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NONOSCILLATORY PROPERTY OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS OF ADVANCED TYPE*

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Abstract. In this paper, we study the nonoscillatory property of the second order nonlinear differential equation of advanced type:

$$(r(t)\phi(x(t))x'(t))' + p(t)f(x(g(t))) = 0.$$

We prove an existence theorem for nonoscillatory solutions and compare the oscillation of nonlinear equations.

1. Introduction

Consider the second order nonlinear differential equation of advanced type:

$$\text{(E)} \qquad (r(t)\phi(x(t))x'(t))' + p(t)f(x(g(t))) = 0, \qquad t \geq t_0 \geq 0,$$

where

(H$_1$) $r(t) \in C^1([t_0, \infty), (0, \infty))$, $\int_t^\infty \frac{ds}{r(s)} = \infty$;

(H$_2$) $p(t) \in C([t_0, \infty), [0, \infty))$, $p \not\equiv 0$;

(H$_3$) $\phi(x) \in C^1([-\infty, \infty), (0, \infty))$, $\phi(x) \neq 0$ $(x \neq 0)$;

(H$_4$) $f(x) \in C^1(-\infty, \infty), (-\infty, \infty))$, $xf(x) > 0$ $(x \neq 0)$;

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(H_5) $G(x) = \frac{f'(x)}{\phi(x)} > 0 \quad (x \neq 0)$; $G(x)$ is nondecreasing in $(0, \infty)$ and nonincreasing in $(-\infty, 0)$.

(H_6) $g(t) \in C([t_0, \infty) \to [0, \infty))$, and $g(t) \geq t$.

A nontrivial solution $x(t)$ is said to be oscillatory if it has arbitrarily large zeros, otherwise $x(t)$ is said to be nonoscillatory.

The aim of this note is to prove the following characterization of the existence of nonoscillatory solutions of (E). The proof is an adaptation of that given in [1], where the special case $g(t) = t$ was considered.

Theorem 1. Under the hypotheses (H_1) – (H_6), (E) *has a nonoscillatory solution if and only if there exists a positive differentiable function* $\varphi(t)$ *defined on* $[t_0, \infty)$ *and some* $a \neq 0$ *such that*

$$\begin{aligned}(1)\qquad &\varphi'(t) + \tfrac{\varphi^2(t)}{r(t)} G\left(F^{-1}\left(a, \int_{t_1}^{t} \frac{\varphi(u)}{r(u)} du\right)\right) \\ &\leq -p(t) \exp\left(\int_{t}^{g(t)} \frac{\varphi(s)}{r(s)} G\left(F^{-1}(a, \int_{t_1}^{s} \frac{\varphi(u)}{r(u)} du\right)\right) ds, \quad t \geq t_1 \geq t_0.\end{aligned}$$

Here F *denote the function* $F(a, x) = \int_a^x \frac{\phi(u)}{f(u)} du,\ a \neq 0$.

It is clear that when $a > 0$, $F(a, x)$ is strictly increasing for $x > 0$; when $a < 0$, $F(a, x)$ is strictly decreasing for $x < 0$ (see [1] for further properties of $F(a, x)$).

By (H_1), (H_5), (H_6), and $\varphi(t) > 0$, we see that the integral $\int_t^{g(t)} \cdots$ is positive, which together with (1) implies that

$$\varphi'(t) + \frac{\varphi^2(t)}{r(t)} G\left(F^{-1}\left(a, \int_{t_1}^{t} \frac{\varphi(u)}{r(u)} du\right)\right) \leq -p(t), \quad t \geq t_1.$$

Then, by applying Theorem 1 to the case $g(t) = t$, we immediately deduce the following corollary.

Corollary 1. *If* (E) *has a nonoscillatory solution, then equation* (E) *with* $g(t) = t$ *also has a nonoscillatory solution.*

According to Theorem 1 (with $g(t) = t$) and Corollary 1, we easily deduce the following corollary which generalizes Hille-Wintner comparison theorem [2, 3] and Theorem 2.3 of [4] to include nonlinear equations of advanced type.

Consider, together with (E), the following equation:

$$(^*) \qquad (\bar{r}(t)\phi(x(t))x'(t))' + \bar{p}(t) f(x(g(t))) = 0.$$

Corollary 2. *Suppose $\bar{r}$ and $\bar{p}$ also satisfy (H_1) and (H_2), and suppose that $0 < r(t) \le \bar{r}(t)$, $0 \le \bar{p}(t) \le p(t)$, $t \ge t_1 \ge t_0$. If equation* (E) *has a nonoscillatory solution, then the equation* (*) *also has a nonoscillatory solution.*

2. Proofs of Results

We will see later that all solutions of (E) oscillate when $\int^{\infty} p(s)ds = \infty$ (Remark 1). Hence we assume that $P(t) = \int_t^{\infty} p(s)ds < \infty$. First, we modify the arguments in the proof of Lemma 1 of [1] (which is for the case $g(t) = t$) to prove the next lemma.

Lemma 1. *If* (E) *has an eventually positive solution* $x(t) > 0\,(t \ge t_1 \ge t_0)$, *then* $\omega(t) > 0$, $\lim_{x\to+\infty} w(t) = 0$ *and for* $t \ge t_1$

$$
\begin{aligned}
w(t) = &\int_t^{\infty} \frac{w^2(s)}{r(s)} G\left(F^{-1}\left(x(t_1), \int_{t_1}^{s} \frac{w(u)}{r(u)} du\right)\right) ds \\
&+ \int_t^{\infty} p(s) \exp\left(\int_s^{g(s)} \frac{w(u)}{r(u)} G\left(F^{-1}\left(x(t_1), \int_{t_1}^{u} \frac{w(v)}{r(v)} dv\right)\right) du\right) ds,
\end{aligned} \tag{2}
$$

where $w(t) = \frac{r(t)\phi(x(t))x'(t)}{f(x(t))}$.

Proof. From (E), we have

$$
w'(t) + \frac{w^2(t)}{r(t)} \frac{f'(x(t))}{\phi(x(t))} + p(t) \frac{f(x(g(t)))}{f(x(t))} = 0, \qquad t \ge t_1. \tag{3}
$$

Since

$$
\frac{w(t)}{r(t)} \frac{f'(x(t))}{\phi(x(t))} = \frac{f'(x(t))x'(t)}{f(x(t))} = (\ell n f(x(t)))',
$$

we have

$$
\frac{f(x(g(t)))}{f(x(t))} = \exp\left(\int_t^{g(t)} \frac{w(s)}{r(s)} \frac{f'(x(s))}{\phi(x(s))} ds\right). \tag{4}
$$

Thus, from (3) and (4), we have

$$
w'(t) + \frac{w^2(t)}{r(t)} \frac{f'(x(t))}{\phi(x(t))} + p(t) \exp\left(\int_t^{g(t)} \frac{w(s)}{r(s)} \frac{f'(x(s))}{\phi(x(s))} ds\right) = 0, \ t \ge t_1. \tag{5}
$$

From (5), we have $w'(t) \le 0$. Integrating the above equation from t to T ($t_1 \le t \le T$), we obtain

$$
\begin{aligned}
w(T) - w(t) + &\int_t^T \frac{w^2(s)}{r(s)} \frac{f'(x(s))}{\phi(x(s))} ds \\
&+ \int_t^T p(s) \exp\left(\int_s^{g(s)} \frac{w(u)}{r(u)} \frac{f'(x(u))}{\phi(x(u))} du \right) ds = 0, \;\; t \geq t_1.
\end{aligned} \tag{6}
$$

Since

$$
\int_{t_1}^t \frac{w(s)}{r(s)} ds = \int_{t_1}^t \frac{\phi(x(s))x'(s)}{f(x(s))} = \int_{x(t_1)}^{x(t)} \frac{\phi(x)}{f(x)} dx, \quad t \geq t_1,
$$

we have

$$
x(t) = F^{-1}\left(x(t_1), \int_{t_1}^t \frac{w(s)}{r(s)} ds \right).
$$

From (6), we have for $t \geq t_1 \geq t_0$

$$
\begin{aligned}
w(t) = &\int_t^\infty \frac{w^2(s)}{r(s)} G\left(F^{-1}\left(x(t_1), \int_{t_1}^s \frac{w(u)}{r(u)} du \right) \right) ds \\
&+ \int_t^\infty p(s) \exp\left(\int_s^{g(s)} \frac{w(u)}{r(u)} G\left(F^{-1}\left(x(t_1), \int_{t_1}^u \frac{w(v)}{r(v)} dv \right) \right) du \right) ds.
\end{aligned}
$$

Finally, the same arguments in Lemma 1 of [1] show that $w(t) > 0$ and $\lim_{x \to +\infty} w(t) = 0$.

Remark. Suppose $\int^\infty p(t)dt = \infty$. If (E) has an eventually positive solution $x(t)$, then from (5) we have $w'(t) + p(t) \leq 0$, hence $w(t) - w(t_1) \leq -\int_{t_1}^t p(t)dt \to -\infty$ as $t \to \infty$, which implies $x'(t) < 0$ eventually. Now following the proof in Lemma 1, we can get the contradictory conclusion that $x(t) < 0$ eventually. Hence every solution of (E) oscillates if $\int^\infty p(t)dt = \infty$ holds.

Proof of Theorem 1. The necessity part has been proved in Lemma 1. We now prove the sufficiency part of the theorem. Without loss of generality, set $a > 0$. The same argument as in Theorem 1 of [1] shows that $\lim_{t \to \infty} \varphi(t) = 0$. Integrating (1) from t to ∞, we have for $t \geq t_1$

$$
\begin{aligned}
&\int_t^\infty \frac{\varphi^2(s)}{r(s)} G\left(F^{-1}\left(a, \int_{t_1}^s \frac{\varphi(u)}{r(u)} du \right) \right) ds \\
&+ \int_t^\infty p(s) \exp\left(\int_s^{g(s)} \frac{\varphi(u)}{r(u)} G\left(F^{-1}\left(a, \int_{t_1}^u \frac{\varphi(v)}{r(v)} dv \right) \right) du \right) ds \leq \varphi(t).
\end{aligned} \tag{7}
$$

Define a mapping T as follows:

$$(8)\quad \begin{aligned}(Ty)(t) = &\int_t^{\infty} \frac{y^2(s)}{r(s)} G\left(F^{-1}\left(a, \int_{t_1}^{s} \frac{y(u)}{r(u)}du\right)\right) ds \\ &+ \int_t^{\infty} p(s) \exp\left(\int_t^{g(t)} \frac{y(u)}{r(u)} G\left(F^{-1}\left(a, \int_{t_1}^{u} \frac{y(v)}{r(v)}dv\right)\right) du\right) ds\end{aligned}$$

for $t \geq t_1$. Let $x_0(t) = 0,\ x_n(t) = (Tx_{n-1})(t),\ n = 1, 2, \ldots$. Then $x_0(t) \leq x_1(t) \leq \cdots \leq x_n(t) \leq \cdots \leq \varphi(t)$. Hence $\lim_{t\to\infty} x_n(t) = w(t) \leq \varphi(t)$. From (8), we have

$$(9)\quad \begin{aligned}x_n(t) = &\int_t^{\infty} \frac{x_{n-1}^2(s)}{r(s)} G\left(F^{-1}\left(a, \int_{t_1}^{s} \frac{x_{n-1}(u)}{r(u)}du\right)\right) ds \\ &+ \int_t^{\infty} p(s) \exp\left(\int_s^{g(s)} \frac{x_{n-1}(u)}{r(u)} G\left(F^{-1}\left(a, \int_{t_1}^{s} \frac{x_{n-1}(v)}{r(v)}dv\right)\right) du\right) ds.\end{aligned}$$

According to Lebesgue's theorem, letting $n \to \infty$ in (9), we get

$$(10)\quad \begin{aligned}w(t) = &\int_t^{\infty} \frac{w^2(s)}{r(s)} G\left(F^{-1}\left(a, \int_{t_1}^{s} \frac{w(u)}{r(u)}du\right)\right) ds \\ &+ \int_t^{\infty} p(s) \exp\left(\int_s^{g(s)} \frac{w(u)}{r(u)} G\left(F^{-1}\left(a, \int_{t_1}^{s} \frac{w(v)}{r(v)}dv\right)\right) du\right) ds.\end{aligned}$$

Let

$$(11)\quad x(t) = F^{-1}\left(a, \int_{t_1}^{t} \frac{w(u)}{r(u)}du\right), \quad t \geq t_1.$$

Then

$$\int_a^{x(t)} F(a, x(t)) = \int_{t_1}^{t} \frac{w(u)}{r(u)}du, \quad t \geq t_1,$$

and so

$$F_x'(a, x(t))x'(t) = \frac{w(t)}{r(t)}, \quad t \geq t_1,$$

$$(12)\quad \frac{\phi(x(t))x'(t)}{f(x(t))} = \frac{w(t)}{r(t)}, \quad t \geq t_1.$$

From (10), we have

$$(13)\quad w'(t) + \frac{w^2(t)}{r(t)} \frac{f'(x(t))}{\phi(x(t))} + p(t) \exp\left(\int_t^{g(t)} \frac{w(s)f'(x(s))}{r(s)\phi(x(s))}ds\right) = 0, \quad t \geq t_1.$$

From (4), (12) and (13), we have

$$(r(t)\phi(x(t))x'(t))' + p(t)f(x(g(t)) = 0, \quad t \geq t_1,$$

which implies that $x(t)$ is a nonoscillatory solution of (E). The proof is completed.

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