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ON THE MINIMUM AREA OF CONVEX LATTICE POLYGONS*

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Abstract. A convex polygon is a polygon whose vertices are points on the integer lattice with interior angles all convex. Let $a(v)$ be the least possible area of a convex lattice polygon with v vertices. It is known that $cv^{2.5} \leq a(v) \leq (15/784)v^3 + o(v^3)$. In this paper, we prove that $a(v) \geq (1/1152)v^3 + O(v^2)$.

A convex lattice polygon is a polygon whose vertices are points on the integer lattice with interior angles strictly less than π radians. A convex lattice polygon with v vertices is called a *v-gon*. The least possible area of a *v-gon* is denoted by $a(v)$. The function $a(v)$ has been studied by Arkinstall [1], Rabinowitz [4], Simpson [5], Colbourn and Simpson [3]. The values of $a(v)$ are known for $v \leq 10$ and $v \in \{12, 13, 14, 16, 18, 20, 22\}$. For example, $a(3) = 1/2, a(4) = 1, a(5) = 5/4, a(6) = 3, \ldots$. For general v, only bounds are known. Rabinowitz [4] proved that $a(2n) \leq \binom{n}{3} - n + 1$. Simpson [5] established that $a(2n) \geq \binom{n}{2}$, and that

$$(1) \qquad [\{a(2n+2) + a(2n)\}/2] + \frac{1}{2} \leq a(2n+1) \leq a(2n+2) - \frac{1}{2}.$$

These together imply that for all v,

$$(1/8)v^2 + o(v^2) \leq a(vc) \leq (1/48)v^3 + o(v^3).$$

In 1992, Colbourn and Simpson [3] proved that

$$cv^{2.5} \leq a(v) \leq (15/784)v^3 + o(v^3)$$

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for some positive constant c. Moreover, they conjecture that $a(v) = c_0 v^3 + o(v^3)$ for a constant c_0.

In this paper, we improve the lower bound on $a(v)$, by proving the following theorem (I announced in [2]).

Theorem. *The minimum area of a convex lattice v-gon, $a(v)$, satisfies*

$$a(v) \geq (1/1152)v^3 + O(v^2).$$

From (1) we need only to treat the cases with v even; therefore, we write $v = 2n$. Now, define an admissible n-sequence V to be a sequence of ordered pairs $\{v_i = (x_i, y_i), 1 \leq i \leq n\}$ satisfying

$$\begin{array}{ll} y_i x_j - x_i y_j > 0 & for\ 1 \leq i < j \leq n, \\ gcd(x_i - y_i) = 1 & for\ 1 \leq i \leq n, \\ y_i \geq x_i > 0 & for\ 1 \leq i \leq n. \end{array}$$

We need the following characterization of $a(2n)$ in determining new lower bound.

Lemma. [5] *One has*

$$a(2n) = \min \sum_{i=1}^{n} \sum_{j=i+1}^{n} (y_i x_j - x_i y_j) \tag{2}$$

where the minimum is taken over all admissible n-sequences.

Proof of Theorem. Suppose that $\{v_1,\ v_2, \ldots, v_n\}$ is an admissible n-sequence. Consider the contribution to equation (2) arising from pairs containing $v_1 = (x_1, y_1)$, let L_i be the set of vectors v_j whose contribution is i, i.e., $y_1 x_j - x_1 y_j = i$, and $l_i = |L_i|$ be the number of the elements in L_i. Then

$$\sum_{i=1}^{\infty} l_i = n - 1. \tag{3}$$

Consider the contribution to (2) arising from pairs between v_1 and the vectors in L_i, this is $i\,l_i$. Furthermore, let (x_0, y_0) be the least pair of positive integers satisfying $y_1 x_0 - x_1 y_0 = i$. Then all the vectors in L_i are in the form $(x_i, y_j) = (x_0 + x_1 t_j, y_0 + y_1 t_j)$, with $t_j \geq 0, 1 \leq j \leq l_i$. If $l_i \geq 2$, consider the contribution to (2) arising from pairs among the vectors in L_i, this is $i\ \sum_{t_j > t'_j} (t_j - t_{j'})$. It is easy to see that the above summation is minimized when $t_j = 1, 2, \ldots, l_i, 1 \leq$

$j \le l_i$, i.e.,

$$\begin{aligned} & i\,\{1(l_i-1)+2(l_i-2)+\ldots+(l_i-1)1\} \\ & \quad = i\,\{l_i(1+2+\ldots+(l_i-1))-(1^2+2^2+\ldots+(l_i-1)^2)\} \\ & \quad = i\; l_i\;(l_i^2-1)/6, \end{aligned}$$

which means that the total contribution to (2) arising from pairs among v_1 and the vectors in L_i is

$$s_i \ge i l_i (l_i^2+5)/6. \tag{4}$$

Using (3) we have

$$\sum_{l_i \ge 1} \frac{1}{\sqrt{i}} \le 2\sqrt{n}.$$

This together with (3), (4) and Hölder's inequality gives us

$$\begin{aligned} n-1 &= \sum_{i=1}^{\infty} l_i = \sum_{l_i \ge 1} (i^{-\frac{1}{3}}, i^{\frac{1}{3}} l_i) \\ &\le \left(\sum_{l_i \ge 1} i^{-\frac{1}{2}}\right)^{\frac{2}{3}} \left(\sum_{i=1}^{\infty} i l_i^3\right)^{\frac{1}{3}} \\ &\le 2^{\frac{2}{3}} n^{\frac{1}{3}} \left(6 \sum_{i=1}^{\infty} s_i\right)^{\frac{1}{3}}. \end{aligned}$$

Therefore

$$S_1 = \sum_{i=1}^{\infty} s_i \ge (1/24) n^2 \left(1-\frac{1}{n}\right)^3 = (1/24)n^2 + O(n).$$

Similarly, we consider the contribution to (2) arising from pairs containing $v_k = (x_k, y_k), k \ge 2$ and $v_j = (x_i, y_j), k < j \le n$. Since $\{v_k, v_{k+1}, \ldots, v_n\}$ is also an admissible $n-k+1$-sequence, the total contribution to (2), S_k, is

$$S_k \ge (1/24)(n-k+1)^2 + O(n-k+1). \tag{5}$$

Let L_i^k be the set of vectors $v_j (j > k)$ whose contribution with v_k are i, i.e., $y_k x_j - x_k y_j = i$. Then

$$|L_j^k \cap L_{j'}^{k'}| \le 1. \tag{6}$$

In fact, if there are $1 \le k < k' < a < b \le n$ such that

$$\begin{cases} x_a y_k - x_k y_a = j, \\ x_b y_k - x_k y_b = j, \end{cases} \tag{7}$$

$$\begin{cases} x_a y'_k - x'_k y_a = j', \\ x_b y'_k - x'_k y_b = j', \end{cases} \tag{8}$$

from (7) and (8), we obtain, respectively,

$$\frac{x_a - x_b}{y_a - y_b} = \frac{x_k}{y_k}, \; \frac{x_a - x_b}{y_a - y_b} = \frac{x'_k}{y'_k},$$

which means that

$$\frac{x_k}{y_k} = \frac{x'_k}{y'_k}.$$

This is impossible. From (6) we conclude that in the preceding way the contribution to (2) arising from any two fixed pairs is at most twice in $S_1 + S_2 + \cdots + S_{n-1}$. Summing (5) over $1 \le k \le n-1$, one has

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=i+1}^{n} (y_i x_j - x_i y_j) &\ge \frac{1}{2} \sum_{k=1}^{n-1} S_k \\ &\ge \frac{1}{48} \sum_{k=1}^{n-1} (n-k+1)^2 + O\left(\sum_{k=1}^{n-1} n-k+1 \right) \\ &= \left(\frac{1}{144} \right) n^3 + O(n^2). \end{aligned}$$

By the Lemma, we complete the proof of the Theorem.

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