TAIWANESE JOURNAL OF MATHEMATICS
Vol. 1, No. 4, pp. 417-431, December 1997

GRAM-SCHMIDT PROCESS OF ORTHONORMALIZATION IN BANACH SPACES*

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Abstract. Gram-Schmidt orthonormalization in Banach spaces is considered. Using this orthonormalization process we can prove that if P is a projection on a reflexive Banach space X with a basis $\{e_n; f_n\}$, then there exists a basis $\{u_n; g_n\}$ of X such that $\{g_n\} \approx \{f_n\}$ and the matrix of P with respect to $\{u_n; g_n\}$ has the property that all but a finite number of entries of each column and each row are zero.

1. Introduction

Hilbert spaces possess many beautiful properties which are derived from Gram-Schmidt Process of Orthonormalization. For instances, every separable closed subspace of a Hilbert space has an unconditional [resp. symmetric, orthogonal, etc] basis; any closed subspace of a separable Hilbert space is complemented; any infinite dimensional closed [complemented] subspace Y of an infinite dimensional separable Hilbert space H is isomorphic to H; $\cdots$ It naturally arises the question to determine Banach spaces which possess similar properties by using Gram-Schmidt Process of Orthonormalization. In this paper, we shall first introduce Gram-Schmidt Process of Orthonormalization in Banach spaces. By virtue of this orthonomalization process we can prove that if P is a projection on a reflexive Banach space X with a basis $\{e_n; f_n\}$, then there exists a basis $\{u_n; g_n\}$ of X such that $\{g_n\} \approx \{f_n\}$ and the matrix of P with respect to $\{u_n; g_n\}$ has the property that all but a finite number

Received March 1, 1997; revised June 4, 1997.
Communicated by P. Y. Wu.
1991 *Mathematics Subject Classification*: 46B25.
Key words and phrases: equivalent bases, process of orthonormalization, reduction of matrices of projections.
* Presented at the International Mathematics Conference '96, National Changhua University of Education, December 13–16, 1996. The Conference was sponsored by the National Science Council and the Math. Soc. of the R.O.C.

of entries of each column and each row are zero. This result will be used to study projections in reflexive Banach spaces with unconditional (symmetric) bases, in particular, to study primary Banach spaces. Recall that a Banach space X is primary if for all projection P on X, X is isomorphic to $P(X)$ or $(I-P)(X)$. For some results on primary Banach spaces, see [1, 3, 4, 5, 6 and 7].

Let $\{x_n\}$ be a sequence in a Banach space X. If there exists a sequence $\{f_n\}$ in X^* such that $f_i(x_j)=\delta_{ij}=\begin{cases} 0, & i\neq j \\ 1, & i=j. \end{cases}$ for all i,j in N, where δ_{ij} is the Kronecker delta, then we say that $\{x_n;f_n\}$ is a biorthogonal system in $X\times X^*$; in this situation, $\{x_n\}$ is called a minimal sequence in X. A sequence $\{x_n\}$ in X is called a (Schauder) basis for X if for every x in X, there exists a unique sequence $\{a_n\}$ of scalars such that $x=\sum\limits_{n=1}^{\infty} a_nx_n$; or, equivalently, if $\{x_n\}$ is a minimal sequence in X, then for every x in X, $x=\sum\limits_{n=1}^{\infty} f_n(x)x_n$, where $\{x_n;f_n\}$ is a biorthogonal system in $X\times X^*$. Hence $\{f_n\}$ is unique. We sometimes call $\{f_n\}$ the coordinate (or biorthogonal , or coefficient) functionals with respect to $\{x_n\}$. For convenience, we also call $\{x_n;f_n\}$ a basis for X. Other notions concerning bases can be seen in [14], [15] and [16]. By a sequence in N we mean an increasing sequence of positive integers unless otherwise stated. Let $\{x_n\}$ and $\{y_n\}$ be bases of Banach spaces X and Y, respectively. Then $\{x_n\}\approx\{y_n\}$ denotes that the bases $\{x_n\}$ and $\{y_n\}$ are equivalent; that is, there is an isomorphism U from X onto Y such that $Ux_n=y_n \quad (n\in N)$. The (closed linear) subspace spanned by the sequence $\{x_n\}$ is denoted by $[x_1,x_2,\ldots,x_n,\ldots]$, or $[\{x_n\}]$, or simply $[x_n]$. Throughout this paper, all subspaces are closed linear subspaces, and all operators are bounded linear operators unless specifically noted.

2. Gram-Schmidt Process of Orthonormalization

Let us characterize a linearly independent sequence in a Banach space X as follows:

Proposition 2.1. *Let $\{x_n\}$ be a (finite or infinite) sequence in X. Then the following are equivalent:*

(a) *$\{x_n\}$ is linearly independent.*

(b) *There exists a sequence $\{f_n\}$ in X^* such that*

$$\begin{aligned} &f_1(x_1)\neq 0;\ f_2(x_1)=0,\ f_2(x_2)\neq 0;\ f_3(x_1)=f_3(x_2)=0,\ f_3(x_3)\neq 0; \\ &f_4(x_1)=f_4(x_2)=f_4(x_3)=0, f_4(x_4)\neq 0;\ldots. \end{aligned} \tag{1}$$

(c) *There exists a sequence* $\{f_n\}$ *in* X^* *such that*

$$(2)\ \ f_1(x_1) \neq 0, \ \begin{vmatrix} f_1(x_1) & f_2(x_1) \\ f_1(x_2) & f_2(x_2) \end{vmatrix} \neq 0, \begin{vmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{vmatrix} \neq 0, \ldots.$$

Proof. "$(a) \Rightarrow (b)$". If $\{x_n\}$ is linearly independent, then $x_1 \neq 0$. Hence by Hahn-Banach Theorem, there exists $f_1 \in X^*$ such that $f_1(x_1) \neq 0$. By the linear independence of $\{x_n\}$, we have $x_2 \notin [x_1]$. Hence by Hahn-Banach Theorem again, there exists $f_2 \in X^*$ such that $f_2(x_1) = 0$ and $f_2(x_2) \neq 0$. Similarly, by the linear independence of $\{x_n\}$, we have $x_3 \notin [x_1, x_2]$.
Hence by Hahn-Banach Theorem again, there exists $f_3 \in X^*$ such that $f_3(x_1) = f_3(x_2) = 0$ and $f_3(x_3) \neq 0; \cdots$

"$(b) \Rightarrow (c)$" is obvious.

"$(c) \Rightarrow (a)$". Assume that condition (c) holds. Suppose that $\{x_n\}$ is linearly dependent. Then there exists $n \in N$ such that $\{x_1, x_2, ..., x_n\}$ is linearly dependent. Thus there are scalars $\alpha_1, \alpha_2, ..., \alpha_n$, not all zero, such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0$. Let k be the largest positive integer such that $\alpha_k \neq 0$. Then $x_k = \frac{-1}{\alpha_k}[\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{k-1} x_{k-1}]$.
Thus the determinant

$$\begin{vmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_k(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_k(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x_k) & f_2(x_k) & \cdots & f_k(x_k) \end{vmatrix} = 0,$$

which is absurd. q.e.d

Now we consider the Gram-Schmidt orthonormalization process as follows: Let $\{x_n\}$ be a (finite or infinite) linearly independent sequence in X and let $\{f_n\}$ in X^* such that (1) holds. Define

$$x_n^{(1)} = x_n - \frac{f_1(x_n)}{f_1(x_1)} x_1 \qquad \text{for } n \in N.$$

Then

$$x_n^{(1)} = \frac{\begin{vmatrix} f_1(x_1) & x_1 \\ f_1(x_n) & x_n \end{vmatrix}}{f_1(x_1)} \ \text{ and } \quad f_1(x_n^{(1)}) = 0 \quad \text{ for all } n \text{ in } N;\ f_2(x_1) = 0, \text{ and}$$

$$f_2(x_2^{(1)}) = \frac{\begin{vmatrix} f_1(x_1) & f_2(x_1) \\ f_1(x_2) & f_2(x_2) \end{vmatrix}}{f_1(x_1)} = f_2(x_2) \neq 0.$$

Moreover, for $n \in N$ and all $f \in X^*$ the determinant

$$
\begin{aligned}
&\begin{vmatrix} f_1(x_1) & f_2(x_1) & f(x_1) \\ f_1(x_2) & f_2(x_2) & f(x_2) \\ f_1(x_n) & f_2(x_n) & f(x_n) \end{vmatrix} \\
(3) \qquad &= \begin{vmatrix} f_1(x_1) & f_2(x_1) & f(x_1) \\ 0 & f_2(x_2) - \frac{f_1(x_2)}{f_1(x_1)} f_2(x_1) & f(x_2) - \frac{f_1(x_2)}{f_1(x_1)} f(x_1) \\ 0 & f_2(x_n) - \frac{f_1(x_n)}{f_1(x_1)} f_2(x_1) & f(x_n) - \frac{f_1(x_n)}{f_1(x_1)} f(x_1) \end{vmatrix} \\
&= \begin{vmatrix} f_1(x_1) & f_2(x_1) & f(x_1) \\ 0 & f_2(x_2^{(1)}) & f(x_2^{(1)}) \\ 0 & f_2(x_n^{(1)}) & f(x_n^{(1)}) \end{vmatrix}
\end{aligned}
$$

Define $x_n^{(2)} = x_n^{(1)} - \frac{f_2(x_n^{(1)})}{f_2(x_2^{(1)})} x_2^{(1)}$ for $n \in N$. Then $f_2(x_n^{(2)}) = 0$ for all n in N; $f_3(x_1) = f_3(x_2) = 0$, and for $n \in N$ and all $f \in X^*$,

$$
\begin{aligned}
f(x_n^{(2)}) &= \frac{\begin{vmatrix} f_2(x_2^{(1)}) & f(x_2^{(1)}) \\ f_2(x_n^{(1)}) & f(x_n^{(1)}) \end{vmatrix}}{f_2(x_2^{(1)})} = \frac{f_1(x_1) \begin{vmatrix} f_2(x_2^{(1)}) & f(x_2^{(1)}) \\ f_2(x_n^{(1)}) & f(x_n^{(1)}) \end{vmatrix}}{f_1(x_1) f_2(x_2^{(1)})} \\
&= \frac{\begin{vmatrix} f_1(x_1) & f_2(x_1) & f(x_1) \\ f_1(x_2) & f_2(x_2) & f(x_2) \\ f_1(x_n) & f_2(x_n) & f(x_n) \end{vmatrix}}{f_1(x_1) f_2(x_2^{(1)})} \text{ (from (3))} .
\end{aligned}
$$

Hence

$$
x_n^{(2)} = \frac{\begin{vmatrix} f_1(x_1) & f_2(x_1) & x_1 \\ f_1(x_2) & f_2(x_2) & x_2 \\ f_1(x_n) & f_2(x_n) & x_n \end{vmatrix}}{f_1(x_1) f_2(x_2^{(1)})} \text{ for } n \text{ in } N.
$$

Thus

$$
f_3(x_3^{(2)}) = \frac{\begin{vmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{vmatrix}}{f_1(x_1) f_2(x_2^{(1)})} = f_3(x_3) \neq 0 \quad \text{(from (1) and } f_1(x_1) \neq 0,
$$

$f_2(x_2^{(1)}) = f_2(x_2) \neq 0)$.

Also $f_3(x_2^{(1)}) = \dfrac{\begin{vmatrix} f_1(x_1) & f_3(x_1) \\ f_1(x_2) & f_3(x_2) \end{vmatrix}}{f_1(x_1)} = f_3(x_2) = 0$. And $f_j(x_n^{(2)}) = 0$ for $j = 1, 2$ and $n \in N$.

By induction, define

$$x_n^{(k)} = x_n^{(k-1)} - \frac{f_k(x_n^{(k-1)})}{f_k(x_k^{(k-1)})} x_k^{(k-1)} \qquad \text{for } n \in N,$$

provided that $x_n^{(k-1)}$ is defined and $k \in N$, with $f_k(x_1) = \cdots = f_k(x_{k-1}^{(k-2)}) = 0$ and $f_k(x_k^{(k-1)}) = f_k(x_k) \neq 0$. Then $f_k(x_n^{(k)}) = 0$ for $n \in N$; $f_k(x_1) = f_k(x_2) = \cdots = f_k(x_{k-1}) = 0$; and

$$x_n^{(k)} = \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_k(x_1) & x_1 \\ \cdots & \cdots & \cdots & \cdots \\ f_1(x_k) & \cdots & f_k(x_k) & x_k \\ f_1(x_n) & \cdots & f_k(x_n) & x_n \end{vmatrix}}{f_1(x_1) f_2(x_2^{(1)}) \cdots f_k(x_k^{(k-1)})}. \tag{4}$$

Hence

$$\begin{aligned} f_{k+1}(x_{k+1}^{(k)}) &= \frac{\begin{vmatrix} f_1(x_1) & \cdots & f_k(x_1) & f_{k+1}(x_1) \\ \cdots & \cdots & \cdots & \cdots \\ f_1(x_k) & \cdots & f_k(x_k) & f_{k+1}(x_k) \\ f_1(x_{k+1}) & \cdots & f_k(x_{k+1}) & f_{k+1}(x_{k+1}) \end{vmatrix}}{f_1(x_1) f_2(x_2^{(1)}) \cdots f_k(x_k^{(k-1)})} \\ &= f_{k+1}(x_{k+1}) \neq 0 \end{aligned} \tag{5}$$

(from (1); and $f_2(x_2^{(1)}) = f_2(x_2), \ldots, f_k(x_k^{(k-1)}) = f_k(x_k)$) and $f_j(x_n^{(k)}) = 0$ for $1 \leq j \leq k$ and $n \in N$.

Therefore $f_1(x_1) \neq 0$, $f_1(x_2^{(1)}) = f_1(x_3^{(2)}) = \cdots = 0$; $f_2(x_1) = 0$, $f_2(x_2^{(1)}) \neq 0, f_2(x_3^{(2)}) = f_2(x_4^{(3)}) = \cdots = 0$ and $f_3(x_1) = f_3(x_2^{(1)}) = 0, f_3(x_3^{(2)}) \neq 0, f_3(x_4^{(3)}) = f_3(x_5^{(4)}) = \cdots = 0; \cdots$ (from (1), (4), (5),...). Note: it is easily seen that $[x_1] = [x_1]$, $[x_1, x_2^{(1)}] = [x_1, x_2]$, $[x_1, x_2^{(1)}, x_3^{(2)}] = [x_1, x_2, x_3], \ldots$.

Consequently, we get a generalized Gram-Schmidt process for orthonormalizing the linearly independent sequence $\{x_1, x_2, \ldots\}$ in X as follows:

Let

$$y_1 = \frac{x_1}{f_1(x_1)}; \quad y_n = \frac{x_n^{(n-1)}}{f_n(x_n^{(n-1)})} \qquad \text{for } n > 1.$$

Then $\{y_n; f_n\}$ is a minimal sequence in X.

Remark. Let $\{x_n\}$ be a (finite or infinite) linearly independent sequence in X and let $\{f_n\}$ in X^* such that (2) holds. Then by the same way as above we can get a minimal sequence $\{y_n; g_n\}$ derived from $\{x_n; f_n\}$. But $\{y_n; f_n\}$ may not be a minimal sequence in general.

3. Reduction of the Matrices of Projections on Reflexive Banach Spaces with a Basis.

Let X be a reflexive Banach space with a basis $\{e_n; f_n\}$. One application of the generalized Gram-Schmidt Process of Orthonormalization is to reduce the matrices of projections on X relative to $\{e_n; f_n\}$. Let P be an operator on X. Set $a_{ij} = f_j(P(e_i))$ for all i, j in N. Then the matrix $[a_{ij}]$ is called the matrix of P with respect to $\{e_n; f_n\}$. The column vector $[a_{1j}, a_{2j}, \cdots]^\top$ is called the $j-th$ column of the matrix, and the row vector $[a_{i1}, a_{i2}, \cdots]$ is called the $i-th$ row of the matrix.

We first show by induction on k that there are three increasing (finite or infinite) sequences $\{r_k\}, \{p_k\}, \{n_k\}$ of positive integers, and $\{e_i^{(k)}\}$ in X, $\{a_{ij}^{(k)}\}$ in R $(k = 0, 1, 2 \cdots)$ $\quad (i, j \in N)$ given by

$$\begin{cases} e_i^{(0)} = e_i & \text{and} \\ e_i^{(k)} = e_i^{(k-1)} - \frac{a_{i,r_k}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} e_{p_k}^{(k-1)} & \text{for } k \geq 1 \end{cases}$$

for each $i \in N$, where

$$\begin{aligned} &a_{ij}^{(0)} = a_{ij} \qquad \text{and} \\ &a_{ij}^{(k)} = f_j(P(e_i^{(k)})) \quad \text{for } k \geq 1 \text{ and all } i, j \in N \quad \text{such that} \end{aligned}$$

(i) r_1 is the smallest positive integer so that $a_{i,r_1} \neq 0$ for some $i \in N$; p_1 is a positive integer so that $a_{p_1,r_1} \neq 0$; n_1 is an integer such that $p_1 < n_1$. Next, define

$$e_i^{(1)} = e_i - \frac{a_{i,r_1}}{a_{p_1,r_1}} e_{p_1},$$

and then $\quad a_{ij}^{(1)} = f_j(P(e_i^{(1)})) = a_{ij} - \frac{a_{i,r_1}}{a_{p_1,r_1}} a_{p_1,j} \quad (i, j, k \in N).$

(ii) r_k is the smallest positive integer so that $a_{i,r_k}^{(k-1)} = f_{r_k}(P(e_i^{(k-1)})) \neq 0$ for some $i \geq n_{k-1}$; p_k is a positive integer so that $a_{p_k,r_k}^{(k-1)} \neq 0$ and $p_k \geq n_{k-1}$; n_k is an integer such that $p_k < n_k$; provided that $r_{k-1}, p_{k-1}, n_{k-1}, e_i^{(k-1)}$ and $a_{ij}^{(k-1)}$ are defined and r_k exists.

Next, define

$$e_i^{(k)} = e_i^{(k-1)} - \frac{a_{i,r_k}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} e_{p_k}^{(k-1)},$$

and then $$a_{ij}^{(k)} = f_j(P(e_i^{(k)})) = a_{ij}^{(k-1)} - \frac{a_{i,r_k}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} a_{p_k,j}^{(k-1)} \quad (i, j \in N).$$

Lemma 3.1. (a) *If dim P(X)$< \infty$, then $\{r_i\}$ is a finite sequence, say $\{r_i\}_{i=1}^k$; and $a_{ij}^{(k)} = f_j(P(e_i^{(k)})) = 0$ for all $j \in N$ and $i \geq n_k$; and hence $P(e_i^{(k)}) = 0$ for $i \geq n_k$. In this case, $\{e_1, e_2, \ldots, e_{n_1-1}; e_{n_1}^{(1)}, e_{n_1+1}^{(1)}, \ldots, e_{n_2-1}^{(1)}; \ldots; e_{n_{k-1}}^{(k-1)}, e_{n_{k-1}+1}^{(k-1)}, \ldots, e_{n_k-1}^{(k-1)}; e_{n_k}^{(k)}, e_{n_k+1}^{(k)}, \ldots, e_m^{(k)}, \ldots\}$ is a basis. Let $\{u_n; g_n\}$ denote this basis. Then $g_n = f_n$ for $n \notin \{p_1, p_2, \ldots, p_k\}$. Hence $\{g_n\} \approx \{f_n\}$. And $P(u_i) = 0$ for $i \geq n_k$.*

(b) *If dim P(X)$= \infty$, then $\{r_n\}$ is an infinite sequence in N; and $f_j(P(e_i^{(k)})) = 0$ for $1 \leq j < r_{k+1}$ and $i \geq n_k$ (see Remark* (2) *and Note below and by the definition of r_n), where $n_0 = 1$.*

(c) *If $P(e_m) = 0$, then $e_m^{(n)} = e_m$ for all $n \in N$.*

Remark. (1) For every k, we first define r_k, next p_k and then n_k. (2) We easily derive from(4) that

$$a_{ij}^{(k)} = \frac{\begin{vmatrix} a_{p_1,r_1} & \cdots & a_{p_1,r_k} & a_{p_1,j} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p_k,r_1} & \cdots & a_{p_k,r_k} & a_{p_k,j} \\ a_{i,r_1} & \cdots & a_{i,r_k} & a_{i,j} \end{vmatrix}}{a_{p_1,r_1} \cdot a_{p_1,r_2}^{(1)} \cdots a_{p_k,r_k}^{(k-1)}} \tag{6}$$

if $a_{ij}^{(k)}$ and $e_i^{(k)}$ exist.

Note: By (4), (6) and Proposition 2.1, it is easy to prove that $\{P(e_{p_n})\}$ is a linearly independent sequence in X and $\{f_{r_n}\}$ is a sequence in X^* corresponding to $\{P(e_{p_n})\}$, satisfying (2) in Proposition 2.1. And $\{P(e_n^{(k)})\}$ is derived from $\{P(e_{p_n})\}$ by the generalized Gram-Schmidt process of orthonormalization.

Lemma 3.2. *Under the same assumptions of Lemma* 3.1*, let $\{a_{ij}\}, \{r_i\}, \{p_i\}, \{n_i\}, \{e_i^{(k)}\}$ and $\{a_{ij}^{(k)}\}$ be defined as in Lemma* 3.1*. Then for each $j \in N$, we have $\sum\limits_{i=1}^{\infty} a_{ij}^{(k-1)} f_i$ converges in norm to an element in $[P^* f_{r_1}, \ldots, P^* f_{r_{k-1}}, P^* f_j]$. In particular, $\sum\limits_{i=1}^{\infty} A_k(i) f_i$ converges in norm to an element in $[P^* f_{r_1}, \ldots, P^* f_{r_{k-1}},$*

$P^* f_{r_k}]$ $(k \in n)$, *where*

$$A_k(i) = \frac{a_{i,r_k}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} \quad (i, k \in N).$$

Proof. Since $P^* f_j = \sum_{i=1}^{\infty} (P^* f_j(e_i)) f_i = \sum_{i=1}^{\infty} a_{ij} f_i$ converges in norm in X^* for every $j \in N$, it follows that the lemma holds for $k = 1$. Assume that $\sum_{i=1}^{\infty} a_{ij}^{(k-1)} f_i \in [P^* f_{r_1}, \ldots, P^* f_{r_{k-1}}, P^* f_j]$ for each $j \in N$. Then

$$\begin{aligned} \sum_{i=1}^{\infty} a_{ij}^{(k)} f_i &= \sum_{i=1}^{\infty} \left[a_{ij}^{(k-1)} - \frac{a_{p_k,j}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} a_{i,r_k}^{(k-1)} \right] f_i \\ &= \sum_{i=1}^{\infty} a_{ij}^{(k-1)} f_i - \frac{a_{p_k,j}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} \sum_{i=1}^{\infty} a_{i,r_k}^{(k-1)} f_i \end{aligned}$$

converges in norm to an element in $[P^* f_{r_1}, \ldots, P^* f_{r_k}, P^* f_j]$ for $j \in N$. This completes the proof. q.e.d.

Remarks. (1) For each $i, k \in N$, we have

$$e_i^{(k)} = e_i - \frac{a_{i,r_1}}{a_{p_1,r_1}} e_{p_1} - \cdots - \frac{a_{i,r_k}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} e_{p_k}^{(k-1)}.$$

(2) For each $i, j, k \in N$, we have

$$a_{ij}^{(k)} = a_{ij} - \frac{a_{i,r_1}}{a_{p_1,r_1}} a_{p_1,j} - \cdots - \frac{a_{i,r_k}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} a_{p_k,j}^{(k-1)}.$$

(2) is a direct consequence of (1), the fact $a_{ij}^{(k)} = f_j(P(e_i^{(k)}))$ and the definition of a_{ij}. Hence it suffices to prove (1) as follows:

It is obvious that (1) holds for $k = 1$ from the definition of $e_i^{(1)}$. Assume that (1) holds for k. Then

$$\begin{aligned} e_i^{(k+1)} &= e_i^{(k)} - \frac{a_{i,r_{k+1}}^{(k)}}{a_{p_{k+1},r_{k+1}}^{(k)}} e_{p_{k+1}}^{(k)} \\ &= e_i - \frac{a_{i,r_1}}{a_{p_1,r_1}} e_{p_1} - \cdots - \frac{a_{i,r_k}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} e_{p_k}^{(k-1)} - \frac{a_{i,r_{k+1}}^{(k)}}{a_{p_{k+1},r_{k+1}}^{(k)}} e_{p_{k+1}}^{(k)} \end{aligned}$$

also holds for $k+1$. Thus (1) follows by induction. q.e.d.

Theorem 3.3. *Let P be a projection on a reflexive Banach space X with a basis $\{e_n; f_n\}$. Then there is a basis $\{u_n; g_n\}$ of X such that $\{g_n\} \approx \{f_n\}$ and the matrix of P with respect to $\{u_n\}$ possesses the property that all but a finite number of entries of each column are zero. If $\{e_n\}$ is an unconditional [resp. symmetric] basis for X, so is $\{u_n\}$. If $P(e_k) = 0$, then $u_k = e_k$.*

Proof. If dim $P(X) < \infty$, then by Lemma 3.1 (a) there exists a basis $\{u_n; g_n\}$ for X such that $\{g_n\} \approx \{f_n\}$, and the matrix of P with respect to $\{u_n\}$ has the required property. Also, if $P(e_k) = 0$, then $u_k = e_k^{(n)} = e_k$ for some $n \le k$. Since $\{g_n\} \approx \{f_n\}$ and X is reflexive, it is clear that $\{u_n\}$ has the required properties in this case. Hence we may assume that dim $P(X) = \infty$. By Lemma 3.2, we have $\sum\limits_{i=1}^{\infty} A_k(i) f_i$ converges in norm in X^* for each $k \in N$. Thus we may let n_1 be the smallest positive integer such that $n_1 > p_1$ and

$$\| \sum_{i=m}^{\infty} A_1(i) f_i \| \ \|e_{p_1}\| < \frac{1}{2^2} \quad \text{for } m \ge n_1.$$

Similarly, for $k > 1$, by Lemma 3.2, $\sum\limits_{i=1}^{\infty} A_n(i) f_i$ converges in X^* for $1 \le n \le k$. Hence we may let n_k be the smallest positive integer such that $n_k > p_k$ and

$$\| \sum_{i=m}^{\infty} A_n(i) f_i \| \max(\|e_{p_n}^{(n-1)}\|, \|e_{p_n}\|) < \frac{1}{2k} \cdot \frac{1}{2^n} \tag{7}$$

for $1 \le n \le k$, and all $m \ge n_k$.

$$\text{Define } \begin{cases} u_n = e_n, \ \text{for } 1 \le n < n_1 \text{ and} \\ u_n = e_n^{(k)}, \ \text{for } n_k \le n < n_{k+1} \quad (k \in N). \end{cases}$$

We claim that $\{u_n\}$ is a basis of X. Let $x \in X$. Put

$$a_n = \begin{cases} f_n(x), & \text{for } n \in N - \{p_1, p_2, \ldots\}, \\ f_{p_k}(x) + \sum\limits_{i=n_k}^{\infty} A_k(i) f_i(x) & \text{for } n = p_k \text{ with some } k \in N. \end{cases}$$

Then we have two possibilities:

Case 1. If $n_k \le n < p_{k+1}$, *then*

$$\begin{aligned}
\sum_{i=1}^{n} a_i u_i &= \sum_{i=1}^{n_1-1} a_i u_i + \sum_{i=n_1}^{n_2-1} a_i u_i + \cdots + \sum_{i=n_{k-1}}^{n_k-1} a_i u_i + \sum_{i=n_k}^{n} a_i u_i \\
&= \sum_{i=1}^{n_1-1} f_i(x) e_i + \sum_{i=n_1}^{\infty} \left(\left(\frac{a_{i,r_1}}{a_{p_1,r_1}} \right) f_i(x) \right) e_{p_1} \\
&\quad + \sum_{i=n_1}^{n_2-1} f_i(x) e_i^{(1)} + \sum_{i=n_2}^{\infty} \left(\left(\frac{a_{i,r_2}^{(1)}}{a_{p_2,r_2}^{(1)}} \right) f_i(x) \right) e_{p_2}^{(1)} + \cdots \\
&\quad + \sum_{i=n_{k-1}}^{n_k-1} f_i(x) e_i^{(k-1)} + \sum_{i=n_k}^{\infty} \left(\left(\frac{a_{i,r_k}^{(k-1)}}{a_{p_k,r_k}^{(k-1)}} \right) f_i(x) \right) e_{p_k}^{(k-1)} \\
&\quad + \sum_{i=n_k}^{n} f_i(x) e_i^{(k)} \\
&= \sum_{i=1}^{n_1-1} f_i(x) e_i + \sum_{i=n_1}^{n} (A_1(i) f_i(x)) e_{p_1} + \sum_{i=n+1}^{\infty} (A_1(i) f_i(x)) e_{p_1} \\
&\quad + \sum_{i=n_1}^{n_2-1} f_i(x)(e_i - A_1(i) e_{p_1}) + \sum_{i=n_2}^{n} (A_2(i) f_i(x)) e_{p_2}^{(1)} \\
&\quad + \sum_{i=n+1}^{\infty} (A_2(i) f_i(x)) e_{p_2}^{(1)} + \sum_{i=n_2}^{n_3-1} f_i(x)(e_i - A_1(i) e_{p_1} - A_2(i) e_{p_2}^{(1)}) \\
&\quad + \sum_{i=n_3}^{n} (A_3(i) f_i(x)) e_{p_3}^{(2)} + \sum_{i=n+1}^{\infty} (A_3(i) f_i(x)) e_{p_3}^{(2)} + \cdots \\
&\quad + \sum_{i=n_{k-1}}^{n_k-1} f_i(x)(e_i - A_1(i) e_{p_1} - A_2(i) e_{p_2}^{(1)} - \cdots - A_{k-1}(i) e_{p_{k-1}}^{(k-2)}) \\
&\quad + \sum_{i=n_k}^{n} (A_k(i) f_i(x)) e_{p_k}^{(k-1)} + \sum_{i=n+1}^{\infty} (A_k(i) f_i(x)) e_{p_k}^{(k-1)} \\
&\quad + \sum_{i=n_k}^{n} f_i(x)(e_i - A_1(i) e_{p_1} - A_2(i) e_{p_2}^{(1)} - \cdots - A_k(i) e_{p_k}^{(k-1)}) \\
&= \sum_{i=n_k}^{n} f_i(x) e_i + \sum_{j=1}^{k} \sum_{i=n+1}^{\infty} A_j(i) f_i(x) e_{p_j}^{(j-1)}.
\end{aligned}$$

(Note: By Remark (1) of Lemma 3.2, $e_i^{(1)} = e_i - A_1(i) e_{p_1}, e_i^{(k)} = e_i - A_1(i) e_{p_1} - A_2(i) e_{p_2}^{(1)} - \cdots - A_k(i) e_{p_k}^{(k-1)}$ for $k > 1$.)

Thus

$$\begin{aligned}
\| \sum_{i=1}^{n} a_i u_i - \sum_{i=1}^{n} f_i(x) e_i \| &\leq \sum_{j=1}^{k} \| \sum_{i=n+1}^{\infty} A_j(i) f_i \| \| e_{p_j}^{(j-1)} \| \| x \| \\
&< \sum_{j=1}^{k} \left(\frac{1}{2k}\right) \frac{1}{2^j} \| x \| \qquad \text{(from (7))} \\
&< \frac{1}{2k} \| x \| < \frac{1}{k} \| x \| .
\end{aligned}$$

Case 2. If $p_{k+1} \leq n < n_{k+1}$, then

$$\begin{aligned}
\sum_{i=1}^{n} a_i u_i &= \sum_{i=1}^{n_1 - 1} a_i u_i + \sum_{i=n_1}^{n_2 - 1} a_i u_i + \cdots + \sum_{i=n_{k-1}}^{n_k - 1} a_i u_i + \sum_{i=n_k}^{n} a_i u_i \\
&= \sum_{i=1}^{n_1 - 1} f_i(x) e_i + \sum_{i=n_1}^{\infty} (A_1(i) f_i(x)) e_{p_1} \\
&\quad + \sum_{i=n_1}^{n_2 - 1} f_i(x) e_i^{(1)} + \sum_{i=n_2}^{\infty} (A_2(i) f_i(x)) e_{p_2}^{(1)} + \cdots \\
&\quad + \sum_{i=n_{k-1}}^{n_k - 1} f_i(x) e_i^{(k-1)} + \sum_{i=n_k}^{\infty} (A_k(i) f_i(x)) e_{p_k}^{(k-1)} \\
&\quad + \sum_{i=n_k}^{n} f_i(x) e_i^{(k)} + \sum_{i=n_{k+1}}^{\infty} (A_{k+1}(i) f_i(x)) e_{p_{k+1}}^{(k)}.
\end{aligned}$$

Hence as in Case 1, we deduce that

$$\begin{aligned}
\sum_{i=1}^{n} a_i u_i &= \sum_{i=1}^{n} f_i(x) e_i + \sum_{j=1}^{k} \sum_{i=n+1}^{\infty} (A_j(i) f_i(x)) e_{p_j}^{(j-1)} \\
&\quad + \sum_{i=n_{k+1}}^{\infty} (A_{k+1}(i) f_i(x)) e_{p_{k+1}}^{(k)}, \quad \text{and whence}
\end{aligned}$$

$$\begin{aligned}
\| \sum_{i=1}^{n} a_i u_i - \sum_{i=1}^{n} f_i(x) e_i \| &\leq \sum_{j=1}^{k} \| \sum_{i=n+1}^{\infty} A_j(i) f_i \| \, \| e_{p_j}^{(j-1)} \| \, \| x \| \\
&\quad + \| \sum_{i=n_k+1}^{\infty} A_{k+1}(i) f_i \| \, \| e_{p_{k+1}}^{(k)} \| \, \| x \| \\
&< \sum_{j=1}^{k} \frac{1}{2k} 2^{-j} \| x \| + \frac{1}{2(k+1)} 2^{-(k+1)} \| x \| \\
&< \left(\frac{1}{2k} + \frac{1}{2k} \right) \| x \| = \tfrac{1}{k} \| x \|.
\end{aligned}$$

Combining both cases, we have

$$\|\sum_{i=1}^{n} a_i u_i - \sum_{i=1}^{n} f_i(x)e_i\| < \frac{1}{k}\|x\| \quad \text{for } n_k \leq n < n_{k+1} \quad (k \in N).$$

Now if $\varepsilon > 0$ is given, then there exist $N_1, k_0 \in N$ (independent of k) such that

$$\frac{1}{k_o}\|x\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|x - \sum_{i=1}^{n} f_i(x)e_i\| < \frac{\varepsilon}{2} \quad \text{for } n \geq N_1.$$

Take $N_o = max\{n_{ko}, N_1\}$. Then N_o is independent of k and if $n \geq N_o$ with $n_k \leq n < n_{k+1}$ for some $k \in N$, then $k \geq k_o$, whence

$$\begin{aligned} \|x - \sum_{i=1}^{n} a_i u_i\| &\leq \|x - \sum_{i=1}^{n} f_i(x)e_i\| + \|\sum_{i=1}^{n} f_i(x)e_i - \sum_{i=1}^{n} a_i u_i\| \\ &< \frac{\varepsilon}{2} + \frac{1}{k}\|x\| \leq \frac{\varepsilon}{2} + \frac{1}{k_o}\|x\| \\ &< \varepsilon, \quad \text{if } n \geq N_o. \end{aligned}$$

This implies that for every $x \in X$, there is a sequence $\{a_i\}$ of scalars such that $x = \sum\limits_{i=1}^{\infty} a_i u_i$.

Moreover, it is easy to verify that $\{u_i; g_i\}$ is a birothogonal system, where $\{g_n\}$ is defined by

$$g_n = \begin{cases} f_n & \text{for } n \in N - \{p_1, p_2, \ldots\}, \\ f_{p_k} + \sum\limits_{i=n_k}^{\infty} A_k(i) f_i & \text{for } n = p_k \text{ with some } k \in N. \end{cases}$$

This proves that $\{u_n; g_n\}$ is a basis for X.

By Lemma 3.1 (b),

$$(8) \qquad \begin{cases} f_j(P(u_i)) = 0 & \text{for } i \geq 1 \text{ and } 1 \leq j < r_1; \\ f_j(P(u_i)) = 0 & \text{for } i \geq n_1 \text{ and } 1 \leq j < r_2; \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ f_j(P(u_i)) = 0 & \text{for } i \geq n_{k-1} \text{ and } 1 \leq j < r_k; \cdots \end{cases}$$

Let $b_{ij} = g_j(P(u_i))$ $(i, j \in N)$. It remains to prove that for each $j \in N$, $b_{ij} = 0$ for all but a finite number of indices i. Let $j \in N$. Then there exists $k \in N$ such that $r_{k-1} \leq j < r_k$.

1^o If $j \notin \{p_1, p_2, \cdots\}$, then it follows from (8) and the definition of g_j that $b_{ij} = g_j(p(u_i)) = f_j(p(u_i)) = 0$ for $i \geq n_{k-1}$.

2^o Suppose $j = p_m$ for some $m \in N$. By the definition of g_{p_m}, we have $g_j = g_{p_m} = f_{p_m} + \sum\limits_{i=n_m}^{\infty} A_m(i) f_i = A_m(p_m) f_{p_m} + \sum\limits_{i=n_m}^{\infty} A_m(i) f_i$.

Since $\sum\limits_{i=1}^{\infty} A_m(i) f_i \in [P^* f_{r_1}, \ldots, P^* f_{r_m}]$, we have $\sum\limits_{i=1}^{\infty} A_m(i) \;\; f_i = \sum\limits_{n=1}^{m} \alpha_n P^* f_{r_n}$ for some scalars $\alpha_s^{'}$. Thus from (8) we have

$$\begin{aligned}\left(\sum_{i=1}^{\infty} A_m(i) f_i)(P(u_t)\right) &= \left(\sum_{n=1}^{m} \alpha_n P^* f_{r_n}\right)(P(u_t)) \sum_{n=1}^{m} \alpha_n P^* f_{r_n}(P(u_t)) \\ &= \sum_{n=1}^{m} \alpha_n f_{r_n}(P(u_t)) \quad \text{(for P is a projection)} \\ &= 0 \quad \text{for } t \geq n_m;\end{aligned}$$

that is,

$$\left(\sum_{\substack{i=1 \\ i \neq p_m}}^{n_m - 1} A_m(i) f_i + g_{p_m}\right)(P(u_t)) = 0 \quad \text{for } t \geq n_m. \tag{9}$$

But $r_{n_m} \geq n_m > n_m - 1$. Thus by (8) we get

$$f_i(P(u_t)) = 0 \quad \text{for } 1 \leq i \leq n_m - 1 < r_{n_m} \text{ and } t \geq n_{n_m} - 1. \tag{10}$$

Take $\eta(m) = \max\{n_m, n_{n_m - 1}\}$. Then $\eta(m)$ is increasing on m. From (9) and (10) we have

$$g_{p_m}(P(u_t)) = 0 \quad \text{for } t \geq \eta(m).$$

This implies that $b_{i,p_m} = 0$ for $i \geq \eta(m)$. Hence from both cases we prove that the matrix of P with respect to $\{u_n\}$ has the property that every column has at most finitely many nonzero entries.

Since by (7) and the defintion of g_n, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \|g_n - f_n\| \;\; \|e_n\| &= \sum_{k=1}^{\infty} \|\sum_{i=n_k}^{\infty} A_k(i) f_i\| \;\; \|e_{p_k}\| \\ &< \sum_{k=1}^{\infty} \frac{1}{2k} 2^{(-k)} \quad < 1,\end{aligned}$$

by the stability theorem ([15], Theorem 10.2, p. 95), $\{g_n\}$ is a basis of X^* and $\{g_n\} \approx \{f_n\}$. Suppose $\{e_n\}$ is an unconditional [resp. symmetric] basis for X, by ([15], Theorem 17.7, p. 524 & Proposition 22.5, p. 595) we have $\{f_n\}$ is also an unconditional [resp. symmetric] basis for X^*. But $\{g_n\} \approx \{f_n\}$. Thus $\{g_n\}$ is an unconditional [resp. symmetric] basis of X^*. Since X is reflexive, we have $\{u_n\}$ is an unconditional [resp.symmetric] basis of X (from [15], Theorem 17.7, p. 524 & Proposition 22.5, p. 595 again).

From Lemma 3.1(c) we can easily prove that if $P(e_k) = 0$, then $u_k = e_k^{(i)} = e_k$ (for some i). Hence we complete the proof of Theorem 3.3 q.e.d.

Remarks. (1) Theorem 3.3 can be improved such that the matrix of P with respect to $\{u_n\}$ possesses the property that all but a finite number of entries of each column and each row are zero.

(2) Although all Banach spaces discussed in this paper are reflexive, our technique also applies for non-reflexive Banach spaces. In fact, we can obtain the same results in a similar manner if the underlying Banach space is only assumed to have a shrinking basis. We also note that all bases of a reflexive Banach space are shrinking. For further development in this line, readers are refered to [13].

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