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CONSTRAINED ARRANGEMENTS OF OBJECTS IN A CYCLE*

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Abstract. An analysis of connectivity reliability in daisy chain computer networks led to a combinatorial problem that is most easily described in terms of necklaces. Basically n beads, some black and some white, are to be arranged into a necklace. The problem is to find the arrangements that satisfy some constraints on the separations between black beads. Thus, no black beads may be allowed to be adjacent and the number of black beads with s beads between them may be specified. Hwang and Wright gave a matrix method of counting those necklaces. Here we use generating functions for similar counts, but with an added condition that all s separating beads must be white. To reduce the set of arrangements to more manageable size we also count necklaces when two that differ only by a rotation are considered the same.

1. Introduction

A number of problems require objects, of d different kinds, to be placed at n locations that form a closed cycle. Thus, n beads of d colors might be arranged into a necklace. Or, in the problème des menages [5, 7, 9], $d = 2$ kinds of objects (men and women) must be seated at n chairs around a circular table. In another seating problem with $d = 2$, each chair is either occupied or left empty. For the sake of concreteness we will often use the terminology of beads and necklaces but other applications will be obvious.

The basic problem will be to count arrangements that satisfy given constraints. A simple constraint might be that no two black beads can be adjacent

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or, in the case of seating unsociable diners, that no two occupied chairs can be adjacent. Other constraints might specify how many beads of certain colors are to be used. With those constraints, rotating an allowed arrangement bodily by $1, 2, \ldots$, or n places produces only arrangements that still satisfy the constraints. Other constraints to be considered will all have that property.

For the Problème des Menages, Kaplansky [5] derived the following Lemma:

Lemma 1. *The number of ways of selecting k objects with no two adjacent, from n objects arranged in a cycle, is*

$$g(k,n) = \frac{n}{n-k}\left[\begin{array}{c} n-k \\ k \end{array}\right].$$

With k black and $n-k$ white beads replacing the selected and unselected objects, Lemma 1 counts necklaces with no adjacent black beads.

Lemma 1 has been extended to count necklaces of black and white beads, having p pairs of adjacent black beads. That extension has interest in the theory of runs because the number of runs of consecutive black beads is just $k-p$.

A second extension of Lemma 1, first studied by Konvalina [7], involves the notion of *separation-s pair*, a pair of black beads with s beads between them. The s beads can have any colors. Two consecutive black beads form a separation-0 pair. A separation-s pair is also a separation-$(n-2-s)$ pair. Hwang [2], also Kirschenhofer and Prodinger [6], used generating functions to derive the number $g(k,n,p,s)$ of necklaces of k black and $n-k$ white beads with p separation-s pairs. In a further generalization, Hwang and Wright [3] counted necklaces with numbers of separation-s pairs prescribed jointly for several separations s.

Here we are mainly interested in a special kind of separation-s pair, one with all s separating beads colored white. We call such a configuration an *$s-$run* because a run of s consecutive white beads separates the black beads. In some problems all separation-s pairs are also s-runs. For example, in necklaces with no 0-runs, 1-runs, $\ldots$, or a-runs, each black bead lies between two white runs of length at least $a+1$; a separation-s pair with $s < 2a+3$ is then also an s-run. In general we seek the number $g(k,n,P)$ of necklaces with k black and $n-k$ white beads that satisfy a set P of constraints specifying numbers of separation-s pairs or s-runs for some given values of s.

There are two kinds of counting problems. They depend on whether two arrangements, obtained from one another by a rotation around the cycle, are counted as the same. The results cited above treat the n positions around the cycle as distinguishable and consider two arrangements the same only if each position is occupied alike in both arrangements. However, for necklaces, it may be more natural to ignore rotations and count the symmetry types (equivalence classes of arrangements induced by rotations). Because a rotation can carry an arrangement into itself, the numbers of types and arrangements do not just differ by a factor n. However, the number $g^*(k,n,P)$ of types of necklaces subject to given constraints P will be expressed simply in terms of numbers $g(k,n,P)$ of arrangements and we will derive $g^*(k,n,P)$ for specific cases of interest. In counting $g^*(k,n,P)$, reflections can transform arrangements to others of different type.

2. Transformation

In the next lemma, $g(k,n,P)$ is the number of arrangements of k black beads and $n-k$ of other colors. P can be any set of constraints that remain satisfied as an arrangement is rotated. The lemma relates $g(k,n,P)$ to the number $g'(k,n,P)$ of these arrangements having a black bead in position 1.

Lemma 2. $ng'(k,n,P) = kg(k,n,P)$.

Proof. Let $g_i'(k,n,P)$ denote the number, of those arrangements counted by $g(k,n,P)$, having a black bead at position i. Write

$$U = \sum_{i=1}^{n} g_i'(k,n,P).$$

Note $g'(k,n,P) = g_1'(k,n,P)$. Also, $g'(k,n,P) = g_i'(k,n,P)$ for every i because a rotation by $i-1$ positions transforms each arrangement with a black bead at 1 into one with a black bead at i. Thus $U = ng'(k,n,P)$. But also $U = kg(k,n,P)$, because each of the $g(k,n,P)$ arrangements is counted k times in the sum U, once for each black bead. ■

As an illustration, Lemma 2 will count the number of necklaces of black and white beads having at least s white beads between each two black beads. Each arrangement contains k blocks of $s+1$ beads with colors $BWW\dots W$, inserted among $n-k(s+1)$ other white beads. If the bead at position 1 is black, the rest of the arrangement is obtained by choosing the locations of the $k-1$ other blocks from $n-ks-1$ possibilities. Then $g'(k,n,P)$ is a binomial coefficient and Lemma 2 shows

$$g(k,n,P) = \frac{n}{k}\begin{bmatrix} n-ks-1 \\ k-1 \end{bmatrix}. \tag{1}$$

With $s=1$, (1) reduces to Kaplansky's formula in Lemma 1.

3. Symmetry Types

The procedure for counting symmetry types will rely on a form of Pólya's theorem [8, 9]. In general, a symmetry group G partitions objects (here arrangements or necklaces) into types or equivalence classes. Two objects are equivalent if some transformation in G carries one object into the other. Here G is cyclic with transformations R^i, the rotations of the beads by i positions. In general, the number of types can be expressed in terms of the numbers $I(T)$ of objects (arrangements) left invariant by each transformation T,

$$\text{Number of types} = \sum I(T)/|G|, \tag{2}$$

where $|G|$ is the order of G and the sum extends over all transformations T. Since the identity in G leaves all arrangements invariant, the number of types is at least $1/|G|$ times the number of arrangements.

Here G is a group of rotations, the cyclic group of order $|G| = n$. An arrangement is specified by an n-tuple $a = (c_1, c_2, \ldots, c_n)$ with c_r the color of the bead in position r. A rotation $T = R^i$ transforms A into $TA = (c'_1, c'_2, \ldots, c'_n)$ with $c'_r = c_r + i$ and leaves A invariant if $c_r = c_r + i$ for all r (subscripts are to be added modulo n). Since $c_r = c_r + n$ also holds, an invariant A has $c_r = c_r + d$ where $d = (i, n)$ is the g. c. d. of i and n. Then A contains the d-tuple $A_d = (c_1, c_2, \ldots, c_d)$ of d initial colors repeated n/d times, $A = (A_d, A_d, \ldots, A_d)$. The constraints P impose a set of constraints, called dP/n, on A_d. Thus, if A has k beads of color c, then dk/n of them belong to the necklace specified by A_d. Or, if A has p s-runs then A_d has dp/n of them. If A has p separation-s pairs, then A_d has dp/n of them but the seperatron-s pairs or A_d must also be considered separation-s pairs if d divides $s - s'$. Of course, A will not be invariant unless k and p are divisible by n/d. The number of symmetry types in (2) is then

$$g^*(k, n, P) = \frac{1}{n} \sum_{i=1}^{n} g(dk/n, d, dP/n)$$

where $d = (i, n)$ and the only non-zero terms are those with integer constraints dP/n and integer dk/n.

For a given divisor d of n, the number of terms i having $(i, n) = d$ is Euler's function $\phi(n/d)$. Combining the terms with like values of d produces

$$g^*(k, n, P) = \frac{1}{n} \sum_{d/n} \phi\left(\frac{n}{d}\right) g(dk/n, d, dP/n).$$

Or, with d replaced by the divisor n/d, we have

Theorem.
$$g^*(k,n,P) = \frac{1}{n}\sum_d \phi(d)g(k/d,n/d,P/d)$$
$$= \frac{1}{k}\sum_d \phi(d)g'(k/d,n/d,P/d).$$

The second line of the theorem follows from the first because of Lemma 2. The only non-zero terms in the sums have values of d that divide n, k, and all other numbers of beads specified by P.

In a simple example, P might specify, for all i, the number k_i of beads of the i^{th} color. Then the theorem applies with

$$g(k/d,n/d,P/d) = \frac{(n/d)!}{(k_1/d)!(k_2/d)!\dots} \tag{3}$$

and $n = k_1 + k_2 + \dots$.

In another example, there are two colors and P specifies all numbers p_i of i-runs. Then there are $k = p_0 + p_1 + \dots$ black beads and $n = k + p_1 + 2p_2 + \dots$ beads total. The example is actually related to (3) if each black bead, together with the run of white beads that follows it, is regarded as one bead of a new kind of color $BWW\dots W$. Now the theorem applies as though there were k beads of the new colors,

$$g^*(k/d,n/d,P/d) = \frac{1}{k}\sum_d \phi(d)\frac{(k/d)!}{(p_0/d)!(p_1/d)!\dots}. \tag{4}$$

For 2-color necklaces with k black and w white beads, and no further restriction, the number of types is

$$f(k,n) = \frac{1}{n}\sum_d \phi(d)\begin{bmatrix} n/d \\ k/d \end{bmatrix}, \tag{5}$$

a result due to Jablonsky [1, 4, 9]. An immediate extension counts 2-color necklaces with each pair of black beads separated by s or more white beads. Regard each block BW^s of one black bead and s white beads as a single bead of a new color. The necklace then becomes a 2-color necklace with k beads of color BW^s and $n-k-sk$ remaining white beads. The number of types is $f(k, n-sk)$. Table I gives numbers of types when $s = 1$. The types themselves, for $n = 12$ and $s = 1$, appear in Table II as k-tuples $(s_1, s_2, \dots, s_k)$ with s_j the number of white beads that separate the j^{th} and $(j+1)^{st}$ black beads. Cyclic permutations of $(s_1, s_2, \dots, s_k)$ represent the same type. Note that certain pairs of types, e. g. (1, 2, 6) and (1, 6, 2), would have counted as one if reflection symmetries had been allowed.

TABLE I. Numbers of symmetry types of necklaces with k black and $n-k$ white beads, no black beads adjacent

n	$k=1$	2	3	4	5	6	7	8	9	10	total
2	1										1
3	1										1
4	1	1									2
5	1	1									2
6	1	2	1								4
7	1	2	1								4
8	1	3	2	1							7
9	1	3	4	1							9
10	1	4	5	3	1						14
11	1	4	7	5	1						18
12	1	5	10	10	3	1					30
13	1	5	12	14	7	1					40
14	1	6	15	22	14	4	1				63
15	1	6	19	30	26	10	1				93
16	1	7	22	43	42	22	4	1			142
17	1	7	26	55	66	42	12	1			210
18	1	8	31	73	99	80	30	5	1		328
19	1	8	35	91	143	132	66	15	1		492
20	1	9	40	116	201	217	132	43	5	1	765

4. Generating Functions

This section counts 2-color necklaces, such as (4) counted but with not all of $p_0, p_1, p_2, \ldots$ specified. Of course, one can always sum (4) over the unspecified p_i but, as (5) illustrated with all p_i unspecified, there may be simpler solutions. Hwang and Wright [3] gave a matrix method to count arrangements by numbers of separation-i pairs. These pairs were separated by i beads of either color. Here we count arrangements by i-runs, pairs of black beads separated by i white beads only. In general we obtain a generating function for $g'(n,k,P)$ in which powers of x indicate numbers of white beads and, for certain specified i, powers of y_i indicate numbers of i- runs.

The procedure will be clear from a special case. Allow no 0-runs, 1-runs, …, or a-runs i. e. $p_0 = p_1 = \ldots = p_a = 0$, and specify $p_s = p$ for some given

TABLE II. Symmetry types of 2-color necklaces of 12 beads

k	Types
1	(11)
2	$(1,9)$, $(2,8)$, $(3,7)$, $(4,6)$, $(5,5)$
3	$(1,1,7)$, $(1,2,6)$, $(1,6,2)$, $(1,3,5)$, $(1,5,3)$
	$(1,4,4)$, $(2,2,5)$, $(2,3,4)$, $(2,4,3)$, $(3,3,3)$
4	$(1,1,1,5)$, $(1,1,2,4)$, $(1,1,4,2)$, $(1,4,1,2)$, $(1,1,3,3)$
	$(1,3,1,3)$, $(1,2,2,3)$, $(1,3,2,2)$, $(1,2,3,2)$, $(2,2,2,2)$
5	$(1,1,1,1,3)$, $(1,1,1,2,2)$, $(1,1,2,1,2)$
6	$(1,1,1,1,1,1)$

$s > a$. All other p_i are left unspecified. The generating function will have two variables, x and $y = y_s$, and must have $g'(k,n,P)$ as the coefficient of $y^p x^{(n-k)}$. Imagine the necklace decomposed into k blocks of form $BWW\ldots W$. The possible contributions from one block are counted by a generating function

$$\begin{aligned}\psi(x,y) &= x^{a+1} + x^{a+2} + \cdots + yx^s + \cdots \\ &= yx^s - x^s + \frac{x^{a+1}}{1-x}.\end{aligned} \tag{6}$$

Then the generating function for the entire necklace is

$$g(x,y) = [\psi(x,y)]^k. \tag{7}$$

With ψ given by (6),

$$\begin{aligned}g(x,y) &= \sum_p \begin{bmatrix} k \\ p \end{bmatrix} y^p x^{sp} \left[\frac{x^{a+1}}{1-x} - x^s\right]^{k-p} \\ &= \sum_{p,q} \begin{bmatrix} k \\ p \end{bmatrix} \begin{bmatrix} k-p \\ q \end{bmatrix} y^p x^{sp} \left[\frac{x^{a+1}}{1-x}\right]^q (-x^s)^{k-p-q}.\end{aligned}$$

Next, expand

$$\left[\frac{1}{1-x}\right]^q = \sum_r \begin{bmatrix} q+r-1 \\ r \end{bmatrix} x^r.$$

Caution is needed here when $q = 0$. To make the expansion hold at $q = 0$ one must interpret $\begin{bmatrix} r-1 \\ -1 \end{bmatrix}$ to mean 1 if $r = 0$, and 0 otherwise. With that convention,

$$g(x,y) = \sum_{p,q,r} \begin{bmatrix} k \\ p \end{bmatrix} \begin{bmatrix} k-p \\ q \end{bmatrix} \begin{bmatrix} q+r-1 \\ q-1 \end{bmatrix} (-1)^{k+p+q} y^p x^{sk+(a+1-s)q+r}.$$

Now $g'(k,n,P)$ is the coefficient of $y^p x^{n-k}$ in $g(x,y)$,

$$(8)\quad g'(k,n,P) = (-1)^{k+p} \begin{bmatrix} k \\ p \end{bmatrix} \sum_q (-1)^q \begin{bmatrix} k-p \\ q \end{bmatrix} \begin{bmatrix} n-(s+1)k+(s-a)q-1 \\ q-1 \end{bmatrix}.$$

The generating function simplifies if $s = a+1$. For, (6) becomes

$$\psi(x,y) = \{y + x/(1-x)\} x^{(a+1)}.$$

The derivation of $g(x,y)$ now needs only two binomial expansions; the coefficient of $y^p x^{(n-k)}$ in $g(x,y)$ becomes

$$(9)\qquad g'(k,n,P) = \begin{bmatrix} k \\ p \end{bmatrix} \begin{bmatrix} n-ak-2k-1 \\ k-p-1 \end{bmatrix}.$$

One can also count arrangements with specified numbers of s-runs for more than one given s by introducing extra variables like y into the generating function. The result will be a formula like (8) but summed over more indices.

Hwang and Wright [3] counted arrangements with given numbers of separation-s pairs. Their Tables 1 and 2 gave numbers $g(k,n,P)$ of necklaces with no black beads adjacent, counted by separation-2 pairs. As Section 1 explained, these separation-2 pairs are also 2-runs. Indeed, one may obtain the same numbers from (8) with the aid of Lemma 2.

The numbers $g'(k,n,P)$ in (8) or (9) can be used in the second form of the Theorem to count symmetry types. Table III gives data on the calculation for 15-bead necklaces with no adjacent black beads ($a = 0$), counted by numbers p of 2-runs (or separation-2 pairs). Numbers of types are in parentheses. The other numbers are numbers $g(k,n,P)$ of arrangements calculated from (8) and Lemma 2; these agree with numbers tabulated in [3]. The numbers of types are conveniently smaller than the numbers of arrangements. Thus, the 155 arrangements with $k = 3$ black beads and $p = 0$ 2-runs fall into only 11 types. These may be exhibited in the notation of Table II as $(1,1,10)$, $(1,3,8)$, $(1,8,3)$, $(1,4,7)$, $(1,7,4)$, $(1,5,6)$, $(1,6,5)$, $(3,3,6)$, $(3,4,5)$, $(3,5,4)$, $(4,4,4)$. In the Theorem, d must divide p as well as n and k. Often, the only such divisor is $d = 1$ and then $g^*(k,n,P)$ reduces to $g(k,n,P)/n$. The calculation requires numbers of arrangements of 3 or 5 beads, also tabulated. Note, with 3 beads, that one black bead forms a 2-run with itself.

TABLE III. Numbers of arrangements and types (in parentheses) of necklaces with p 2-runs (or separation-2 pairs) and no adjacent black beads.

n	k	$p=0$	1	2	3	4	5
3	1		3 (1)				
5	1	5 (1)					
	2		5 (1)				
15	1	15 (1)					
	2	75 (5)	15 (1)				
	3	155 (11)	105 (7)	15 (1)			
	4	150 (10)	195 (13)	90 (6)	15 (1)		
	5	75 (5)	150 (10)	90 (6)	60 (4)		3 (1)
	6	15 (1)	75 (5)		50 (4)		
	7		15 (1)				

The number of types could have been reduced by allowing reflection as well as rotation symmetries as equivalences. In (2), G would be the dihedral group of order $|G| = 2n$. Reflections applied to the necklaces of same rotational type usually produce necklaces of a different rotational type. The union of the two types is a dihedral type. Other rotational types contain necklaces that have a reflection symmetry; these types are already dihedral types. Thus, of the 11 rotational types in Table 3 with $n = 15,\ k = 3,\ a = 0,\ s = 2$, $(1,1,10)$, $(3,3,6)$, and $(4,4,4)$ contain necklaces with reflection symmetries and are already dihedral types. The remaining 8 rotational types pair off into 4 dihedral types; then there are 7 dihedral types total. Necklaces having reflection symmetries can also be counted by generating functions but the result is more complicated than (8); using a dihedral group would reduce the number of types by a factor 2 at most.

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