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NONLINEAR MEAN ERGODIC THEOREMS*

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Abstract. Let C be a nonempty subset (not necessarily closed and convex) of a Hilbert space, and $T : C \to C$ be a nonlinear mapping (not necessarily asymptotically nonexpansive). In this paper, we study the convergence of $(1/n)\sum_{i=0}^{n-1} T^i x (x \in C)$ as $n \to \infty$.

Introduction

Let X be a Banach space, C a subset of X and $T : C \to C$ be a nonlinear mapping. We are concerned with the convergence of $(1/n)\sum_{i=0}^{n-1} T^i x (x \in C)$ as $n \to \infty$). In order to investigate such problem it is usually assumed that C is closed and convex and T is *nonexpansive* on C, i.e., T satisfies

$$\|Tu - Tv\| \leq \|u - v\| \quad for \quad u, v \in C. \tag{0.1}$$

Under such conditions on C and T, Baillon [1] established the following nonlinear ergodic theorem: If X is a Hilbert space and T has a fixed point, then for every $x \in C, \{T^n x\}$ is weakly almost-convergent to a fixed point y of T, i.e., w - $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} T^{i+k} x = y$ uniformly in $k \geq 0$, and $Ty = y$. This result has been extended to the case that X is a uniformly convex Banach space with Fréchet differentiable norm by Bruck [6] and Reich [14] and a uniformly convex Banach space satisfying the Opial condition by Hirano [8]. On the other hand, Baillon [2] also proved the following strong ergodic theorem:

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If T is nonexpansive on a closed convex subset C of a Hilbert space and it is *odd*, i.e.,

$$-C = C \quad \text{and} \quad T(-u) = -Tu \quad \text{for} \quad u \in C, \tag{0.2}$$

then for every $x \in C, \{T^n x\}$ is strongly almost-convergent to a fixed point of T. Brezis and Browder [4] showed that the Baillon result remains true even if the oddness of T is weakened as follows:

$$\begin{aligned} 0 \in C \text{ and } \|Tu + Tv\|^2 \leq \ & \|u+v\|^2 + c[\|u\|^2 - \|Tu\|^2 \\ & + \|v\|^2 - \|Tv\|^2] \text{ for } u, v \in C, \end{aligned} \tag{0.3}$$

where c is a nonnegative constant. We see that if T is a nonexpansive mapping on a closed convex subset C of a Hilbert space and (0.3) is satisfied, then T satisfies

$$\text{for} \quad u, v \in C, \lim_{n\to\infty} \|T^{n+i}u - T^n v\| \quad \text{exists uniformly in } i \geq 0. \tag{0.4}$$

Bruck [5] proved that if C is a closed convex subset of a Hilbert space X and $T : C \to C$ is a nonexpansive mapping satisfying (0.4) and has a fixed point then for every $x \in C, \{T^n x\}$ is strongly almost-convergent to a fixed point of T. Later, Kobayasi and Miyadera [10] showed that the Bruck result remains true even if X is a uniformly convex Banach space. Hirano and Takahashi [9] and Oka [11, 12] showed that the nonexpansivity of T in the above-mentioned ergodic theorems can be weakened as follows:

$$\|T^k u - T^k v\| \leq a_k \|u - v\| \quad \text{for} \quad u, v \in C \quad \text{and} \quad k \geq 0, \tag{0.5}$$

where a_k are nonnegative constants with $\lim_{k\to\infty} a_k = 1$. A mapping T satisfying (0.5) is said to be *asymptotically nonexpansive* (in the usual sense). If $T : C \to C$ is asymptotically nonexpansive and odd then it satisfies

$$\|T^k u + T^k v\| \leq a_k \|u + v\| \quad \text{for} \quad u, v \in C \quad \text{and} \quad k \geq 0, \tag{0.6}$$

where a_k are nonnegative constants with $\lim_{k\to\infty} a_k = 1$.

Recently, Wittmann [15] proved the following interesting theorem: If C is a subset of a Hilbert space and $T : C \to C$ satisfies condition (0.6), then $\{(1/n)\sum_{i=0}^{n-1} T^i x\}$ is strongly convergent for every $x \in C$. It should be noted here that the closedness and convexity of C and the asymptotic nonexpansivity of T are not assumed.

In Section 1 of this paper we first show that condition (0.6) in the Wittman theorem can be replaced by a weaker condition (a_1). (See Theorem 1.1.) The condition (0.3) due to Brezis and Browder also implies condition (a_1). It is proved that if C is a subset of a Hilbert space and $T : C \to C$ satisfies condition (a_1) then for every $x \in C, \{T^n x\}$ is strongly almost-convergent to its asymptotic center. (See Theorem 1.1.) We shall also establish strong ergodic theorems for mappings satisfying condition (a_2) of asymptotically nonexpansive type or condition (a_3) of asymptotically noncontractive type. (See Theorems 1.2 and 1.3.) In Section 2 we deal with weak ergodic theorems, and we prove that if C is a subset of a Hilbert space, and $T : C \to C$ satisfies condition (a_2) and has a fixed point, then for every $x \in C, \{T^n x\}$ is weakly almost-convergent to its asymptotic center. (See Theorem 2.1.) The key to our ergodic theorems is Propositions 1.4 and 2.3. It is interesting that these propositions can be extended to L^p spaces, where p is an integer with $p \geq 2$. The results are stated in Section 5. Section 3 is devoted to applications to the space L^4, and examples are given in Section 4.

1. Strong Ergodic Theorems

Throughout this section, let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|, C$ a subset of H and let $T : C \to C$ be a mapping. It should be noted that the closedness and convexity of C are not assumed here. The set of fixed points of T will be denoted by $F(T)$. The main results in this section are stated as follows:

Theorem 1.1 *Suppose that for every bounded set $B \subset C$ and integer $k \geq 0$ there exists a $\delta_k(B) \geq 0$ with $\lim_{k\to\infty} \delta_k(B) = 0$ such that*

$$(a_1)\qquad \begin{aligned}\|T^k u + T^k v\|^p \leq\ & a_k\|u + v\|^p + c[a_k\|u\|^p \\ & -\|T^k u\|^p + a_k\|v\|^p - \|T^k v\|^p] + \delta_k(B)\end{aligned}$$

for $u, v \in B$, where a_k, c and p are nonnegative constants independent of B such that $\lim_{k\to\infty} a_k = 1$ and $p \geq 1$. Then for every $x \in C, \{T^n x\}$ is strongly almost-convergent to its asymptotic center y, i.e., $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} T^{i+k}x = y$ uniformly in $k \geq 0$.

In particular we have

Theorem 1.1′**.** *If T satisfies*

$$(a'_1)\quad \|T^k u + T^k v\|^p \leq a_k\|u + v\|^p + c[a_k\|u\|^p - \|T^k u\|^p + a_k\|v\|^p - \|T^k v\|^p]$$

for $u, v \in C$ and $k \geq 0$, where a_k, c and p are the same constants as in condition (a_1)*, then for every $x \in C, \{T^n x\}$ is strongly almost-convergent to its asymptotic center.*

Theorem 1.2. *Suppose that for every bounded set $B \subset C$ and integer $k \geq 0$ there exists a $\delta_k(B) \geq 0$ with $\lim_{k\to\infty} \delta_k(B) = 0$ such that*

$$(\mathrm{a}_2) \qquad \begin{aligned} \|T^k u - T^k v\|^p \leq\ & a_k\|u-v\|^p + c[a_k\|u\|^p \\ & -\|T^k u\|^p + a_k\|v\|^p - \|T^k v\|^p] + \delta_k(B) \end{aligned}$$

for $u, v \in B$, where a_k, c and p are the same constants as in condition (a_1). *If either $F(T) \neq \emptyset$ or $c > 0$ in* (a_2), *and if $x \in C$ satisfies*

$$(1.1) \qquad \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i\geq 0}[\|T^{m+i}x - T^m x\|^2 - \|T^{n+i}x - T^n x\|^2] \leq 0,$$

then $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

In particular we have

Theorem 1.2′. *Suppose that T satisfies*

$$(\mathrm{a}'_2) \quad \|T^k u - T^k v\|^p \leq a_k\|u-v\|^p + c[a_k\|u\|^p - \|T^k u\|^p + a_k\|v\|^p - \|T^k v\|^p]$$

for $u, v \in C$ and $k \geq 0$, where a_k, c and p are the same constants as in condition (a_1). *If either $F(T) \neq \emptyset$ or $c \geq 0$ in* (a'_2), *and if $x \in C$ satisfies* (1.1), *then $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.*

Theorem 1.3. *Suppose that T satisfies*

$$(\mathrm{a}_3) \quad \|u - v\|^p \leq a_k\|T^k u - T^k v\|^p + c[a_k\|T^k u\|^p - \|u\|^p + a_k\|T^k v\| - \|v\|^p]$$

for $u, v \in C$ and $k \geq 0$, where a_k, c and p are the same constants as in condition (a_1).

(I) *If $x \in C$ and $\{\|T^n x\|\}$ is convergent, then $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.*

(II) *If either $F(T) \neq \emptyset$ or $c > 0$ in* (a_3), *then for every $x \in C$, either* $\lim_{n\to\infty}\|T^n x\| = \infty$ *or $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.*

Remark. Let $\{x_n\}$ be a bounded sequence in H. It is known that there exists a unique element $y \in H$ such that $\overline{\lim}_{n\to\infty}\|x_n - y\| < \overline{\lim}_{n\to\infty}\|x_n - z\|$ for every $z \in H\backslash\{y\}$. The element y is called the *asymptotic center of* $\{x_n\}$.

The Wittmann condition (0.6) is the case of (a_1) with $c = 0$ and $\delta_k(B) \equiv 0$ (i.e., the case of (a'_1) with $c = 0$). There exists a mapping $T : C \to C$ such that C is closed, convex and T satisfies condition (a'_1) but does not (0.6). (See

Example 4.3 (ii).) Condition (0.3) due to Brezis and Browder also implies (a'_1) with $a_k \equiv 1$ and $p = 2$. Therefore, Theorem 1.1 extends [15, Theorem 2.2] and [4, Theorem 2]. We note that the limit y in Theorem 1.1 is not necessarily a fixed point of T even if C is closed and convex. (See [15, Example 3.2].) Condition (a_2) in Theorem 1.2 covers the notion of asymptotical nonexpansivity for mappings. Indeed, by the definition, a mappimg $T : C \to C$ is *asymptotically nonexpansive* (in the usual sense) if and only if it satisfies condition (a'_2) with $c = 0$. It is also seen that if $T : C \to C$ is *asymptotically nonexpansive in the intermediate sense* (see [7] and [8]), then it satisfies condition (a_2) with $a_k \equiv 1, p = 1$ and $c = 0$. Moreover, (1.1) is satisfied if $\lim_{n\to\infty} \|T^{n+i}x - T^n x\|^2$ exists uniformly in $i \geq 0$. So, Theorem 1.2 improves on [5, Theorem 2.1]. (We remark here that if C is closed and convex, $T : C \to C$ is asymptotically nonexpansive and $F(T) \neq \emptyset$, then for every $x \in C$ the asymptotic center of $\{T^n x\}$ is a fixed point of T.) Theorem 1.3 is a strong ergodic theorem for mappings T satisfying condition (a_3) of asymptotically noncontractive type.

It is interesting that the above-mentioned theorems can be proved in a unified way. In fact, the proofs of Theorems 1.1, 1.2 and 1.3 are based on the following

Proposition 1.4. *Let* $\{x_n\}$ *be a sequence in* H. *The following conditions* (i), (ii) *and* (iii) *are mutually equivalent:*

(i) $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty} \sup_{i\geq 0}[(x_{n+i}, x_n) - (x_{m+i}, x_m)] \leq 0$;

(ii) $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty} \sup_{i\geq 0}[\|x_{n+i} + x_n\|^2 - \|x_{m+i} + x_m\|^2] \leq 0$;

(iii) $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty} \sup_{i\geq 0}[\|x_{m+i} - x_m\|^2 - \|x_{n+i} - x_n\|^2] \leq 0$ *and* $\{\|x_n\|\}$ *is convergent.*

Moreover, if $\{x_n\}$ *satisfies the equivalent conditions above, then it is strongly almost-convergent to its asymptotic center.*

Proof. The equivalence of (i), (ii) and (iii) follows easily from the identity $\|x_{n+i} \pm x_n\|^2 = \|x_{n+i}\|^2 \pm 2(x_{n+i}, x_n) + \|x_n\|^2$ and the convergence of $\|x_n\|$. By virtue of [3, Lemma 3], (i) implies that $\{x_n\}$ is strongly almost-convergent to its asymptotic center. □

Remarks. 1) Let $\{x_n\}$ be a sequence in a normed space with norm $\|\cdot\|$. We see that if $\{x_n\}$ satisfies

$$\overline{\lim_{k,m,n\to\infty}}(\|x_{k+m} + x_{k+n}\|^2 - \|x_m + x_n\|^2) \leq 0, \tag{1.2}$$

then it satisfies (ii) in Proposition 1.4. 2) Wittmann [15, Theorem 2.3] has proved that if a sequence $\{x_n\}$ in H satisfies

$$(1.3)\quad \|x_{k+m}+x_{k+n}\|^2 \le \|x_m+x_n\|^2+\delta_k \text{ for } k,m,n\ge 0 \text{ with } \lim_{k\to\infty}\delta_k=0,$$

then $\{(1/n)\sum_{i=0}^{n-1} x_i\}$ is strongly convergent. Clearly, (1.3) implies (1.2). But the converse does not hold. For example, consider the sequence $\{1-1/(n+1)\}$ in $R^1=(-\infty,\infty)$. So, Proposition 1.4 improves on [15, Theorem 2.3].

As a direct cosequence of Proposition 1.4 we have

Proposition 1.5. *Let $x\in C$ and $f\in H$.*

(I) *If $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\sup_{i\ge 0}[\|T^{n+i}x+T^nx+2f\|^2-\|T^{m+i}x+T^mx+2f\|^2]\le 0$, then $\{T^nx\}$ is strongly almost-convergent to its asymptotic center.*

(II) *If $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\sup_{i\ge 0}[\|T^{m+i}x-T^mx\|^2-\|T^{n+i}x-T^nx\|^2]\le 0$, and $\{\|T^nx-f\|\}$ is convergent, then $\{T^nx\}$ is strongly almost-convergent to its asymptotic center.*

Proof. (I) Setting $x_n=T^nx+f$ for $n\ge 0,\{x_n\}$ satisfies condition (ii) in Proposition 1.4. Therefore $\{T^nx+f\}$ is strongly almost-convergent to its asymptotic center z, which implies that $\{T^nx\}$ is strongly almost-convergent to $z-f$ and $z-f$ is the asymptotic center of $\{T^nx\}$. (II) Setting $x_n=T^nx-f$ for $n\ge 0,\{x_n\}$ satisfies condition (iii) in Proposition 1.4. Therefore $\{T^nx-f\}$ is strongly almost-convergent to its asymptotic center z, which means that $\{T^nx\}$ is strongly almost-convergent to its asymptotic center $z+f$. □

Proof of Theorem 1.1. Let $x\in C$. By condition (a_1) with $B=\{x\}$ we have $\|T^kx\|^p\le a_k\|x\|^p+\delta_k(\{x\})/(2^p+2c)$ for $k\ge 0$. Therefore $\{T^nx;n\ge 0\}$ is bounded. Now, set $B=\{T^nx;n\ge 0\}$. By virtue of condition (a_1) with $u=v=T^nx$, we obtain $\|T^{k+n}x\|^p\le a_k\|T^nx\|^p+\delta_k(B)/(2^p+2C)$ for $k,n\ge 0$, which implies $\overline{\lim}_{k\to\infty}\|T^kx\|^p\le\|T^nx\|^p$ for $n\ge 0$. So that

$$(1.4)\qquad \{\|T^nx\|\}\quad\text{is convergent.}$$

Let $n>m\ge 0$. By condition (a_1) with $k=n-m, u=T^{m+i}x$ and $v=T^mx$ we obtain

$$\begin{aligned}\|T^{n+i}x+T^nx\|^p &\le \|T^{m+i}x+T^mx\|^p+[(2M)^p+2cM^p]|a_{n-m}-1|\\ &\quad +c(\|T^{m+i}x\|^p-\|T^{n+i}x\|^p+\|T^mx\|^p-\|T^nx\|^p)+\delta_{n-m}(B)\end{aligned}$$

for $i\ge 0$, where $M=\sup_{\ell\ge 0}\|T^\ell x\|$. Combining this with (1.4) we obtain

$$\overline{\lim_{m\to\infty}}\;\overline{\lim_{n\to\infty}}\sup_{i\ge 0}[\|T^{n+i}x+T^nx\|^p-\|T^{m+i}x+T^mx\|^p]\le 0,$$

which implies

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i\geq 0}[\|T^{n+i}x + T^n x\|^2 - \|T^{m+i}x + T^m x\|^2] \leq 0.$$

Therefore the theorem follows from Proposition 1.5 (I). □

Proof of Theorem 1.2. We first consider the case when $c = 0$ in condition (a_2), i.e., for every bounded set $B \subset C$ and integer $k \geq 0$ there exists a $\delta_k(B) \geq 0$ with $\lim_{k\to\infty} \delta_k(B) = 0$ such that

$$\|T^k u - T^k v\|^p \leq a_k \|u - v\|^p + \delta_k(B) \tag{1.5}$$

for $u, v \in B$, where $\lim_{k\to\infty} a_k = 1$ and $p \geq 1$. Let $x \in C$ and take an $f \in F(T)$. Considering the bounded set $B_0 = \{x, f\}$, we have $\|T^k x - f\|^p \leq a_k \|x - f\|^p + \delta_k(B_0)$ for $k \geq 0$, which shows that $\{T^n x; n \geq 0\}$ is a bounded set. Now, set $B = \{T^n x; n \geq 0\} \cup \{f\}$. By (1.5) we have

$$\|T^{k+n}x - f\|^p \leq a_k \|T^n x - f\|^p + \delta_k(B) \quad \text{for} \quad k, n \geq 0.$$

Letting $k \to \infty, \overline{\lim}_{k\to\infty} \|T^k x - f\| \leq \|T^n x - f\|$ for $n \geq 0$, which shows that $\{\|T^n x - f\|\}$ is convergent. By virtue of Proposition 1.5 (II), $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

We next consider the case when $c > 0$ in condition (a_2). Let $x \in C$. By (a_2) with $B = \{x\}$ we have $\|T^k x\|^p \leq a_k \|x\|^p + \delta_k(\{x\})/2c$ for $k \geq 0$. Therefore $\{T^n x; n \geq 0\}$ is a bounded set. Set $B = \{T^n x; n \geq 0\}$. By (a_2) with $u = v = T^n x$ we have $\|T^{k+n}x\|^p \leq a_k \|T^n x\|^p + \delta_k(B)/2c$ for $k, n \geq 0$. This shows that $\{\|T^n x\|\}$ is convergent. By virtue of Proposition 1.5 (II) again, $\{T^n x\}$ is strongly almost-convergent to its asymptotic center. □

Proof of Theorem 1.3. (I) Let $n > m \geq 0$. Using condition (a_3) with $u = T^{m+i}x, v = T^m x$ and $k = n - m$ we have

$$\begin{aligned}\|T^{m+i}x - T^m x\|^p &\leq \|T^{n+i}x - T^n x\|^p + [(2M)^p + 2cM^p]|a_{n-m} - 1| \\ &\quad + c[\|T^{n+i}x\|^p - \|T^{m+i}x\|^p + \|T^n x\|^p - \|T^m x\|^p] \quad \text{for} \quad i \geq 0,\end{aligned}$$

where $M = \sup_{\ell\geq 0} \|T^\ell x\|$. Combining this with the convergence of $\{\|T^n x\|\}$ we obtain $\overline{\lim}_{m\to\infty} \overline{\lim}_{n\to\infty} \sup_{i\geq 0} [\|T^{m+i}x - T^m x\|^p - \|T^{n+i}x - T^n x\|^p] \leq 0$, which implies

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i\geq 0}[\|T^{m+i}x - T^m x\|^2 - \|T^{n+i}x - T^n x\|^2] \leq 0.$$

It follows from Proposition 1.5 (II) that $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

(II) Let $x \in C$ and let $\underline{\lim}_{n\to\infty}\|T^n x\| < \infty$. We first consider the case when $c = 0$ in condition (a_3), i.e., $\|u - v\|^p \le a_k\|T^k u - T^k v\|^p$ for $u, v \in C$ and $k \ge 0$. Take an $f \in F(T)$. Using the above inequality with $u = T^n x$ and $v = f$, we have

$$\|T^n x - f\|^p \le a_k\|T^{k+n}x - f\|^p \quad \text{for} \quad k, n \ge 0.$$

Noting $\underline{\lim}_{k\to\infty}\|T^k x - f\| < \infty$, we see that $\{\|T^n x - f\|\}$ is convergent. Since

$$\begin{aligned}\|T^{m+i}x - T^m x\|^p &\le a_{n-m}\|T^{n+i}x - T^n x\|^p\\ &\le \|T^{n+i}x - T^n x\|^p + |a_{n-m} - 1|(2M)^p\end{aligned}$$

for $n > m \ge 0$ and $i \ge 0$, where $M = \sup_{\ell\ge 0}\|T^\ell x\|$, we have $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\sup_{i\ge 0}[\|T^{m+i}x - T^m x\|^p - \|T^{n+i}x - T^n x\|^p] \le 0$ and hence

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\sup_{i\ge 0}[\|T^{m+i}x - T^m x\|^2 - \|T^{n+i}x - T^n x\|^2] \le 0.$$

Therefore by Proposition 1.5 (II), $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

We next consider the case when $c > 0$ in condition (a_3). By condition (a_3) with $u = v = T^n x$ we have $\|T^n x\|^p \le a_k\|T^{k+n}x\|$ for $k, n \ge 0$, which implies that $\{\|T^n x\|\}$ is convergent. So by part (I), $\{T^n x\}$ is strongly almost-convergent to its asymptotic center. □

2. Weak Ergodic Theorems

Let H, C, T and $F(T)$ be as in Section 1. The main results in this section are the follwing theorems.

Theorem 2.1. *Suppose that for every bounded set $B \subset C$ and integer $k \ge 0$ there exists a $\delta_k(B) \ge 0$ with $\lim_{k\to\infty}\delta_k(B) = 0$ such that*

$$(a_2)\qquad \begin{aligned}\|T^k u - T^k v\|^p \le\, & a_k\|u - v\|^p + c[a_k\|u\|^p\\ & -\|T^k u\|^p + a_k\|v\|^p - \|T^k v\|^p] + \delta_k(B)\end{aligned}$$

for $u, v \in B$, where a_k, c and p are the same constants as in condition (a_1) *in Theorem* 1.1. *If either $F(T) \ne \emptyset$ or $c > 0$ in* (a_2), *then for every $x \in C$ $\{T^n x\}$ is weakly almost-convergent to its asymptotic center.*

In particular we have

Theorem 2.1′. *Suppose that T satisfies condition* (a'_2) *in Theorem 1.2′. If either $F(T) \ne \emptyset$ or $c > 0$ in* (a'_2), *then for every $x \subset C$ $\{T^n x\}$ is weakly almost-convergent to its asymptotic center.*

Remarks. 1) Theorem 2.1′ (and then Theorem 2.1) extends a main result in [9] because if T is asymptotically nonexpansive (in the usual sense) then it satisfies condition (a_2') with $c = 0$. As shown in Section 4 there exists a mapping $T : C \to C$ such that C is closed convex, T satisfies condition (a'_2) and $F(T) \neq \emptyset$ but it is not asymptotically nonexpansive (in the usual sense). (See Example 4.3.) 2) It follows from Theorem 2.1 that if $T : C \to C$ is asymptotically nonexpansive in the intermediate sense and $F(T) \neq \emptyset$, then for every $x \in C$, $\{T^n x\}$ is weakly almost-convergent to its asymptotic center.

Next, corresponding to Theorem 1.3 we have

Theorem 2.2. *Suppose that T satisfies*

$$(a_4) \qquad \begin{aligned} \|u+v\|^p \le\ & a_k\|T^k u + T^k v\|^p \\ & +c[a_k\|T^k u\|^p - \|u\|^p + a_k\|T^k v\|^p - \|v\|^p] \end{aligned}$$

for $u, v \in C$ and $k \ge 0$, where a_k, c and p are the same constants as in condition (a_1) *in Theorem* 1.1. *Then for every $x \in C$, either $\lim_{n\to\infty} \|T^n x\| = \infty$ or $\{\|T^n x\|\}$ is weakly almost-convergent to its asymptotic center.*

The proofs of Theorems 2.1 and 2.2 are based on the following propositions.

Proposition 2.3. *Let $\{x_n\}$ be a sequence in H and let $\{\|x_n\|\}$ be convergent. The following* (i), (ii) *and* (iii) *are mutually equivalent:*

(i) $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\overline{\lim}_{i\to\infty}[(x_{m+i}, x_m) - (x_{n+i}, x_n)] \le 0$;

(ii) $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\overline{\lim}_{i\to\infty}[\|x_{m+i} + x_m\|^2 - \|x_{n+i} + x_n\|^2] \le 0$;

(iii) $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\overline{\lim}_{i\to\infty}[\|x_{n+i} - x_n\|^2 - \|x_{m+i} - x_m\|^2] \le 0$.

Moreover, if $\{x_n\}$ satisfies the equivalent conditions above and $\{\|x_n\|\}$ is convergent then $\{x_n\}$ is weakly almost-convergent to its asymptotic center.

Proof. The equivalence of (i), (ii) and (iii) is a direct consequence of the identity $\|x_{n+i} \pm x_n\|^2 = \|x_{n+i}\|^2 \pm 2(x_{n+i}, x_n) + \|x_n\|^2$ and the convergence of $\|x_n\|$. By [5, Lemma 1.3], (i) and the convergence of $\|x_n\|$ imply that $\{x_n\}$ is weakly almost-convergent to its asymptotic center. □

Proposition 2.4. *Suppose that T satisfies condition* (a_2). *If $x \in C$ and $\{\|T^n x\|\}$ is convergent, then we have*

$$(2.1) \qquad \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i\ge 0}[\|T^{n+i}x - T^n x\|^2 - \|T^{m+i}x - T^m x\|^2] \le 0.$$

Proof. Set $B = \{T^n x; n \geq 0\}$. By condition (a_2) we have

$$\begin{aligned}\|T^{n+i}x - T^n x\|^p \leq\; & a_{n-m}\|T^{m+i}x - T^m x\|^p + c[a_{n-m}\|T^{m+i}x\|^p \\ & -\|T^{n+i}x\|^p + a_{n-m}\|T^m x\|^p - \|T^n x\|^p] + \delta_{n-m}(B) \\ \leq\; & \|T^{m+i}x - T^m x\|^p + |a_{n-m} - 1|[(2M)^p + 2cM^p] \\ & +c[\|T^{m+i}x\|^p - \|T^{n+i}x\|^p + \|T^m x\|^p - \|T^n x\|^p] \\ & +\delta_{n-m}(B)\end{aligned}$$

for $n > m \geq 0$ and $i \geq 0$, where $M = \sup_{\ell \geq 0} \|T^\ell x\|$. Combining this with the convergence of $\|T^n x\|$ we get

$$\overline{\lim_{m\to\infty}}\;\overline{\lim_{n\to\infty}}\sup_{i\geq 0}[\|T^{n+i}x - T^n x\|^p - \|T^{m+i}x - T^m x\|^p] \leq 0,$$

which implies (2.1). □

Proof of Theorem 2.1. We first consider the case when $c = 0$ in condition (a_2). Let $x \in C$ and take an $f \in F(T)$. Similarly as in the proof of Theorem 1.2, we see that $\{\|T^n x - f\|\}$ is convergent. Set $B = \{T^n x; n \geq 0\}$ and use (1.5) with $u = T^{m+i}x, v = T^m x$ and $k = n - m$. We see that if $n > m \geq 0$ then $\|T^{n+i}x - T^n x\|^p \leq a_{n-m}\|T^{m+i}x - T^m x\|^p + \delta_{n-m}(B)$ for $i \geq 0$. Therefore we have

$$\overline{\lim_{n\to\infty}}\sup_{i\geq 0}[\|T^{n+i}x - T^n x\|^p - \|T^{m+i}x - T^m x\|^p] \leq 0. \;\text{ for }\; m \geq 0,$$

which implies (2.1) and a fortiori

$$\overline{\lim_{m\to\infty}}\;\overline{\lim_{n\to\infty}}\;\overline{\lim_{i\to\infty}}[\|T^{n+i}x - T^n x\|^2 - \|T^{m+i}x - T^m x\|^2] \leq 0. \tag{2.2}$$

Thus $x_n \equiv T^n x - f$ satisfy condition (iii) in Proposition 2.3 and hence $\{T^n x - f\}$ is weakly almost-convergent to its asymptotic center z. So that $\{T^n x\}$ is weakly almost-convergent to its asymptotic center $z + f$.

We next consider the case when $c > 0$ in condition (a_2). Let $x \in C$. Similarly as in the proof of Theorem 1.2, it is shown that $\{\|T^n x\|\}$ is convergent. So, by Proposition 2.4 we have (2.1) and a fortiori (2.2). Using Proposition 2.3 with $x_n = T^n x$ we see that $\{T^n x\}$ is weakly almost-convergent to its asymptotic center. □

Proof of Theorem 2.2. Let $x \in C$ and suppose $\underline{\lim}_{n\to\infty}\|T^n x\| < \infty$. By condition ($a_4$) with $u = v = T^n x$ we have $\|T^n x\|^p \leq a_k\|T^{k+n}x\|^p$ for $k, n \geq 0$, which implies that $\{\|T^n x\|\}$ is convergent. Next, similarly as in the proof of Theorem 1.3 (I) we have $\overline{\lim}_{m\to\infty}\overline{\lim}_{n\to\infty}\sup_{i\geq 0}[\|T^{m+i}x+T^m x\|^2-\|T^{n+i}x+T^n x\|^2]\leq 0$ and a fortiori

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ \overline{\lim_{i\to\infty}}[\|T^{m+i}x+T^m x\|^2-\|T^{n+i}x+T^n x\|^2]\leq 0.$$

Therefore it follows from Proposition 2.3 that $\{T^n x\}$ is weakly almost-convergent to its asymptotic center. □

3. Applications

Our argument in the preceding sections can be applied to the real space $L^4(\Omega)$ with norm $\|\cdot\|$, where Ω is a measure space with measure μ.

We start with the following

Proposition 3.1. *If $\{x_n\}$ is a sequence in $L^4(\Omega)$ satisfying*

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\sup_{i\geq 0}[\|x_{n+i}+x_n\|^4-\|x_{m+i}+x_m\|^4]\leq 0 \tag{3.1}$$

and

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\sup_{i\geq 0}[\|x_{n+i}-x_n\|^4-\|x_{m+i}-x_m\|^4]\leq 0, \tag{3.2}$$

then $\{x_n^2\}$ is strongly almost-convergent to its asymptotic center in $L^2(\Omega)$.

Proof. We see from (3.1) that $\{\|x_n\|^4\}$ is convergent. Together with (3.1), (3.2) and [15, Lemma 3.3] we obtain

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\sup_{i\geq 0}[\|x_{n+i}^2+x_n^2\|_2^2-\|x_{m+i}^2+x_m^2\|_2^2]\leq 0,$$

where $\|\cdot\|_2$ denotes the norm in the space $L^2(\Omega)$. So the conclusion follows from Proposition 1.4. □

Theorem 3.2. *Let C be a subset of $L^4(\Omega)$ and $T: C\to C$ satisfy*

$$\begin{aligned}\|T^k u+T^k v\|^4\leq\ & a_k\|u+v\|^4+c[a_k\|u\|^4\\ &-\|T^k u\|^4+a_k\|v\|^4-\|T^k v\|^4]\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\|T^k u-T^k v\|^4\leq\ & a_k\|u-v\|^4+b[a_k\|u\|^4\\ &-\|T^k u\|^4+a_k\|v\|^4-\|T^k v\|^4]\end{aligned} \tag{3.4}$$

for $u, v \in C$ and $k \geq 0$, where a_k, c and b are constants such that $\lim_{k\to\infty} a_k = 1$, $c \geq 0$ and $b \geq 0$. Then for every $x \in C, \{(T^n x)^2\}$ is strongly almost-convergent to its asymptotic center in $L^2(\Omega)$.

Proof. Let $x \in C$. It follows from (3.3) that $\{\|T^n x\|\}$ is convergent. Let $n > m \geq 0$. Putting $u = T^{m+i}x, v = T^m x$ and $k = n - m$ in (3.3) we have $\|T^{n+i}x + T^n x\|^4 \leq \|T^{m+i}x + T^m x\|^4 + c[\|T^{m+i}x\|^4 - \|T^{n+i}x\|^4 + \|T^m x\|^4 - \|T^n x\|^4] + [(2M)^4 + 2cM^4]|a_{n-m} - 1|$ for $i \geq 0$. Combining this with the convergence of $\{\|T^n x\|\}$ we obtain

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i\geq 0}[\|T^{n+i}x + T^n x\|^4 - \|T^{m+i}x + T^m x\|^4] \leq 0.$$

Similarly we see from (3.4) that

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i\geq 0}[\|T^{n+i}x - T^n x\|^4 - \|T^{m+i}x - T^m x\|^4] \leq 0.$$

So by Proposition 3.1 we have the desired conclusion. □

The results above improve on [15, Corollaries 3.4 and 3.5].

Proposition 3.3. *If $\{x_n\}$ is a bounded sequence in $L^4(\Omega)$ satisfying*

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i\geq 0}[\|x_{m+i} + x_m\|^4 - \|x_{n+i} + x_n\|^4] \leq 0 \tag{3.5}$$

and

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i\geq 0}[\|x_{m+i} - x_m\|^4 - \|x_{n+i} - x_n\|^4] \leq 0, \tag{3.6}$$

then $\{x_n^2\}$ is weakly almost-convergent to its asymptotic center in $L^2(\Omega)$.

Proof. The boundedness of $\{x_n\}$ and (3.5) imply that $\{\|x_n\|\}$ is convergent. Combining this with (3.5), (3.6) and [15, Lemma 3.3] we obtain

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i\geq 0}[\|x_{m+i}^2 + x_m^2\|_2^2 - \|x_{n+i}^2 + x_n^2\|_2^2] \leq 0.$$

Since $\{\|x_n^2\|_2^2\}$ is convergent, the conclusion follows from Proposition 2.3. □

Theorem 3.4. *Let C be a subset of $L^4(\Omega)$ and $T : C \to C$ satisfy*

$$\begin{aligned} \|u + v\|^4 \leq\ & a_k\|T^k u + T^k v\|^4 \\ & + b[a_k\|T^k u\|^4 - \|u\|^4 + a_k\|T^k v\|^4 - \|v\|^4] \end{aligned} \tag{3.7}$$

and

$$\|u-v\|^4 \le a_k\|T^k u - T^k v\|^4 + c[a_k\|T^k u\|^4 - \|u\|^4 + a_k\|T^k v\|^4 - \|v\|^4] \tag{3.8}$$

for $u, v \in C$ and $k \ge 0$, where a_k, b and c are the same constants as in Theorem 3.2. Then for every $x \in C$, either $\lim_{n\to\infty}\|T^n x\| = \infty$ or $\{(T^n x)^2\}$ is weakly almost-convergent to its asymptotic center in $L^2(\Omega)$.

Proof. Let $x \in C$ and $\underline{\lim}_{n\to\infty}\|T^n x\| < \infty$. Taking $u = T^{m+i}x, v = T^m x$ and $k = n - m$ in (3.7) we have

$$\begin{aligned}\|T^{m+i}x + T^m x\|^4 \le & a_{n-m}\|T^{n+i}x + T^n x\|^4 + b[a_{n-m}\|T^{n+i}x\|^4 \\ & -\|T^{m+i}x\|^4 + a_{n-m}\|T^n x\|^4 - \|T^m x\|^4] \\ & \text{for} \quad n > m \ge 0 \text{ and } i \ge 0.\end{aligned} \tag{3.9}$$

In particular, taking $i = 0$ in (3.9) we obtain $\|T^m x\| \le (a_{n-m})^{1/4}\|T^n x\|$ for $n > m \ge 0$. Hence $\{\|T^n x\|\}$ is convergent. By (3.9) and the convergence of $\|T^n x\|$ we have

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ \sup_{i\ge 0}[\|T^{m+i}x + T^m x\|^4 - \|T^{n+i}x + T^n x\|^4] \le 0.$$

Similarly it follows from (3.8) and the convergence of $\|T^n x\|$ that

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ \sup_{i\ge 0}[\|T^{m+i}x - T^m x\|^4 - \|T^{n+i}x - T^n x\|^4] \le 0.$$

By virtue of Proposition 3.3, $\{(T^n x)^2\}$ is weakly almost-convergence to its asymptotic center in $L^2(\Omega)$. □

4. Examples

Example 4.1. Let C be a subset of a real Hilbert space H with norm $\|\cdot\|$ and $T : C \to C$ be a mapping.

I. If T satisfies

$$\|Tu + Tv\|^p \le \|u+v\|^p + c[\|u\|^p - \|Tu\|^p + \|v\|^p - \|Tv\|^p] \quad \text{for} \quad u, v \in C, \tag{4.1}$$

where $c \ge 0$ and $p \ge 1$ are constants, then for every $x \in C$, $\{T^n x\}$ is strongly almost- convergent to its asymptotic center.

II. Let T satisfy

$$\tag{4.2} \begin{aligned} \|Tu - Tv\|^p \le\ & \|u - v\|^p + c[\|u\|^p - \|Tu\|^p \\ & + \|v\|^p - \|Tv\|^p] \quad \text{for} \quad u, v \in C, \end{aligned}$$

where $c \ge 0$ and $p \ge 1$ are constants. If either $F(T) \neq \emptyset$ or $c > 0$ in (4.2), and if $x \in C$ satisfies (1.1),then $\{T^n x\}$ is strongly almost-convergent to its asymptotic center.

III. If T satisfies

$$\tag{4.3} \begin{aligned} \|u - v\|^p \le\ & \|Tu - Tv\|^p + c[\|Tu\|^p - \|u\|^p \\ & + \|Tv\|^p - \|v\|^p] \quad \text{for} \quad u, v \in C, \end{aligned}$$

where $c \ge 0$ and $p \ge 1$ are constants, then (I) and (II) of Theorem 1.3 hold true.

In fact, let T satisfy (4.1). Then by considering $T^j u$ and $T^j v$ instead of u and v in (4.1) we have $\|T^{j+1}u + T^{j+1}v\|^p \le \|T^j u + T^j v\|^p + c[\|T^j u\|^p - \|T^{j+1}u\|^p + \|T^j v\|^p - \|T^{j+1}v\|^p]$ for $j \ge 0$. Adding these inequalities for $j = 0, 1, \cdots, k-1$ we have $\|T^k u + T^k v\|^p \le \|u + v\|^p + c[\|u\|^p - \|T^k u\|^p + \|v\|^p - \|T^k v\|^p]$ for $u, v \in C$ and $k \ge 0$, i.e., T satisfies condition (a$'_1$) with $a_k \equiv 1$. The result follows from Theorem 1.1$'$. Similarly, (4.2) and (4.3) imply (a$'_2$) and (a$_3$) with $a_k \equiv 1$ respectively. So, II and III are direct consequences of Theorems 1.2$'$ and 1.3 respectively.

Example 4.2. Let C and T be as in Example 4.1.

IV. If T satisfies (4.2) and if either $F(T) \neq \emptyset$ or $c > 0$ in (4.2), then for every $x \in C$, $\{T^n x\}$ is weakly almost-convergent to its asymptotic center.

V. If T satisfies

$$\tag{4.4} \begin{aligned} \|u + v\|^p \le\ & \|Tu + Tv\|^p + c[\|Tu\|^p - \|u\|^p \\ & + \|Tv\|^p - \|v\|^p] \quad \text{for} \quad u, v \in C, \end{aligned}$$

where $c \ge 0$ and $p \ge 1$ are constants, then for every $x \in C$, either $\lim_{n\to\infty} \|T^n x\| = \infty$ or $\{T^n x\}$ is weakly almost-convergent to its asymptotic center.

In fact, IV and V follow from Theorem 2.1$'$ and Theorem 2.2 respectively.

Finally we give examples of mappings $T : C \to C$ such that C are closed, convex, T are continuous and satisfy condition (a$'_2$) with $F(T) \neq \emptyset$ but they are not asymptotically nonexpansive (in the usual sense). The mapping T in our second example also satisfies condition (a$'_1$) but does not the Wittmann condition (0.6).

Example 4.3. Let φ be the Cantor ternary function. Define $f : [0,1] \to [0,1]$ and $g : [-1,1] \to [-1,1]$ by

$$f(s) = s(0 \leq s \leq 1/2), = \varphi(s)(1/2 < s \leq 1)$$

and

$$g(s) = -f(-s)(-1 \leq s \leq 0), = f(s)(0 < s \leq 1).$$

(i) Set $C = [0,1] \times [0,1] (\subset R^2 = (-\infty, \infty) \times (-\infty, \infty))$ and define $T : C \to C$ by $T(u,v) = (u, f(v))$ for $(u,v) \in C$. Then T is continuous, $F(T) \neq \emptyset$ and T is not astmptotically nonexpansive (in the usual sense). A simple computation yields

$$\begin{aligned} \|T(u_1,v_1) - T(u_2,v_2)\|^2 \leq\ & \|(u_1,v_1) - (u_2,v_2)\|^2 + [\|(u_1,v_1)\|^2 \\ & -\|T(u_1,v_1)\|^2 + \|(u_2,v_2)\|^2 - \|T(u_2,v_2)\|^2] \end{aligned} \tag{4.5}$$

for $(u_1,v_1),(u_2,v_2) \in C$, where $\|(u,v)\|^2 = u^2 + v^2$ for $(u,v) \in R^2$. Hence T satisfies condition $(\mathrm{a'}_2)$ with $a_k \equiv 1, p = 2$ and $c = 1$.

(ii) Set $C = [-1,1] \times [-1,1] (\subset R^2)$ and define $T : C \to C$ by $T(u,v) = (u, g(v))$ for $(u,v) \in C$. Then T is continuous and odd, $F(T) \neq \emptyset$ and T is not asymptotically nonexpansive (in the usual sense). Moreover, T satisfies (4.5) and then condition $(\mathrm{a'}_2)$ with $a_k \equiv 1, p = 2$ and $c = 1$. We also see that T satisfies condition $(\mathrm{a'}_1)$ with $a_k \equiv 1, p = 2$ and $c = 1$ but it does not satisfy the Wittmann condition (0.6).

5. Concluding Remarks

The key of our argument in the preceding sections is Propositions 1.4 and 2.3. We may generalize these propositions to the spaces $L^p(\Omega)$ with positive integers p in the following forms, where Ω is a measure space with measure μ.

Proposition 5.1. *Let $\{x_n\}$ be a sequence of real-valued functions in $L^p(\Omega)$. Suppose*

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sup_{i_1,i_2,\ldots,i_{p-1}\geq 0} \Big[\int_\Omega x_{n+i_1}(s)x_{n+i_2}(s)\cdots x_{n+i_{p-1}}(s)x_n(s)d\mu - \int_\Omega x_{m+i_1}(s)x_{m+i_2}(s)\cdots x_{m+i_{p-1}}(s)x_m(s)d\mu\Big] \leq 0.$$

(I) *If p is even, then $\{(l/n)\sum_{i=0}^{n-1} x_i\}$ is strongly convergent to the unique element of clco W of minimum norm, where W is the set of weak subsequential limits of $\{x_n\}$ and clco W denotes the closed convex hull of W.*

(II) *If p is odd and each x_n is nonnegative, then $\{(1/n)\sum_{i=0}^{n-1} x_i\}$ is strongly convergent to the unique element of clco W of minimum norm.*

Remark. We shall expect that the conclusion of Proposition 5.1 can be strengthened such form as $\{x_n\}$ is strongly almost-convergent to the unique element of clco W of minimum norm.

Proposition 5.2. *Let $\{x_n\}$ be a bounded sequence of real-valued functions in $L^p(\Omega)$. Suppose*

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ \overline{\lim_{i_1,i_2,\ldots,i_{p-1}\to\infty}} \left[\int_\Omega x_{m+i_1}(s)x_{m+i_2}(s)\cdots x_{m+i_{p-1}}(s)x_m(s)d\mu - \int_\Omega x_{n+i_1}(s)x_{n+i_2}(s)\cdots x_{n+i_{p-1}}(s)x_n(s)d\mu\right] \leq 0.$$

(I) *If p is even, then $\{x_n\}$ is weakly almost-convergent to the unique element of clco W of minimum norm, where W and clco W are the same as in Proposition* 5.1.

(II) *If p is odd and each x_n is nonnegative, then $\{x_n\}$ is weakly almost-convergent to the unique element of clco W of minimum norm.*

Added in Proof. (a_1) in Theorem 1.1, (a_2) in Theorem 1.2 and (a_2) in Theorem 2.1 can be replaced by the following weaker conditions.

(β_1) for every bounded set $B \subset C, v \in C$ and integer $k \geq 0$ there exists a $\delta_k(B,v) \geq 0$ with $\lim_{k\to\infty} \delta_k(B,v) = 0$ such that $\|T^k u + T^k v\|^p \leq a_k \|u+v\|^p + c[a_k\|u\|^p - \|T^k u\|^p + a_k\|v\|^p - \|T^k v\|^p] + \delta_k(B, v)$ for $u \in B$, where a_k, c and p are nonnegative constants independent of B and v such that $\lim_{k\to\infty} a_k = 1$ and $p \geq 1$,

(α_2) for every $u, v \in C$ and integer $k \geq 0$ there exists a $\delta_k(u,v) \geq 0$ with $\lim_{k\to\infty} \delta_k(u,v) = 0$ such that $\|T^k u - T^k v\|^p \leq a_k\|u-v\|^p + c[a_k\|u\|^p - \|T^k u\|^p + a_k\|v\|^p - \|T^k v\|^p] + \delta_k(u,v)$, where a_k, c and p are non-negative constants independent of u and v such that $\lim_{k\to\infty} a_k = 1$ and $p \geq 1$,

(β_2) for every bounded set $B \subset C,\ v \in C$ and integer $k \geq 0$ there exists a $\delta_k(B,v) \geq 0$ with $\lim_{k\to\infty} \delta_k(B,v) = 0$ such that $\|T^k u - T^k v\|^p \leq a_k \|u-v\|^p + c[a_k\|u\|^p - \|T^k u\|^p + a_k\|v\|^p - \|T^k v\|^p] + \delta_k(B,v)$ for $u \in B$, where a_k, c and p are the same constants as in condition (β_1) above,

respectively. Similarly, (a_3) in Theorem 1.3 and (a_4) in Theorem 2.2 can be also replaced by some weaker conditions.

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