\tilde\alpha|+|\tilde\beta|}x_{(\tilde\beta,1)}\right]\left[\sum_{\tilde{\mathbf{0}}\le\tilde\gamma\le\tilde\alpha}\Big((-1)^{|\tilde\gamma|}y_{(\tilde\gamma,0)}-(-1)^{|\tilde\gamma|}y_{(\tilde\gamma,1)}\Big)\right]\Bigg\}$$

$$=\sum_{\mathbf{0}\le\alpha\le\mathbf{1}}\left(\sum_{\alpha\le\beta\le\mathbf{1}}(-1)^{|\alpha|+|\beta|}x_\beta\right)\left(\sum_{\mathbf{0}\le\gamma\le\alpha}(-1)^{|\gamma|}y_\gamma\right).$$

This shows that (2.3) holds for the case n. The proof is complete. ■

Denote by D_+^n (respectively D_-^n) the class consisting of all $\lambda\in\mathcal{S}_I$ so that each λ_k dilates at infinity in the sense of (2.4) (respectively (2.4*)):

(2.4) $$\liminf_{t\to\infty}\frac{\lambda_k(t)}{t}>1\qquad(k=1,\cdots,n),$$

(2.4*) $$\liminf_{t\to\infty}\frac{t}{\lambda_k(t)}>1\qquad(k=1,\cdots,n).$$

For a fixed $\alpha=(\alpha_1,\cdots,\alpha_n)\in\{0,1\}^n$, $\lambda\in D_+^n$, and a weight $\omega:\mathbb{R}_+^n\to\mathbb{C}$, define

(2.5) $$\triangle_\lambda^\alpha W(x)=\int_{E_1\times\cdots\times E_n}\omega(y)dy,$$

where

(2.6) $$E_k=\begin{cases}[0,x_k] & \text{if } \alpha_k=0,\\ [x_k,\lambda_k(x_k)] & \text{if } \alpha_k=1.\end{cases}$$

If $\lambda\in D_-^n$, (2.6) will be changed to (2.6*):

(2.6*) $$E_k=\begin{cases}[0,\lambda_k(x_k)] & \text{if } \alpha_k=0,\\ [\lambda_k(x_k),x_k] & \text{if } \alpha_k=1.\end{cases}$$

Consider the subclass $st\text{-}D_+^n(\omega)$ of D_+^n and the subclass $st\text{-}D_-^n(\omega)$ of D_-^n. We write $\lambda\in st\text{-}D_+^n(\omega)$ (respectively $\lambda\in st\text{-}D_-^n(\omega)$) if $\lambda\in D_+^n$ (respectively $\lambda\in D_-^n$) and both of (2.1) and (2.7) hold:

(2.7) $$st\text{-}\limsup_{x\to\infty}\left|\frac{\triangle_\lambda^\alpha W(x)}{\triangle_\lambda^{\mathbf{1}}W(x)}\right|<\infty \text{ for each } \mathbf{0}\le\alpha\le\mathbf{1}.$$

Here $st\text{-}\limsup_{x\to\infty}\phi(x)$ is defined as the supremum of those r satisfying

$$\lim_{\mathbf{a}\to\infty}\frac{1}{a_1\cdots a_n}\left|\{u:\mathbf{0}\le u\le\mathbf{a},\phi(u)>r\}\right|\ne 0$$

(*cf.* [7, 11]). For $\lambda \in D^n_+$, we have $W(\lambda(x)) = \sum_{\mathbf{0}\le\alpha\le\mathbf{1}} \triangle^\alpha_\lambda W(x)$. Thus, for such λ and positive ω, (2.7) can be replaced by the following equivalent condition:

(2.7*) $$st\text{-}\limsup_{x\to\infty} \frac{W(\lambda(x))}{\triangle^{\mathbf{1}}_\lambda W(x)} < \infty.$$

Obviously, (2.7*) is the statistical version of the integral form of [16, Eq.(2.8)]. Hence, $st\text{-}D^n_+(\omega)$ can be regarded as a substitute of Λ_u given in [16]. Analogously, for $\lambda \in D^n_-$ and positive ω, (2.7) is equivalent to (2.7**):

(2.7**) $$st\text{-}\limsup_{x\to\infty} \frac{W(x)}{\triangle^{\mathbf{1}}_\lambda W(x)} < \infty,$$

which is the statistical version of the integral form of [16, Eq.(2.9)]. This indicates that $st\text{-}D^n_-(\omega)$ is a substitute of Λ_ℓ defined in [16]. From §6, we shall see

$$st\text{-}D^n_+(\omega) \supseteqq \{\lambda_\rho : \rho > 1\} \text{ and } st\text{-}D^n_-(\omega) \supseteqq \{\lambda_\rho : 0 < \rho < 1\},$$

where $\omega \in st$-SVA and λ_ρ denotes the mapping $x \mapsto \rho x$ (see §6 for details).

The following theorem shows the convergence property of deferred means obtained from the $(\overline{N}, \omega)$ summability. This result is an integral analogue of [1, Theorem 3.1]. It plays an important role in the proofs of Theorems 4.1 and 4.2.

Theorem 2.3. *Let* $f(x) \xrightarrow{st} l\ (\overline{N}, \omega)$. *Then for each* $\lambda \in st\text{-}D^n_+(\omega)$,

(2.8) $$\left(\triangle^{\mathbf{1}}_\lambda W(x)\right)^{-1} \int_{[x_1,\lambda_1(x_1)]\times\cdots\times[x_n,\lambda_n(x_n)]} f(y)\omega(y)dy \xrightarrow{st} l,$$

and for each $\lambda \in st\text{-}D^n_-(\omega)$,

(2.8*) $$\left(\triangle^{\mathbf{1}}_\lambda W(x)\right)^{-1} \int_{[\lambda_1(x_1),x_1]\times\cdots\times[\lambda_n(x_n),x_n]} f(y)\omega(y)dy \xrightarrow{st} l.$$

Proof. We suppose $\lambda \in st\text{-}D^n_+(\omega)$ and the proof for $\lambda \in st\text{-}D^n_-(\omega)$ will be carried out in a similar way. For each $\mathbf{0} \le \alpha \le \mathbf{1}$, let $x_\alpha = W(\lambda_1^{1-\alpha_1}(x_1), \cdots, \lambda_n^{1-\alpha_n}(x_n))$ and $y_\alpha = T_\omega f(\lambda_1^{1-\alpha_1}(x_1), \cdots, \lambda_n^{1-\alpha_n}(x_n))$, where λ^ℓ_k is defined by (2.2). Then the left side of (2.3) becomes $\int_{[x_1,\lambda_1(x_1)]\times\cdots\times[x_n,\lambda_n(x_n)]} f(y)\omega(y)dy$. On the other hand, the readers can prove that $\sum_{\alpha\le\beta\le\mathbf{1}} (-1)^{|\alpha|+|\beta|} x_\beta = \triangle^{\mathbf{1}-\alpha}_\lambda W(x)$. From

(2.3), we obtain

$$\begin{aligned}&\left(\triangle_\lambda^{\mathbf{1}}W(x)\right)^{-1}\int_{[x_1,\lambda_1(x_1)]\times\cdots\times[x_n,\lambda_n(x_n)]} f(y)\omega(y)dy\\ &=\sum_{\mathbf{0}\le\alpha\le\mathbf{1}}\left(\frac{\triangle_\lambda^{\mathbf{1}-\alpha}W(x)}{\triangle_\lambda^{\mathbf{1}}W(x)}\right)\left(\sum_{\mathbf{0}\le\gamma\le\alpha}(-1)^{|\gamma|}T_\omega f(\lambda_1^{1-\gamma_1}(x_1),\cdots,\lambda_n^{1-\gamma_n}(x_n))\right)\\ &=T_\omega f(\lambda_1(x_1),\cdots,\lambda_n(x_n))+\sum_{\alpha\ne\mathbf{0}}\left\{\cdots\right\},\ \text{say.}\end{aligned}$$

For $\alpha \neq \mathbf{0}$, the term "$\sum_{\mathbf{0}\le\gamma\le\alpha}(\cdots)$" in (2.9) tends to 0 statistically as $\min(x_1, \cdots, x_n) \to \infty$. This can be proved by using Theorem 2.1 and the linearity of the statistical convergence. From (2.9), we get (2.8). This finishes the proof. ■

3. TAUBERIAN CONDITIONS FROM $f(x) \xrightarrow{st} l$ TO $f(x) \to l$

The following gives a Tauberian result from $f(x) \xrightarrow{st} l$ to $f(x) \to l$ and generalizes [14, Theorem 2].

Theorem 3.1. *Let $f(x) \xrightarrow{st} l$. If (3.1) or (3.1*) holds, then $f(x) \to l$, where*

$$\inf_{\lambda\in D_+^n}\left\{\limsup_{x\to\infty}\left(\sup_{x<u<\lambda(x)}\left|f(u)-f(x)\right|\right)\right\}=0, \tag{3.1}$$

$$\inf_{\lambda\in D_-^n}\left\{\limsup_{x\to\infty}\left(\sup_{\lambda(x)<u<x}\left|f(x)-f(u)\right|\right)\right\}=0. \tag{3.1*}$$

Proof. We adopt the same proof of [2, Theorem 2.1]. It is easy to see that $\lambda \in D_+^n \iff \lambda^{-1} \in D_-^n$, where λ^{-1} denotes the inverse function of λ. This indicates that (3.1) $\iff$ (3.1*). Hence, it suffices to prove the case of (3.1). Let $\epsilon > 0$. By (3.1), we can find $\lambda \in D_+^n$ and $N_1 > 0$ such that

$$\min(x_1,\cdots,x_n)\ge N_1 \Longrightarrow \sup_{x<u<\lambda(x)}\left|f(u)-f(x)\right|<\epsilon. \tag{3.2}$$

We have assumed that $f(x) \xrightarrow{st} l$. Thus, there exists $N_2 > 0$ so that for $\min(a_1, \cdots, a_n) \ge N_2$,

$$\frac{1}{a_1\cdots a_n}\left|\left\{u:\mathbf{0}\le u\le\mathbf{a},|f(u)-l|\ge\epsilon\right\}\right|<\left(1-\frac{1}{K}\right)^n, \tag{3.3}$$

where

$$K = \min\left(\inf_{t \geq t_0} \frac{\lambda_1(t)}{t}, \cdots, \inf_{t \geq t_0} \frac{\lambda_n(t)}{t}\right) > 1 \quad \text{for some} \quad t_0 > 0.$$

Set $N_0 = \max(N_1, N_2, t_0)$. For $\min(x_1, \cdots, x_n) \geq N_0$, we have $\min(\lambda_1(x_1), \cdots, \lambda_n(x_n)) \geq N_2$, and so (3.3) tells us that

$$\left|\left\{u : \mathbf{0} \leq u \leq \lambda(x), |f(u) - l| \geq \epsilon\right\}\right| < \left\{\prod_{k=1}^{n}\left(1 - \frac{x_k}{\lambda_k(x_k)}\right)\right\}\left(\prod_{k=1}^{n} \lambda_k(x_k)\right)$$

$$= \prod_{k=1}^{n}\left(\lambda_k(x_k) - x_k\right).$$

This enables us to find $u^* = (u_1^*, \cdots, u_n^*)$ with the properties: $x < u^* < \lambda(x)$ and $|f(u^*) - l| < \epsilon$. Putting this with (3.2) together yields

$$|f(x) - l| \leq |f(u^*) - l| + \sup_{x < u < \lambda(x)} \left|f(u) - f(x)\right| < 2\epsilon. \qquad \blacksquare$$

Theorem 3.1 is an integral analogue of [2, Theorem 2.1]. We know that $\lambda(x) = \rho x$ with $\rho > 1$ is in D_+^n, so (3.1) can be replaced by (1.4). We indicate that the λ in (3.1) can not be relaxed to those of the form $\lambda(x) = (\lambda_1(x_1), \cdots, \lambda_n(x_n))$ with the property:

$$\liminf_{t \to \infty} \frac{\lambda_k(t)}{t} = 1 \qquad (k = 1, \cdots, n).$$

This is illustrated by the functions $f : [0, \infty) \to [0, 1]$ and $\lambda : [0, \infty) \to [0, \infty)$, defined in the following way:

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = n \text{ or } x = n + \frac{1}{n^2} \text{ for some } n \in \mathbb{N}, \\ 1 & \text{if } x = n + \frac{1}{2n^2} \text{ for some } n \in \mathbb{N}, \end{cases}$$

f is linear on each subinterval, and

$$\lambda(t) = \begin{cases} \frac{17}{16}t & \text{if } t \in [0, 2), \\ t + \frac{1}{t^3} & \text{if } t \in [2, \infty). \end{cases}$$

In this case, $f(x) \xrightarrow{st} 0$, $f(x) \nrightarrow 0$, $\liminf\limits_{t \to \infty} \dfrac{\lambda(t)}{t} = 1$, and $\sup\limits_{x < u < x + \frac{1}{x^3}} |f(u) - f(x)| < \dfrac{2(n+1)^2}{n^3}$ for $2 \leq n \leq x < n + 1$. Hence,

$$\limsup_{x \to \infty}\left(\sup_{x < u < \lambda(x)} \left|f(u) - f(x)\right|\right) \leq \lim_{n \to \infty} \frac{2(n+1)^2}{n^3} = 0.$$

For real-valued f, we have the following corresponding result of Theorem 3.1, in which (3.1) and (3.1*) are replaced by the combination of (3.4) and (3.4*):

$$\sup_{\lambda\in D_+^n}\left\{\liminf_{x\to\infty}\left(\inf_{x<u<\lambda(x)}(f(u)-f(x))\right)\right\}\geq 0, \tag{3.4}$$

$$\sup_{\lambda\in D_-^n}\left\{\liminf_{x\to\infty}\left(\inf_{\lambda(x)<u<x}(f(x)-f(u))\right)\right\}\geq 0. \tag{3.4*}$$

Theorem 3.2. *Let f be real-valued. If $f(x)\overset{st}{\to} l$ and one of (3.4)-(3.4*) holds, then $f(x)\to l$.*

Proof. It is easy to see that (3.4) $\Longleftrightarrow$ (3.4*). By (3.4), we have the following fact instead of (3.2):

$$\min(x_1,\cdots,x_n)\geq N_1 \Longrightarrow \inf_{x<u<\lambda(x)}(f(u)-f(x))>-\epsilon. \tag{3.5}$$

The proof of Theorem 3.1 with this change leads us to

$$-\epsilon<\inf_{x<u<\lambda(x)}(f(u)-f(x))\leq(f(u^*)-l)-(f(x)-l)<\epsilon-(f(x)-l),$$

where u^* is defined there. This implies $\sup\limits_{\min(x_1,\cdots,x_n)\geq N_0}(f(x)-l)\leq 2\epsilon$, and therefore, $\limsup\limits_{x\to\infty}(f(x)-l)\leq 0$. To replace (3.4) by (3.4*), we see that a similar proof to the above also lead us to $\liminf\limits_{x\to\infty}(f(x)-l)\geq 0$. Therefore, $f(x)\to l$. ■

Theorem 3.2 is an integral analogue of [2, Theorem 2.3]. It generalizes [14, Theorem 1]. The same functions f and λ given after Theorem 3.1 indicate that the λ in (3.4) (respectively (3.4*)) can not be relaxed to those with equality sign instead of the inequality sign in (2.4) (respectively (2.4*)).

4. Tauberian Conditions from $f(x)\overset{st}{\to} l\ (\overline{N},\omega)$ to $f(x)\overset{st}{\to} l$

The $(\overline{N},\omega)$ summability can be related to the original convergence by the use of controlling the magnitudes of $M_\lambda^+f(x;\omega)$ and $M_\lambda^-f(x;\omega)$, which are defined below:

$$\begin{aligned}
&M_\lambda^+f(x;\omega)\\
&=\left(\triangle_\lambda^{\mathbf{1}}W(x)\right)^{-1}\int_{[x_1,\lambda_1(x_1)]\times\cdots\times[x_n,\lambda_n(x_n)]}f(y)\omega(y)dy-f(x)\\
&=\left(\triangle_\lambda^{\mathbf{1}}W(x)\right)^{-1}\int_{[x_1,\lambda_1(x_1)]\times\cdots\times[x_n,\lambda_n(x_n)]}(f(y)-f(x))\,\omega(y)dy\quad(\lambda\in st\text{-}D_+^n(\omega)),
\end{aligned}$$

$$\begin{aligned}&M_\lambda^- f(x;\omega)\\ &= f(x) - \left(\triangle_\lambda^{\mathbf{1}} W(x)\right)^{-1} \int_{[\lambda_1(x_1),x_1]\times\cdots\times[\lambda_n(x_n),x_n]} f(y)\omega(y)dy\\ &= \left(\triangle_\lambda^{\mathbf{1}} W(x)\right)^{-1} \int_{[\lambda_1(x_1),x_1]\times\cdots\times[\lambda_n(x_n),x_n]} (f(x)-f(y))\,\omega(y)dy \quad (\lambda\in st\text{-}D_-^n(\omega)).\end{aligned}$$

The following is an integral analogue of [1, Theorem 3.2]. It generalizes [5, Theorem 2].

Theorem 4.1. *Let* $f(x) \xrightarrow{st} l\ (\overline{N},\omega)$. *The following four assertions hold:*

(i) *Suppose* $st\text{-}D_+^n(\omega) \neq \emptyset$. *Then* $f(x) \xrightarrow{st} l$ *if and only if for all* $\epsilon > 0$,

$$\inf_{\lambda\in st\text{-}D_+^n(\omega)} \limsup_{\mathbf{a}\to\infty} \frac{1}{a_1a_2\cdots a_n}\left|\left\{x : \mathbf{0} \le x \le \mathbf{a}, |M_\lambda^+ f(x;\omega)| \ge \epsilon\right\}\right| = 0. \tag{4.1}$$

(ii) *Suppose* $st\text{-}D_-^n(\omega) \neq \emptyset$. *Then* $f(x) \xrightarrow{st} l$ *if and only if for all* $\epsilon > 0$,

$$\inf_{\lambda\in st\text{-}D_-^n(\omega)} \limsup_{\mathbf{a}\to\infty} \frac{1}{a_1a_2\cdots a_n}\left|\left\{x : \mathbf{0} \le x \le \mathbf{a}, |M_\lambda^- f(x;\omega)| \ge \epsilon\right\}\right| = 0. \tag{4.1*}$$

(iii) *Condition (4.1) can be replaced by* $M_\lambda^+ f(x;\omega) \xrightarrow{st} 0$ *for some* $\lambda \in st\text{-}D_+^n(\omega)$, *and condition (4.1*) can be replaced by* $M_\lambda^- f(x;\omega) \xrightarrow{st} 0$ *for some* $\lambda \in st\text{-}D_-^n(\omega)$.

(iv) *Moreover, if* $M_\lambda^+ f(x;\omega) \xrightarrow{st} 0$ *holds for some* $\lambda \in st\text{-}D_+^n(\omega)$, *then it holds for all* $\lambda \in st\text{-}D_+^n(\omega)$. *The same situation happens to* $M_\lambda^- f(x;\omega) \xrightarrow{st} 0$ *with* $\lambda \in st\text{-}D_-^n(\omega)$.

Proof. Consider (i). Assume that $f(x) \xrightarrow{st} l$. Let $\lambda \in st\text{-}D_+^n(\omega)$. From (2.8) and the linearity of the statistical convergence, we get

$$\begin{aligned}M_\lambda^+ f(x;\omega) &= \left(\triangle_\lambda^{\mathbf{1}} W(x)\right)^{-1} \int_{[x_1,\lambda_1(x_1)]\times\cdots\times[x_n,\lambda_n(x_n)]} f(y)\omega(y)dy - f(x)\\ &\xrightarrow{st} l - l = 0.\end{aligned}$$

Hence, (4.1) follows. For the converse, write $f(x) = \tilde{M}_\lambda^+ f(x;\omega) - M_\lambda^+ f(x;\omega)$, where

$$\tilde{M}_\lambda^+ f(x;\omega) = \left(\triangle_\lambda^{\mathbf{1}} W(x)\right)^{-1} \int_{[x_1,\lambda_1(x_1)]\times\cdots\times[x_n,\lambda_n(x_n)]} f(y)\omega(y)dy.$$

We have

$$\{x : \mathbf{0} \le x \le \mathbf{a}, |f(x) - l| \ge \epsilon\}$$
$$\subseteq \left\{x : \mathbf{0}\le x\le \mathbf{a}, |\tilde{M}^+_\lambda f(x;\omega) - l| \ge \frac{\epsilon}{2}\right\} \bigcup \left\{x : \mathbf{0}\le x\le \mathbf{a}, |M^+_\lambda f(x;\omega)| \ge \frac{\epsilon}{2}\right\}.$$

By (2.8) and (4.1), we infer that $f(x) \stackrel{st}{\to} l$. This completes the proof of (i). The above argument also verifies both of the first parts in (iii) and (iv). As for $\lambda \in st$-$D^n_-(\omega)$, it can be carried out in a similar way. We leave it to the readers. ■

Theorem 4.1 indicates that any of (4.1) and (4.1*) is a Tauberian condition from $f(x) \stackrel{st}{\to} l\ (\overline{N}, \omega)$ to $f(x) \stackrel{st}{\to} l$. However, the example that $f(x) = 1$ for $x_1 = \cdots = x_n > 0$ and 0 otherwise tells us that it is no longer the case, whenever "$f(x) \stackrel{st}{\to} l$" is replaced by $f(x) \to l$. It is easy to see that for such f, (4.1) and (4.1*) hold, $f(x) \stackrel{st}{\to} 0\ (\overline{N}, \omega)$, $f(x) \stackrel{st}{\to} 0$, but $f(x) \nrightarrow 0$.

For real-valued f, we have the following analogue of Theorem 4.1. It is an integral analogue of [1, Theorem 4.1] and generalizes [5, Theorem 1].

Theorem 4.2. *Let $f(x) \stackrel{st}{\to} l\ (\overline{N}, \omega)$, where f and ω are real-valued, st-$D^n_+(\omega) \neq \emptyset$, and st-$D^n_-(\omega) \neq \emptyset$. Then $f(x) \stackrel{st}{\to} l$ if and only if both of (4.2)-(4.2*) are satisfied for all $\epsilon > 0$:*

$$\text{(4.2)} \qquad \inf_{\lambda \in st\text{-}D^n_+(\omega)} \limsup_{\mathbf{a}\to\infty} \frac{1}{a_1 a_2 \cdots a_n} \left|\left\{x : \mathbf{0} \le x \le \mathbf{a}, M^+_\lambda f(x;\omega) \le -\epsilon\right\}\right| = 0$$

and

$$\text{(4.2*)} \qquad \inf_{\lambda \in st\text{-}D^n_-(\omega)} \limsup_{\mathbf{a}\to\infty} \frac{1}{a_1 a_2 \cdots a_n} \left|\left\{x : \mathbf{0} \le x \le \mathbf{a}, M^-_\lambda f(x;\omega) \le -\epsilon\right\}\right| = 0.$$

Proof. Suppose $f(x) \stackrel{st}{\to} l$. Let $\epsilon > 0$. For $\lambda \in st$-$D^n_+(\omega)$, we have

$$\{x : \mathbf{0} \le x \le \mathbf{a}, M^+_\lambda f(x;\omega) \le -\epsilon\} \subseteq \{x : \mathbf{0} \le x \le \mathbf{a}, |M^+_\lambda f(x;\omega)| \ge \epsilon\}.$$

By Theorem 4.1, (4.2) holds. A similar argument also applies to (4.2*). Conversely, assume that both of (4.2) and (4.2*) hold. Write $f(x) - l = (\tilde{M}^+_\lambda f(x;\omega) - l) - M^+_\lambda f(x;\omega)$, where $\tilde{M}^+_\lambda f(x;\omega)$ is defined in the proof of Theorem 4.1. By (4.2) and (2.8), for $\epsilon > 0$ and $\delta > 0$, there exists $\lambda \in st$-$D^n_+(\omega)$ such that

$$\text{(4.3)} \qquad \limsup_{\mathbf{a}\to\infty} \frac{1}{a_1 a_2 \cdots a_n} \left|\left\{x : \mathbf{0} \le x \le \mathbf{a}, M^+_\lambda f(x;\omega) \le -\frac{\epsilon}{2}\right\}\right| < \delta$$

and

$$\limsup_{\mathbf{a}\to\infty} \frac{1}{a_1 a_2 \cdots a_n}\left|\left\{x : \mathbf{0} \le x \le \mathbf{a}, \tilde{M}_\lambda^+ f(x;\omega) - l \ge \frac{\epsilon}{2}\right\}\right| = 0. \tag{4.4}$$

Putting (4.3)-(4.4) together first and then letting $\delta \searrow 0$ yields

$$\limsup_{\mathbf{a}\to\infty} \frac{1}{a_1 a_2 \cdots a_n}\left|\left\{x : \mathbf{0} \le x \le \mathbf{a}, f(x) - l \ge \epsilon\right\}\right| = 0. \tag{4.5}$$

On the other hand, consider the expression $f(x)-l = (\tilde{M}_\lambda^- f(x;\omega)-l)+M_\lambda^- f(x;\omega)$, where

$$\tilde{M}_\lambda^- f(x;\omega) = \left(\triangle_\lambda^{\mathbf{1}} W(x)\right)^{-1} \int_{[\lambda_1(x_1),x_1]\times\cdots\times[\lambda_n(x_n),x_n]} f(y)\omega(y)dy.$$

To modify the above proof by changing (4.2) and (2.8) to (4.2*) and (2.8*), respectively, we see that

$$\limsup_{\mathbf{a}\to\infty} \frac{1}{a_1 a_2 \cdots a_n}\left|\left\{x : \mathbf{0} \le x \le \mathbf{a}, f(x) - l \le -\epsilon\right\}\right| = 0. \tag{4.6}$$

Putting (4.5) and (4.6) together, we get $f(x) \xrightarrow{st} l$. This completes the proof. ∎

5. Tauberian Conditions from $f(x) \xrightarrow{st} l$ $(\overline{N},\omega)$ to $f(x) \to l$

We have seen in §4 that (4.1) and (4.1*) are not Tauberian conditions of (1.2). The purpose of this section is to find conditions under which such an implication holds. Consider the following slow oscillation conditions:

$$\inf_{\lambda\in st\text{-}D_+^n(\omega)} \left\{\limsup_{x\to\infty}\left(\sup_{x<u<\lambda(x)} \left|f(u)-f(x)\right|\right)\right\} = 0 \tag{5.1}$$

and

$$\inf_{\lambda\in st\text{-}D_-^n(\omega)} \left\{\limsup_{x\to\infty}\left(\sup_{\lambda(x)<u<x} \left|f(x)-f(u)\right|\right)\right\} = 0. \tag{5.1*}$$

Clearly, (5.1) $\Longrightarrow$ (3.1) and (5.1*) $\Longrightarrow$ (3.1*). For $\omega \in st$-SVA (see §6 for the definition), we have (1.4) $\Longrightarrow$ (5.1) and (1.4) $\Longrightarrow$ (5.1*). The following is an integral analogue of [2, Corollary 2.2]. Our result not only extends [6, Corollary 2] and [12, Corollary 2] from 1-dimensional case to n-dimensional case, but also relaxes the $(\overline{N},\omega)$ summability to its statistical version.

Theorem 5.1. *Let* $f(x) \stackrel{st}{\to} l\ (\overline{N}, \omega)$*, where* $\omega \geq 0$*. If* $st\text{-}D_+^n(\omega) \neq \emptyset$ *(respectively* $st\text{-}D_-^n(\omega) \neq \emptyset$*) and (5.1) (respectively (5.1*)) holds, then* $f(x) \to l$*.*

Proof. We show the case of (5.1) and leave (5.1*) to the readers. It is clear that

$$|M_\lambda^+ f(x;\omega)| \leq \sup_{x<u<\lambda(x)} |f(u) - f(x)|.$$

Thus, (5.1) $\Longrightarrow$ (4.1). By Theorem 4.1(i), $f(x) \stackrel{st}{\to} l$. Putting this with Theorem 3.1, we get the desired result. ∎

Next, assume that f and ω are real-valued. Instead of (5.1)-(5.1*), we consider the following slow decrease conditions:

$$\sup_{\lambda \in st\text{-}D_+^n(\omega)} \left\{ \liminf_{x\to\infty} \left(\inf_{x<u<\lambda(x)} (f(u) - f(x)) \right) \right\} \geq 0 \tag{5.2}$$

and

$$\sup_{\lambda \in st\text{-}D_-^n(\omega)} \left\{ \liminf_{x\to\infty} \left(\inf_{\lambda(x)<u<x} (f(x) - f(u)) \right) \right\} \geq 0. \tag{5.2*}$$

It is clear that (5.2) $\Longrightarrow$ (3.4) and (5.2*) $\Longrightarrow$ (3.4*). The following is an integral analogue of [2, Corollary 2.4]. It extends [6, Corollary 1] and [12, Corollary 1] from 1-dimensional case to n-dimensional case, and relaxes the $(\overline{N}, \omega)$ summability to its statistical version.

Theorem 5.2. *Let* f *be real-valued. Assume that* $f(x) \stackrel{st}{\to} l\ (\overline{N}, \omega)$ *and (5.2)-(5.2*) hold, where* $\omega \geq 0$*,* $st\text{-}D_+^n(\omega) \neq \emptyset$*, and* $st\text{-}D_-^n(\omega) \neq \emptyset$*. Then* $f(x) \to l$*.*

Proof. The inequality

$$M_\lambda^+ f(x;\omega) \geq \inf_{x<u<\lambda(x)} (f(u) - f(x)),$$

where $\omega \geq 0$, shows the fact that (5.2) $\Longrightarrow$ (4.2). Similarly, (5.2*) $\Longrightarrow$ (4.2*). By Theorem 4.2, $f(x) \stackrel{st}{\to} l$. We have (5.2) $\Longrightarrow$ (3.4) and (5.2*)$\Longrightarrow$(3.4*), so Theorem 3.2 ensures that $f(x) \to l$. ∎

6. Other Tauberian Conditions

In §3-§5, the Tauberian conditions introduced there involve the classes: $\mathcal{S}_I, D_+^n$, D_-^n, $st\text{-}D_+^n(\omega)$, and $st\text{-}D_-^n(\omega)$. We have

$$st\text{-}D_+^n(\omega) \subseteq D_+^n \subseteq \mathcal{S}_I \quad \text{and} \quad st\text{-}D_-^n(\omega) \subseteq D_-^n \subseteq \mathcal{S}_I.$$

In the following, we shall further investigate the subclasses of st-$D^n_+(\omega)$ and st-$D^n_-(\omega)$, and then derive new types of Tauberian conditions.

Following [1, 2, 3], we write $\omega \in st$-SVA if $\omega(x) = \omega_1(x_1)\cdots\omega_n(x_n)$ and

$$st\text{-}\liminf_{t\to\infty}\left|\frac{W_k(\rho t)}{W_k(t)} - 1\right| > 0 \text{ for all } \rho > 0 \text{ with } \rho \neq 1 \qquad (k = 1, \cdots, n),$$

where $W_k(t) = \int_0^t \omega_k(z)dz$. For the definition of "st-$\liminf$", we refer the readers to [7, 11]. It is obvious that st-SVA is the n-dimensional statistical version of SVA defined in [1, 2, 3]. For $\omega \in st$-SVA,

$$st\text{-}D^n_+(\omega) \supseteqq \{\lambda_\rho : \rho > 1\} \text{ and } st\text{-}D^n_-(\omega) \supseteqq \{\lambda_\rho : 0 < \rho < 1\},$$

where λ_ρ denotes the mapping $x \mapsto \rho x$. Moreover, the following result is true.

Corollary 6.1. *We have* $(1.4) \Longrightarrow (5.1) \Longrightarrow (3.1)$ *and* $(6.1) \Longrightarrow (5.1^*) \Longrightarrow (3.1^*)$, *where* $\omega \in st$-*SVA and*

$$\inf_{0<\rho<1}\left\{\limsup_{x\to\infty}\left(\sup_{\rho x<u<x}\big|f(x) - f(u)\big|\right)\right\} = 0. \tag{6.1}$$

Hence, the conclusions of Theorems 3.1 and 5.1 remain true, if the Tauberian conditions involved there are replaced by any of (1.4) *and* (6.1).

It is easy to see that for $\rho > 1$,

$$\begin{aligned}\sup_{x<u<\rho x}|f(u) - f(x)| &\leq \sup_{x<u<\rho x}\left(\sum_{k=1}^n|\tilde f(x,k,u) - \tilde f(x,k+1,u)|\right)\\ &\leq \sum_{k=1}^n\sup_{y\geq x}\left(\sup_{\substack{y_k<v_k<\rho y_k\\ v_\ell=y_\ell\ for\ \ell\neq k}}|f(v) - f(y)|\right),\end{aligned} \tag{6.2}$$

where $\tilde f(x,k,u) = f(x_1,\cdots,x_{k-1},u_k,u_{k+1},\cdots,u_n)$. This leads us to the following consequence of Corollary 6.1.

Corollary 6.2. *We have* $(1.3) \Longrightarrow (1.4)$ *and* $(6.3) \Longrightarrow (6.1)$, *where*

$$\inf_{0<\rho<1}\left\{\limsup_{x\to\infty}\left(\sup_{\substack{\rho x_k<u_k<x_k\\ u_\ell=x_\ell\ for\ \ell\neq k}}\big|f(x) - f(u)\big|\right)\right\} = 0 \qquad (k = 1,\cdots,n). \tag{6.3}$$

Hence, for $\omega \in st$-*SVA, the conclusions of Theorems 3.1 and 5.1 remain true, if the Tauberian conditions involved there are replaced by any of* (1.3) *and* (6.3).

It should be noticed that (1.3) $\Longleftrightarrow$ (6.3). Moreover, Corollary 6.2 is an integral analogue of [2, Corollaries 3.1 & 3.2] for Schmidt-type condition. Our result extends [9, Corollary 3], [14, Theorem 4] and [15, Corollary 3] from $\omega_k = 1$ to general ω, and relaxes the $(\overline{N}, \omega)$ summability to its statistical version.

For $\rho > 1$ and $x_k < s < \rho x_k$, set

$$f^*(x, k, s) = f(x_1, \cdots, x_{k-1}, s, x_{k+1}, \cdots, x_n).$$

By the Mean-Value Theorem, there exists $t \in (x_k, s)$ such that

$$\begin{aligned} |f^*(x,k,s) - f(x)| &\le 2t \left| \left(\frac{\partial f}{\partial x_k} \right)^* (x,k,t) \right| (\rho - 1) \\ &\le 2M(\rho - 1) \quad \text{ for } \min(s, x_1, \cdots, x_n) \ge N_0, \end{aligned}$$

where M is a suitable constant satisfying

$$\text{(6.4)} \qquad s \left| \left(\frac{\partial f}{\partial x_k} \right)^* (x,k,s) \right| \le M \quad (\min(s, x_1, \cdots, x_n) \ge N_0;\ k = 1, \cdots, n).$$

Hence, Corollary 6.2 has the following consequence.

Corollary 6.3. *We have* $(6.4) \Longrightarrow (1.3)$. *Hence, for* $\omega \in st$*-SVA, the conclusions of Theorems 3.1 and 5.1 remain true, if the Tauberian conditions involved there are replaced by* (6.4).

Corollary 6.3 is an integral analogue of [2, Corollaries 3.1 & 3.2] for the Hardy-type condition. It extends [9, Corollary 4] and [14, Corollary 4] from $\omega_k = 1$ to general ω, and relaxes the $(\overline{N}, \omega)$ summability to its statistical version.

For real-valued f, the Tauberian conditions (1.4) and (6.1) are replaced by (6.5) and (6.5*), stated below:

$$\text{(6.5)} \qquad \sup_{\rho > 1} \left\{ \liminf_{x \to \infty} \left(\inf_{x < u < \rho x} (f(u) - f(x)) \right) \right\} \ge 0,$$

$$\text{(6.5*)} \qquad \sup_{0 < \rho < 1} \left\{ \liminf_{x \to \infty} \left(\inf_{\rho x < u < x} (f(x) - f(u)) \right) \right\} \ge 0.$$

Like Corollary 6.1, we have the following result.

Corollary 6.4. *Let* f *be real-valued. Then* $(6.5) \Longrightarrow (5.2) \Longrightarrow (3.4)$ *and* $(6.5^*) \Longrightarrow (5.2^*) \Longrightarrow (3.4^*)$, *where* $\omega \in st$*-SVA. Hence, the conclusions of Theorems 3.2 and 5.2 remain true, if the Tauberian conditions involved there are replaced by* (6.5) *and* (6.5^*), *respectively.*

For real-valued f, (6.2) is changed to

$$\inf_{x<u<\rho x}(f(u)-f(x)) \geq \sum_{k=1}^{n} \inf_{y\geq x}\Bigg(\inf_{\substack{y_k<v_k<\rho y_k \\ v_\ell=y_\ell \ for \ \ell\neq k}} (f(v)-f(y))\Bigg).$$

This lead us to the following consequence of Corollary 6.4.

Corollary 6.5. *Let f be real-valued. Then* $(6.6) \Longrightarrow (6.5)$ *and* $(6.6^*) \Longrightarrow (6.5^*)$*, where*

$$\text{(6.6)} \quad \sup_{\rho>1}\Bigg\{\liminf_{x\to\infty}\Bigg(\inf_{\substack{x_k<u_k<\rho x_k \\ u_\ell=x_\ell \ for \ \ell\neq k}} \big(f(u)-f(x)\big)\Bigg)\Bigg\} \geq 0 \qquad (k=1,\cdots,n),$$

$$\text{(6.6*)} \quad \sup_{0<\rho<1}\Bigg\{\liminf_{x\to\infty}\Bigg(\inf_{\substack{\rho x_k<u_k<x_k \\ u_\ell=x_\ell \ for \ \ell\neq k}} \big(f(x)-f(u)\big)\Bigg)\Bigg\} \geq 0 \qquad (k=1,\cdots,n).$$

Hence, for $\omega \in st$-SVA, the conclusions of Theorems 3.2 and 5.2 remain true, if the Tauberian conditions involved there are replaced by (6.6) *and* (6.6^*)*, respectively.*

Conditions (6.6) and (6.6*) are known as Landau-type conditions. The readers can check that (6.7) implies both of (6.6) and (6.6*), where

$$\text{(6.7)} \quad \begin{gathered} s\left\{\left(\frac{\partial f}{\partial x_k}\right)^*(x,k,s)\right\} \geq -M \\ (\min(s,x_1,\cdots,x_n) \geq N_0;\ k=1,\cdots,n), \end{gathered}$$

where $N_0 > 0$ and $M > 0$ are suitable constants. This gives the following result.

Corollary 6.6. *Let f be real-valued. Then* $(6.7) \Longrightarrow (6.6)$ *and* $(6.7) \Longrightarrow (6.6*)$. *Hence, for $\omega \in st$-SVA, the conclusions of Theorems 3.2 and 5.2 remain true, if the Tauberian conditions involved there are replaced by* (6.7).

Corollaries 6.5 and 6.6 are the integral analogues of [2, Corollaries 3.4 & 3.5]. Our results extend [9, Corollaries 1 & 2], [14, Theorem 3] and [15, Corollaries 1 & 2] from $\omega_k = 1$ to general ω, and relax the $(\overline{N},\omega)$ summability to its statistical version.

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