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OPTIMALITY CONDITIONS FOR SEMI-PREINVEX PROGRAMMING*

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Abstract. We consider a semi-preinvex programming as follows:

$$(\mathrm{P}) \quad \begin{cases} \inf \ f(x) \\ \text{subject to } x \in K \subseteq X, \ g(x) \in -D, \end{cases}$$

where K is a semi-connected subset; $f : K \to (Y, C)$ and $g : K \to (Z, D)$ are semi-preinvex maps; while (Y, C) and (Z, D) are ordered vector spaces with order cones C and D, respectively. If f and g are arc-directionally differentiable semi-preinvex maps with respect to a continuous map: $\gamma : [0, 1] \to K \subseteq X$ with $\gamma(0) = 0$ and $\gamma'(0^+) = u$, then the necessary and sufficient conditions for optimality of (P) is established. It is also established that a solution of an unconstrained semi-preinvex optimization problem is related to a solution of a semi-prevariational inequality .

1. Introduction

In general, an optimization problem or mathematical programming problem is considered as the following form:

$$(\mathrm{P}) \quad \begin{cases} \inf \ f(x) \\ \text{subject to } g(x) \leq 0 \ \text{ and } \ x \in K \subset X \end{cases}$$

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where $f : K \to Y$ and $g : K \to Z$ are either differentiable, convex, or nonsmooth and nonconvex functions. Here Y, Z are ordered vector lattices with some order cones. Many authors investigated the optimality conditions for problem (P). Kuhn and Tucker first established the necessary condition for programming problem (P) concerning the differentiable functions with $Y = \mathbb{R}$ and $Z = \mathbb{R}^m$. Henceforth, results for more general cases are often referred to as Kuhn-Tucker type theorems. Theorems on multiobjective programming problems are offen referred to as Fritz John type theorems. For example, in Lai and Ho [8, Theorem 3.1], the Pareto optimality condition for convex continuous functions $f : X \to \mathbb{R}^n$ and $g : X \to \mathbb{R}^m$ is investigated. Moreover, one can consider X to be a measure space (X, Γ, μ) and replace f and g by the convex set functions or set mappings, like

$$f : S \subset \Gamma \to \mathbb{R}^n \text{ (or } Y\text{)} \quad \text{and} \quad g : S \subset \Gamma \to \mathbb{R}^m \text{ (or } Z\text{)}$$

where S is a convex family of measurable subsets in X. In Lai and Lin [9, Theorems 11 and 12], the Pareto optimality conditions are established (cf also [10, Theorem 4.1] for weak minimum). In [11, Theorem 3.1 and Corollary 3.1], Lai and Szilagyi studied the programming with convex set functions and proved that the alternative theorem is valid for convex set functions defined on convex subfamily S of measurable subsets in X, and shows that if the system

$$\begin{cases} f(\Omega) <<_C \theta, \\ g(\Omega) <_D \theta \end{cases}$$

has no solution where θ stands for zero vector in a topological vector space , then there exists a nonzero continuous linear function $(y^*, z^*) \in C^* \times D^*$ such that

$$\langle f(\Omega),\ y^*\rangle + \langle g(\Omega),\ z^*\rangle \geq 0 \quad \text{for all } \Omega \in S.$$

Generalized convexity of functions are investigated by many mathematician, see [2-7] and [12-17]. For example, functions are invex, d-invex, preinvex, arc-connected convex, or convex like etc. These nonconvex functions also have good behavior like the convex case.

The purpose of this paper is studying such class of noncovex functions for constrained semi-preinvex programming problems. In Section 2, we give some definitions of generalized convexity, and show some elementary results and theorem of alternatives. In Section 3, the necessary and sufficient optimality conditions for semi-preinvex programming problems are established. Finally, the semi-prevariational inequality is discussed. It is related to the optimality for semi-preinvex programming problem.

2. Preliminaries, Semi-preinvex Mapping

Throughout the paper we let X, Y and Z be locally convex topological vector spaces over real field $\mathbb{R}$. Let $C \subset Y$ and $D \subset Z$ be pointed closed convex cones with nonempty interior ($C^\circ \neq \emptyset$, $D^\circ \neq \emptyset$) which determine the order complete vector lattice for Y and Z, respectively. In order to connect the order structure and topological structure, it requires furthermore that there be a neighborhood system $\{V\}$ of the origin θ in a topological vector space with an order cone such that

$$V = (V + C) \cap (V - C). \tag{2.1}$$

A cone C with condition (2.1) is called *normal*. We assume throughout that C and D are normal cones in the order-complete vector lattices Y and Z, respectively. The dual cone C^* of a cone C is given by

$$C^* = \{y^* \in Y^* | < y^*, y > \geq 0 \text{ for all } y \in C\}.$$

We employ the convention of partial order in a topological vector space. For $y_1, y_2 \in Y$, we write

$$\begin{array}{lll} y_1 \leq_C y_2 & \text{if} & y_2 - y_1 \in C; \\ y_1 <_C y_2 & \text{if} & y_2 - y_1 \in C - \{\theta\}; \\ y_1 <<_C y_2 & \text{if} & y_2 - y_1 \in C^\circ. \end{array}$$

A point $y_0 \in B \subset Y$ is called a *minimal* [resp. *weakly minimal*] element (point) of B, denoted by $y_0 \in \min B$ [resp. $y_0 \in$ w-min B], if there does not exist $y \in B$ such that $y <_C y_0$ [resp. $y <<_C y_0$].

Let $f : K \subset X \to Y$ be *differentiable* at $x_0 \in K$. Then there is a linear operator $A = f'(x_0) \in L(X, Y)$, the space of all linear operators from X into Y, such that

$$\lim_{\alpha \to 0} \frac{f((1-\alpha)x_0 + \alpha x) - f(x_0)}{\alpha} = f'(x_0)(x - x_0). \tag{2.2}$$

Let $f : K \subset X \to (Y, C)$ be also *C-convex* on a convex set $K \subset X$, i.e., if for any $x_0, x \in K$ and $\alpha \in [0, 1] \subset \mathbb{R}$, we have

$$f((1-\alpha)x_0 + \alpha x) \leq_C (1-\alpha) f(x_0) + \alpha f(x). \tag{2.3}$$

or

$$\frac{f((1-\alpha)x_0 + \alpha x) - f(x_0)}{\alpha} \leq_C f(x) - f(x_0). \tag{2.4}$$

It follows from (2.2) and (2.4) that

$$f'(x_0)(x - x_0) \leq_C f(x) - f(x_0). \tag{2.5}$$

In 1981, Hanson [5] introduced a generalized convexity on $\mathbb{R}^n$, namely *invex*, that is, $x - x_0$ is replaced by a vector $\tau(x_0, x) \in X$ in (2.5), or

$$f'(x_0)\tau(x_{0,}, x) \leq_C f(x) - (x_0). \tag{2.6}$$

Thus an invex function is something a generalization of a convex differentiable function.

In the following context we do not assume $f : X \to Y$ to be differentiable. We need more definition as follows.

Definition 2.1. (1) A subset $K \subset X \to Y$ is said to be *vector τ-connected* if for any $x, y \in K$ and $\alpha \in$ [0,1], there is a vector $\tau(x, y) \in X$ such that

$$x + \alpha\tau(x, y) \in K. \tag{2.7}$$

(2) A map $f : K \subset X \to Y$ is said to be *preinvex* on a vector τ-connected subset K if for any $x, y \in K$, there is a vector $\tau(x, y) \in X$ such that for $\alpha \in [0, 1]$,

$$f(x + \alpha\tau(x, y)) \leq_C (1 - \alpha)f(x) + \alpha f(y). \tag{2.8}$$

Definition 2.2. (1) A set $K \subset X$ is said to be *arcwise-connected* (or *arc-connected* for brevity) if for any $x, y \in K$, there is a continuous map

$$\gamma(x, y; \cdot) : [0, 1] \to K$$

such that

$$\begin{cases} \gamma(x, y; 0) = x, \\ \gamma(x, y, 1) = y, \\ \gamma(x, y; \alpha) \in K \quad \text{for all } \alpha \in [0, 1]. \end{cases} \tag{2.9}$$

(2) A map $f : X \to Y$ is said to be *arcwise connected convex* (or *arc-connected convex* for brevity) on an arc-connected set $K \subset X$ if for any $x, y \in K$ and $\alpha \in [0, 1]$, we have

$$f(\gamma(x, y; \alpha)) \leq_C (1 - \alpha)f(\gamma(x, y; 0)) + \alpha f(\gamma(x, y; 1)),$$

that is,

$$f(\gamma(x, y; \alpha)) \leq_C (1 - \alpha)f(x) + \alpha f(y). \tag{2.10}$$

Definition 2.3. (1) A set $K \subset X$ is said to be *semi-connected* if for any $x, y \in K$ and $\alpha \in [0,1]$, there is a vector $\tau(x,y,\alpha) \in X$ such that

$$x + \alpha\tau(x,y,\alpha) \in K. \tag{2.11}$$

(2) A map $f : X \to Y$ is said to be *semi-preinvex* on a semi-connected subset $K \subset X$ if each $(x,y,\alpha) \in K \times K \times [0,1]$ corresponds a vector $\tau(x,y,\alpha) \in X$ such that

$$\begin{cases} f(x + \alpha\tau(x,y,\alpha)) \leq_C (1-\alpha)f(x) + \alpha f(y) \\ \text{and } \lim\limits_{\alpha \downarrow 0} \alpha\tau(x,y,\alpha) = \theta \end{cases} \tag{2.12}$$

where θ stands for the zero vector (of X).

Remark. By (2.12), it is obvious that a semi-preinvex mapping is upper semi-continuous (u.s.c.). Note that an arc-connected set need not be a vector τ-connected and vice versa. So an arc-connected convex map is not a preinvex map. But both arc-connected convex map and preinvex map are semi-preinvex.

Theorem 2.1. (1) *A preinvex map is semi-preinvex.* (2) *An arc-connected convex map is semi-preinvex.*

Proof. (1) Let K be a vector t-connected set. Then for any $x, y \in K$, we put

$$\tau(x,y,\alpha) = \tau(x,y) \qquad \text{for all } \alpha \in [0,1],$$

so that $x + \alpha\tau(x,y,\alpha) \in K$. It follows that K is also a semi-connected set. Therefore, a preinvex map is also a semi-preinvex.

(2) Let K be an arc-connected subset of X. That is, for any $x, y \in K$, there is a continuous vector value mapping :

$$\gamma(x,y,\cdot) : [0,1] \to K$$

with boundary conditions $\gamma(x,y,0) = x, \;\; \gamma(x,y,1) = y$. If f is arc-connected convex on K, then

$$f(\gamma(x,y,\alpha)) \leq_C (1-\alpha)f(x) + \alpha f(y), \qquad \alpha \in [0,1].$$

Taking $\tau(x,y,\alpha) = (1/\alpha)[\gamma(x,y,\alpha) - x]$, we have

$$\lim_{\alpha \downarrow 0} \alpha\tau(x,y,\alpha) = \lim_{\alpha \downarrow 0}[\gamma(x,y,\alpha) - x] = \gamma(x,y,0) - x = 0.$$

Hence for any $\alpha \in [0,1]$,

$$\begin{aligned} f(x+\alpha\tau(x,y,\alpha)) &= f(\gamma(x,y,\alpha)) \\ &\leq_C (1-\alpha)f(\gamma(x,y,0)) + \alpha f(\gamma(x,y,1)) \\ &= (1-\alpha)f(x) + \alpha f(y). \end{aligned}$$

This shows that f is a semi-preinvex map. □

Like a convex function, any locally minimum is also global minimum. We state this property with respect to a semi-preinvex map as follows.

Theorem 2.2. *Let $K \subset X$ be a semi-connected subset and $f : K \to Y$ a semi-preinvex map. Then any local minimum of f is also a global minimum of f over K.*

Proof. Let x_0 be a local minimum of f on K. Then there is a neighborhood $N(x_0)$ of x_0 such that

$$f(x) \not<_C f(x_0) \text{ for all } x \in N(x_0) - \{x_0\}.$$

We want to show that there is no $x \in K$ which satisfies the inequality

$$f(x) <_C f(x_0). \tag{1}$$

For if there were a point $y \in K$ satisfying (1), then it would deduce a contradiction.

Indeed if f is a semi-preinvex map on a semi-connected set K, we can find a vector $\tau(x_0, y, \alpha) \in X$ for any $a \in [0,1]$ such that

$$f(x_0 + \alpha\tau(x_0,y,\alpha)) \leq_C (1-\alpha)f(x_0) + \alpha f(y).$$

It follows that

$$f(x_0 + \alpha\tau(x_0,y,\alpha)) - f(x_0) \leq_C \alpha[f(y) - f(x_0)] <_C \theta. \tag{2}$$

As α is near 0, we see that $\tilde{x} = x_0 + \alpha\tau(x_0,y,\alpha) \in N(x_0)$. Thus if x_0 is a local minimum for f, then the inequality (2) implies that

$$f(\tilde{x}) <_C f(x_0).$$

This contradicts the fact of x_0 minimizing f locally in K. Therefore a local minimum point x_0 is also a global minimum point. □

Definition 2.4. A mapping $f : X \to Y$ is said to be *convex-like* on an arbitrary nonempty subset K of X if for any $x_1, x_2 \in K$ and a constant $\alpha \in [0,1]$ there exists a vector

$$x = x(x_1, x_2, \alpha) \in K \text{ such that } \quad f(x) \leq_C (1-\alpha)f(x_1) + \alpha f(x_2).$$

Jeyakumar [7] proved that the theorem of alternatives is valid for the convex-like function $f : \mathbb{R}^n \to \mathbb{R}$. We see easily that a semi-preinvex map $f : X \to Y$ is also a convex-like. It follows that the theorem of alternatives is valid for the case of semi-preinvex mapping which we state as follows.

Theorem 2.3. (Theorem of alternatives) *Let $h : X \to Y$ be a semi-preinvex map on a semi-connected subset K in X. Then exactly one of the following two systems of inequality is solvable.*

(a) *There exists $x \in K$ and $h(x) <<_C \theta$,*
(b) *There exists a nonzero $y^* \in C^*$ such that $\langle h(x), y^* \rangle > 0$ for all $x \in K$.*

3. Mathematical Programming with Semi-Preinvex Mappings

Let $f : X \to (Y, C)$ and $g : X \to (Z, D)$ be semi-preinvex maps on a semi-connected subset K in X. Consider a constrained programming problem as follows

$$(\mathrm{P}) \begin{cases} \inf\ f(x) \\ \text{subject to } \ x \in K \subset X \\ \text{and } g(x) \in -D. \end{cases}$$

Definition 3.1. A mapping $f : K \subset X \to Y$ is said to be *arcwise* (*arc*) *directionally differentiable* at $x_0 \in K$ with respect to a continuous arc $\beta : [0,1] \to K \subset X$ if $x_0 + \beta(t) \in K$ for $t \in [0,1]$ with

$$\beta(0) = \theta \ \text{ and } \ \beta'(0^+) = u \quad (\text{in } X), \tag{3.1}$$

that is, the continuous function β is differentiable from right at 0, and the limit

$$\lim_{t \downarrow 0} \frac{f(x_0 + \beta(t)) - f(x_0)}{t} \cong f'(x_0; u) \quad \text{exists.} \tag{3.2}$$

Note that the arc directional derivative $f'(x_0; \cdot)$ is a mapping from X into Y. If for any $x, y \in K$ and $t \in [0,1]$, we choose a vector

$$\tau(x, y, t) = \frac{\beta(t)}{t} = \frac{\beta(t) - \beta(0)}{t - 0}, \tag{3.3}$$

then

$$\lim_{t\downarrow 0}\tau(x,y,t)=\beta'(0^+)=u \tag{3.4}$$

and

$$\frac{d}{dt}[t\tau(x,y,t)]\,|_{t=0^+}=\beta'(0^+)=u. \tag{3.5}$$

The following Fritz John type theorem is essential in this section for programming problem (P).

Theorem 3.1. (Necessary optimality condition). *Suppose that f and g are arc-directionally differentiable with respect to a continuous arc β defined as in Definition* 3.1. *If x_0 minimizes locally for the semi-preinvex programming problem* (P), *then there exist y^* $y^* \in C^*$ such that*

$$\langle f'(x_0;u),y^*\rangle+\langle g'(x;u),z^*\rangle\geq 0, \tag{3.6}$$

where $u=\beta'(0^+)$ and

$$< g(x_0),z^* >= 0. \tag{3.7}$$

Proof. Since f is a semi-preinvex map in (P), the local minimal solution to (P) is also a global minimal solution to (P) (see Theorem 2.2). It follows that the system

$$\begin{cases} f(x)-f(x_0)<_C \theta, \\ g(x)<_D \theta \end{cases} \tag{3.8}$$

has no solution in K. By the Theorem of alternatives (Theorem 2.3), there exists a nonzero element $(y^*,z^*)\in C^*\times D^*$ such that

$$\langle f(x)-f(x_0),y^*\rangle+\langle g(x),z^*\rangle\geq 0 \tag{3.9}$$

for all $x\in K$. Putting $x=x_0$ in (3.9), we get

$$\langle g(x_0),z^*\rangle\geq 0.$$

Since $z^*\in D^*$ and $g(x_0)\in -D$, it follows that

$$\langle g(x_0),z^*\rangle=0.$$

This proves (3.7).

As K is a semi-connected set, for any $x \in K$ and $t \in [0,1]$, we have

$$x_0 + t\tau(x_0, x, t) \in K.$$

For $t \neq 0$, the point $\tilde{x} = x_0 + t\tau(x_0, x, t) \neq x_0$ does not solve the system (3.8). So substituting $\tilde{x}$ in (3.9) and using the result (3.7), we obtain

$$\langle f(x_0 + t\tau(x_0, x, t)) - f(x_0), y^* \rangle + \langle g(x_0 + t\tau(x_0, x, t)) - g(x_0), z^* \rangle \geq 0. \tag{3.10}$$

Since f and g are arc-directionally differentiable with respect to β, choose a vector $\tau(x, y, t)$ as (3.3), so that (3.4) and (3.5) hold. It follows that if we divide (3.10) by $t \neq 0$ and take the limit as $t \downarrow 0$, then we have

$$\langle f'(x_0; u), y^* \rangle + \langle g'(x_0; u), z^* \rangle \geq 0.$$

This proves (3.6), and the proof of theorem is complete. □

If the mappings f and g are preinvex on a τ-connected set $K \subset X$, then by setting $\tau(x, y, t) = \tau(x, y)$ for all $t \in [0,1]$, we have

$$\frac{d}{dt}[t\tau(x, y, t)]\,|_{t=0^+} = \tau(x, y) = u.$$

Hence the following corollary holds.

Corollary 3.2. *Let the mappings f and g in* (P) *be preinvex on a τ-connected set $K \subset X$. Then the Fritz John type Theorem* 3.1 *becomes to the result in the case of preinvex programming problem.*

If the mappings f and g are arc-connected convex on an arc-connected set $K \subset X$, that is, for $x, y \in K$, there is a continuous arc-connected map

$$\gamma(x, y, \cdot) : [0,1] \subset \mathbb{R} \to K \text{ with } \gamma(x, y, 0) = x \text{ and } \gamma(x, y, 1) = y,$$

we can choose $\beta(t) = \gamma(x, y, t) - x$ for any $x, y \in K$, then

$$\frac{d}{dt}[\gamma(x, y, t)]\,|_{\tau=0^+} = \beta'(0^+) = u.$$

Hence we also have the Fritz John type Theorem in the case of arc-connected convex programming problem. That is,

Corollary 3.3. *Let the mappings f and g be arc-connected convex on an arc-connected subset $K \subset X$. Then the Fritz John type Theorem* 3.1 *reduces to the result in the case of arc-connected convex programming problem.*

Remark. If $Y = \mathbb{R}^p$, $Z = \mathbb{R}^m$ and the functions f and g in the programming problem (P) are d-invex, then the Fritz John type Theorem 3.1 extends the results of Ye [17, Theorems 3.2 and 3.3]. A function $f : X \to \mathbb{R}^p$ is said to be *d-invex* on an open set $K \subset X$ at a point $x_0 \in K$ if for any $x \in K$, there exists a vector $\tau = \tau(x, x_0) \in X$ such that

$$f^+(x_0; \tau) \leq f(x) - f(x_0),$$

where $f^+(x_0; \tau)$ is the directional derivative defined by

$$f^+(x_0; \tau) = \lim_{t \downarrow 0^+} \frac{f(x_0 + t\tau(x, x_0)) - f(x_0)}{t}.$$

Evidently, if f is differentiable (in the Gateaux sense), then a d-invex function is invex. So the d-invex map is regarded as a generalization of invex map as well as an arc-directional derivative.

The converse of Theorem 3.1 is also valid. That we state as follows:

Theorem 3.4. (Sufficient optimality condition) *Let f and g be arc-directionally differentiable at $x_0 \in K$ with respect to a continuous arc β defined in Definition* 3.1. *If there exist $y^* \in C^*$ and $z^* \in D^*$ satisfying*

$$\langle f'(x_0; u), y^* \rangle + \langle g'(x_0; u), z^* \rangle \geq 0 \tag{3.11}$$

with $u = \beta'(0^+)$, and

$$\langle g(x_0), z^* \rangle = 0. \tag{3.12}$$

Then x_0 is an optimal solution for problem (P).

Proof. Suppose to the contrary that x_0 were not optimal for problem (P). Then there is an $x \in F$, the feasible solutions of (P) satisfying (3.11) and (3.12) such that

$$f(x) <_C f(x_0) \quad \text{and} \quad g(x) \leq_D \theta.$$

It then have

$$\begin{aligned} &\langle f(x) - f(x_0), y^* \rangle < 0, \\ &\langle g(x) - g(x_0), z^* \rangle \leq 0 \ \text{ since } \langle g(x_0), z^* \rangle = 0 \end{aligned}$$

for any $y^* \in C^*$ and $z^* \in D^*$, thus

$$\langle f(x) - f(x_0), y^* \rangle + \langle g(x) - g(x_0), z^* \rangle < 0. \tag{3.13}$$

Since the semi-preinvex maps f and g are arc-directionally differentiable, it follows that for $(x, x_0, t) \in K \times K \times [0,1]$ there corresponds a vector $\tau(x, x_0, t) \in X$ such that

$$\begin{cases} f(x_0 + t\tau(x, x_0, t)) \leq_C \ (1-t)f(x_0) + tf(x) \\ g(x_0 + t\tau(x, x_0, t)) \leq_D \ (1-t)g(x_0) + tg(x) \end{cases}$$

with $\lim\limits_{t \downarrow 0} \tau(x, x_0, t) = \beta'(0^+) = u$, and so

$$\begin{cases} \dfrac{f(x_0 + t\tau(x, x_0, t)) - f(x_0)}{t} \leq_C f(x) - f(x_0), \\ \dfrac{g(x_0 + t\tau(x, x_0, t)) - g(x_0)}{t} \leq_D g(x) - g(x_0). \end{cases}$$

Letting $t \downarrow 0^+$, the above inequalities imply

$$\text{(a)} \qquad \begin{cases} f'(x_0; u) \leq_C \ f(x) - f(x_0), \\ g'(x_0 + u) \leq_D \ g(x) - g(x_0). \end{cases}$$

Hence for any $y^* \in C^*$ and $z^* \in D^*$, (a) implies

$$\text{(b)} \qquad \begin{cases} \langle f'(x_0; u), y^* \rangle \leq \langle f(x) - f(x_0), y^* \rangle, \\ \langle g'(x_0; u), z^* \rangle \leq \langle g(x) - g(x_0), z^* \rangle. \end{cases}$$

Consequently, from (3.13) and (b), we obtain

$$\langle f'(x_0; u), y^* \rangle + \langle g'(x_0; u), z^* \rangle < 0.$$

This contradicts the fact of (3.11). Therefore x_0 is an optimal solution of problem (P). □

4. Applications in Semi-prevariational Inequality

Consider an operator (not linear) mapping

$$T : X \to L(X, Y).$$

where $L(X, Y)$ stands for the space of continuous linear operators from X into Y. Then a *variational inequality problem* is given by

(VI) To find an $x_0 \in K$ such that there does not exist $x \in K$ satisfying

$$T(x_0)(x - x_0) <_C \theta. \tag{4.1}$$

Note that (4.1) is different from

$$T(x_0)(x - x_0) \geq_C \theta \quad \text{for all } x \in K$$

except when Y is totally ordered.

If $Y = \mathbb{R}$, then $L(X, Y) = X^*$ and $T : X \to X^*$ is a conjugate map. It follows that no $x \in K$ to satisfy (4.1) is equivalent to

$$\langle x - x_0,\ T(x_0)\rangle \geq 0 \quad \text{for all } x \in K.$$

This concept is extended to a *prevariational inequality* problem as follows

(PVI) To find a vector $x_0 \in K$, a vector τ-connected set in X, such that there does not exist an $x \in K$ which satisfies the inequality

$$T(x_0)\tau(x_0, x) <_C f(x_0) - f(x), \tag{4.2}$$

where $\tau(x_0, x)$ is a vector in X and $f : K \to (Y, C)$ is any preinvex mapping.

If $f \cong 0$, then (PVI) becomes to find $x_0 \in K$ such that there does not exist an $x \in K$ which satisfies the inequality

$$T(x_0)\tau(x_0, x) <_C \theta. \tag{4.3}$$

In particular, if $\tau(x_0, x) = x - x_0$, then (4.3) becomes a variational problem related to (4.1).

Now if we consider a preinvex programming problem (P) of differentiable maps f and g with respect to a continuous vector map $\beta(t) = t\tau(x, y)$, and when $x_0 \in K$ is a solution of the preinvex problem (P), then the Fritz John type Theorem 3.1 shows that there exists a nonzero element $(y^*, z^*) \in C^* \times D^*$ satisfying the inequality

$$\langle f'(x_0; \tau(x_0, x)), y^*\rangle + \langle g'(x_0; \tau(x_0, x)), z^*\rangle \geq 0. \tag{4.4}$$

Since f and g are assumed to be differentiable,

$$f'(x_0; \tau(x_0, x)) = (f'x_0)\tau(x_0, x)$$

and

$$g'(x_0; \tau(x_0, x)) = g'(x_0)\tau(x_0, x).$$

It follows from (4.4) that

$$(y^* \circ f'(x_0) + z^* \circ g'(x_0))\tau(x_0, x) \geq 0.$$

Define

$$T(x_0) = y^* \circ f'(x_0) + z^* \circ g'(x_0).$$

Since $f'(x_0) \in L(X, Y)$ and $g'(x_0) \in L(X, Z)$, we see that $T(x_0)$ is a continuous linear functional on X, so that the optimal solution x_0 of (P) is related to the solution of the variational inequality :

$$T(x_0)\tau(x_0, x) \geq 0 \quad \text{for all } x \in K.$$

The question arises from the above concept that whether the solution of a semi-preinvex programming problem (P) is related to a variational inequality by a nonlinear operator map. At first we consider an unconstraint semi-preinvex optimization problem

$$(\tilde{P}) \quad \begin{cases} \inf \ f(x) \\ \text{subject to } x \in K \subset X. \end{cases}$$

where $K \subset X$ is a semi-connected subset and $f : K \to (Y, C)$ is a semi-preinvex map.

For $x_0 \in K$, we consider a continuous arc $\beta : [0, 1] \to K \subset X$ such that $x_0 + \beta(t) \in K$ for $t \in [0, 1]$ with

$$\frac{\beta(t)}{t} = \frac{\beta(t) - \beta(0)}{t} = \tau(x_0, x, t) \in X, x \in K;$$

$$\beta(0) = \theta \quad \text{and} \quad \beta'(0^+) = u(x_0, x) \text{ in } X.$$

We formally formulate a variational inequality problem as follows:

(SPVI) To find a vector $x_0 \in K$, a semi-connected set in X, such that there does not exist $x \in K$ which satisfies the inequality

$$T(x_0)u(x_0, x) <_C f(x_0) - f(x). \tag{4.5}$$

where $f : K \to (Y, C)$ is a semi-preinvex map.

The essential task of this section is to establish an optimal solution of $(\tilde{P})$ related to the solution of (SPVI). It is equivalent to say that the inequality

$$T(x_0)u(x_0, x) <_C \theta \tag{4.6}$$

can not hold for all $x \in K$. We state the result as the following theorem.

Theorem 4.1. *Let $K \subset X$ be a semi-connected subset, $f : K \to (Y, C)$ an arc-directional differentiable semi-preinvex map. Then the following two statements hold.*

(1) *If x_0 solves $(\tilde{P})$, then x_0 solves problem* (SPVI) *related to the inequality* (4.6).

(2) *If x_0 solves problem* (SPVI) *related to the inequality* (4.6), *then x_0 solves the problem $(\tilde{P})$.*

Proof. (1) Let x_0 be an optimal solution of $(\tilde{P})$. For any $x \in K,\ t \in [0,1]$, we have $x_0 + t\tau(x_0, x, t) \in K$. Then there is no $y \in K$ which satisfies

$$f(x_0 + t\tau(x_0, y, t)) - f(x_0) <_C \theta. \tag{4.7}$$

Dividing the above expression (4.7) by $t > 0$, and then letting $t \downarrow O^+$, we have

$$f'\,(x_0;\, u(x_0, y)) \leq_C \theta \tag{4.8}$$

since f is arc-directional differentiable. That is, the inequality

$$T\,(x_0,\, u(x_0, y)) = f'\,(x_0;\, u(x_0, y)) \leq_C \theta \tag{4.9}$$

can not hold for any $y \in K$. Therefore x_0 solves problem (SPVI) related to the inequality (4.8).

(2) If f is a semi-preinvex map on K , then for any $x \in K$, there is a vector

$$u(x_0, x) = \lim_{t \downarrow 0^+} \tau(x_0, x, t) = \frac{d}{dt} t\tau(x_0, x, t)\,|_{t=0} \quad \text{in } X$$

such that

$$f'(x_0; u(x_0, x)) \leq_C f(x) - f(x_0),\ \ f'(x_0; u(x_0, x)) + f(x_0) \leq_C\ f(x) \text{ or}$$

(a) $T'(x_0)u(x_0, x) + f(x_0) \leq_C\ f(x)$.

Since x_0 solves problem (SPVI) related to the inequality (4.8), there does not exist $y \in K$ satisfing

$$T(x_0)u(x_0, y) <_C \theta. \tag{b}$$

It follows from (a) and (b) that there does not exist $y \in K$ satisfying

$$f(y) <_C\ f(x_0).$$

Hence x_0 solves the problem ($\tilde{\text{P}}$). □

If f is preinvex and differentiable at x_0, then for $0 < \alpha < 1$,

$$f(x_0 + \alpha u(x_0, x)) \leq_C (1 - \alpha)f(x_0) + \alpha f(x),$$

$$\frac{f(x_0 + \alpha u(x_0, x)) - f(x_0)}{\alpha} \leq_C f(x) - f(x_0).$$

Letting $\alpha \downarrow 0^+$ we then have

$$f'(x_0)u(x_0, x) \leq_C f(x) - f(x_0)$$

where $f'(x_0) = T(x_0) \in L(X, Y)$. Hence from Theorem 4.1 we conclude that

Corollary 4.2. *If f is a differentiable preinvex map of $K \subset X$ into (Y, C), then f satisfies the inequality*

$$f'(x_0)u(x) \leq_C f(x) - f(x_0)$$

with $u(x) = u(x_0, x)$ if and only if x_0 solves problem (PVI) *related to the inequality* (4.3).

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