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FIXED POINT THEOREMS AND WEAK CONVERGENCE THEOREMS FOR GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first consider a broad class of nonlinear mappings containing the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Then, we deal with fixed point theorems and weak convergence theorems for these nonlinear mappings in a Hilbert space.

1. INTRODUCTION

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$. From Baillon [1] we know the following first nonlinear ergodic theorem in a Hilbert space.

Theorem 1.1. *Let C be a nonempty bounded closed convex subset of H and let $T : C \to C$ be nonexpansive. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping F is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [3] and Goebel and Kirk [5]. It is known that a firmly nonexpansive mapping F can be deduced from an equilibrium

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problem in a Hilbert space; see, for instance, [2] and [4]. Recently, Kohsaka and Takahashi [11], and Takahashi [16] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $T : C \to C$ is called *nonspreading* [11] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. Similarly, a mapping $T : C \to C$ is called *hybrid* [16] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [10] and Iemoto and Takahashi [8]. Very recently, Takahashi and Yao [19] proved the following nonlinear ergodic theorem.

Theorem 1.2. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself such that $F(T)$ is nonempty. Suppose that T satisfies one of the following conditions:*

(i) *T is nonspreading;*

(ii) *T is hybrid;*

(iii) $2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$

Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

In this paper, motivated by Takahashi and Yao [19], we introduce a broad class of mappings $T : C \to C$ such that for some $\alpha, \beta \in \mathbb{R}$,

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a class contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Then, we prove fixed point theorems for such nonlinear mappings in a Hilbert space. Furthermore, we obtain a nonlinear ergodic theorem of Baillon's type for this class of mappings which generalizes Theorems 1.1 and 1.2 in a Hilbert space. Finally, we prove a weak convergence theorem of Mann's type [12] for this class of mappings.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a (real) Hilbert space with inner product

$\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [15], we know the following basic equalities. For $x, y, u, v \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.1}$$

and

$$2\langle x-y, u-v\rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.2}$$

From (2.2), we also have the following equality.

$$\begin{aligned} \|x-y+u-v\|^2 &= \|x-y\|^2 + \|u-v\|^2 + 2\langle x-y, u-v\rangle \\ &= \|x-y\|^2 + \|u-v\|^2 + \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \end{aligned} \tag{2.3}$$

Let C be a nonempty closed convex subset of H and let T be a mapping from C into itself. Then, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [9]. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0,1]$, put $z = \alpha x + (1-\alpha)y$. Then, we have from (2.1) that

$$\begin{aligned} \|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\ &= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\ &= \alpha(1-\alpha)^2\|x-y\|^2 + (1-\alpha)\alpha^2\|x-y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\ &= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|x-y\|^2 \\ &= 0. \end{aligned}$$

This implies $Tz = z$. So, $F(T)$ is convex.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n x_n \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n x_n = a$. For a proof of existence of a Banach limit and its other elementary properties, see [14]. Using Banach limits, Takahashi and Yao [19] proved the following fixed point theorem.

Theorem 2.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself. Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and*

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2, \quad \forall y \in C$$

for some Banach limit μ. Then, T has a fixed point in C.

Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for all $x \in H$ and $u \in C$; see [15] for more details.

3. Fixed Point Theorems

In this section, we start with defining a broad class of nonlinear mappings containing the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Then, a mapping $T : C \to C$ is called *generalized hybrid* if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{3.4}$$

for all $x, y \in C$. We call such a mapping an (α, β)-*generalized hybrid* mapping. We observe that the mapping above covers several well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. We can also show that if $x = Tx$, then for any $y \in C$,

$$\alpha \|x - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|x - y\|^2 + (1 - \beta)\|x - y\|^2$$

and hence $\|x - Ty\| \leq \|x - y\|$. This means that an ($\alpha$, β)-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Now, we prove a fixed point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

Proof. Since $T : C \to C$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.5}$$

for all $x, y \in C$. If $F(T) \neq \emptyset$, then $\{T^n z\} = \{z\}$ for $z \in F(T)$. So, $\{T^n z\}$ is bounded. We show the reverse. Take $z \in C$ such that $\{T^n z\}$ is bounded. Let μ be a Banach limit. Then, for any $y \in C$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}\alpha\|T^{n+1}z - Ty\|^2 &+ (1-\alpha)\|T^n z - Ty\|^2 \\ &\le \beta\|T^{n+1}z - y\|^2 + (1-\beta)\|T^n z - y\|^2\end{aligned}$$

for any $y \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Then, we have

$$\begin{aligned}\mu_n(\alpha\|T^{n+1}z - Ty\|^2 &+ (1-\alpha)\|T^n z - Ty\|^2) \\ &\le \mu_n(\beta\|T^{n+1}z - y\|^2 + (1-\beta)\|T^n z - y\|^2).\end{aligned}$$

So, we obtain

$$\begin{aligned}\alpha\mu_n\|T^{n+1}z - Ty\|^2 &+ (1-\alpha)\mu_n\|T^n z - Ty\|^2 \\ &\le \beta\mu_n\|T^{n+1}z - y\|^2 + (1-\beta)\mu_n\|T^n z - y\|^2\end{aligned}$$

and hence

$$\begin{aligned}\alpha\mu_n\|T^n z - Ty\|^2 &+ (1-\alpha)\mu_n\|T^n z - Ty\|^2 \\ &\le \beta\mu_n\|T^n z - y\|^2 + (1-\beta)\mu_n\|T^n z - y\|^2.\end{aligned}$$

This implies

$$\mu_n\|T^n z - Ty\|^2 \le \mu_n\|T^n z - y\|^2$$

for all $y \in C$. By Theorem 2.1, we have a fixed point in C. ∎

As a direct consequence of Theorem 3.1, we have the following result.

Theorem 3.2. *Let C be nonempty bounded closed convex subset of a Hilbert space H and let T be a generalized hybrid mapping from C to itself. Then T has a fixed point.*

Using Theorem 3.1, we can also prove the following well-known fixed point theorems. We first prove a fixed point theorem for nonexpansive mappings in a Hilbert space.

Theorem 3.3. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.1, a (1, 0)-generalized hybrid mapping of C into itself is nonexpansive. By Theorem 3.1, T has a fixed point in C. ∎

The following is a fixed point theorem for nonspreading mappings in a Hilbert space.

Theorem 3.4. ([11]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.1, a (2, 1)-generalized hybrid mapping of C into itself is nonspreading. By Theorem 3.1, T has a fixed point in C. ∎

The following is a fixed point theorem for hybrid mappings by Takahashi [16] in a Hilbert space.

Theorem 3.5. ([16]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a hybrid mapping, i.e.,*

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.1, a ($\frac{3}{2}$, $\frac{1}{2}$)-generalized hybrid mapping of C into itself is hybrid in the sense of Takahashi [16]. By Theorem 3.1, T has a fixed point in C. ∎

We can also prove the following fixed point theorem in a Hilbert space.

Theorem 3.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a mapping such that*

$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, T has a fixed point in C.

Proof. In Theorem 3.1, a (1, $\frac{1}{2}$)-generalized hybrid mapping of C into itself is the mapping in our theorem. By Theorem 3.1, T has a fixed point in C. ∎

Let C be a nonempty closed convex subset of a Hilbert space H. A mapping $S : C \to C$ is called *super hybrid* if there are $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma \geq 0$ such that

$$\begin{aligned} &\alpha\|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2 \\ &\leq \big(\beta + (\beta - \alpha)\gamma\big)\|Sx - y\|^2 + \big(1 - \beta - (\beta - \alpha - 1)\gamma\big)\|x - y\|^2 \\ &\quad + (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2 \end{aligned} \tag{3.6}$$

for all $x, y \in C$. We call such a mapping an (α, β, γ)-*super hybrid* mapping. We notice that an (α, β, 0)-super hybrid mapping is (α, β)-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings.

Theorem 3.7. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. If a mapping $S : C \to C$ is (α, β, γ)-super hybrid, then the mapping $\frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$ is an (α, β)-generalized hybrid mapping of C into itself.*

Proof. Put $\lambda = \frac{1}{1+\gamma} \neq 0$ and $T = \lambda S + (1-\lambda)I$. Let us consider

$$k := -\alpha\|Tx - Ty\|^2 - (1-\alpha)\|x - Ty\|^2 + \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2.$$

Since $T = \lambda S + (1-\lambda)I$, we have

$$\begin{aligned} k = -\alpha\|\lambda(Sx - Sy) + (1-\lambda)(x-y)\|^2 - (1-\alpha)\|\lambda(x - Sy) + (1-\lambda)(x-y)\|^2 \\ + \beta\|\lambda(Sx - y) + (1-\lambda)(x-y)\|^2 + (1-\beta)\|x-y\|^2. \end{aligned}$$

Applying the identity (2.1), we get

$$\begin{aligned} k = -\alpha\left\{\lambda\|Sx - Sy\|^2 + (1-\lambda)\|x-y\|^2 - \lambda(1-\lambda)\|Sx - Sy - x + y\|^2\right\} \\ - (1-\alpha)\left\{\lambda\|x - Sy\|^2 + (1-\lambda)\|x-y\|^2 - \lambda(1-\lambda)\|y - Sy\|^2\right\} \\ +\beta\left\{\lambda\|Sx - y\|^2 + (1-\lambda)\|x-y\|^2 - \lambda(1-\lambda)\|x - Sx\|^2\right\} + (1-\beta)\|x-y\|^2. \end{aligned}$$

Adding four terms $\|x-y\|^2$ due to $-\alpha - (1-\alpha) + \beta + (1-\beta) = 0$ and dividing by λ, we obtain

$$\begin{aligned} \lambda^{-1}k &= -\alpha\left\{\|Sx - Sy\|^2 - \|x-y\|^2 - (1-\lambda)\|Sx - Sy - x + y\|^2\right\} \\ &\quad -(1-\alpha)\left\{\|x - Sy\|^2 - \|x-y\|^2 - (1-\lambda)\|y - Sy\|^2\right\} \\ &\quad +\beta\left\{\|Sx - y\|^2 - \|x-y\|^2 - (1-\lambda)\|x - Sx\|^2\right\}. \end{aligned}$$

So, we have

$$\begin{aligned} \lambda^{-1}k &= -\alpha\|Sx - Sy\|^2 - (1-\alpha)\|x - Sy\|^2 \\ &\quad +\beta\|Sx - y\|^2 + (1-\beta)\|x-y\|^2 - \beta(1-\lambda)\|x - Sx\|^2 \\ &\quad +(1-\alpha)(1-\lambda)\|y - Sy\|^2 + \alpha(1-\lambda)\|Sx - Sy - x + y\|^2. \end{aligned}$$

Dividing by λ, we have from $\lambda^{-1} = \gamma + 1$ that

$$\begin{aligned} \lambda^{-2}k = -\alpha(\gamma+1)\|Sx - Sy\|^2 - (1-\alpha)(\gamma+1)\|x - Sy\|^2 \\ + \beta(\gamma+1)\|Sx - y\|^2 + (\gamma+1)(1-\beta)\|x-y\|^2 - \beta\gamma\|x - Sx\|^2 \\ + (1-\alpha)\gamma\|y - Sy\|^2 + \alpha\gamma\|Sx - Sy - x + y\|^2. \end{aligned}$$

We know from (2.3) that

$$\begin{aligned}\|Sx-Sy-x+y\|^2 &= \|Sx-Sy\|^2-\|x-Sy\|^2-\|Sx-y\|^2\\ &\quad+\|x-y\|^2+\|Sx-x\|^2+\|Sy-y\|^2.\end{aligned}$$

So, we obtain

$$\begin{aligned}\lambda^{-2}k &= -\alpha\|Sx-Sy\|^2-\{(1-\alpha)+\gamma\}\|x-Sy\|^2\\ &\quad+\{\beta+(\beta-\alpha)\gamma\}\|Sx-y\|^2+\{1-\beta-\gamma(\beta-\alpha-1)\}\|x-y\|^2\\ &\quad+(\alpha-\beta)\gamma\|x-Sx\|^2+\gamma\|y-Sy\|^2.\end{aligned}$$

Since $\lambda^{-2}k \geq 0$ and $\lambda^{-2} > 0$, we obtain $k \geq 0$. This completes the proof. ■

Using Theorem 3.7, we have the following fixed point theorem for super hybrid mappings in a Hilbert space.

Theorem 3.8. *Let C be a nonempty closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be an (α, β, γ)-super hybrid mapping and suppose that C is bounded. Then, S has a fixed point in C.*

Proof. Since $S : C \to C$ is (α, β, γ)-super hybrid, we know from Theorem 3.7 that the mapping $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I : C \to C$ is (α, β)-generalized hybrid. Using Theorem 3.2, we have that T has a fixed point in C. From $F(T) = F(S)$, S has a fixed point in C. ■

4. Nonlinear Ergodic Theorem

In this section, using the technique developed by Takahashi [13], we prove a nonlinear ergodic theorem of Baillon's type [1] for generalized hybrid mappings in a Hilbert space.

Theorem 4.1. *Let H be a Hilbert space and let C be a closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the mertic projection of H onto $F(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element p of $F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Proof. Since $T : C \to C$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2 \tag{4.1}$$

for all $x, y \in C$. Since T is an (α, β)-generalized hybrid mapping, T is quasi-nonexpansive. So, we have that $F(T)$ is closed and convex. Let $x \in C$ and let P be the metric projection of H onto $F(T)$. Then, we have

$$\begin{aligned}\|PT^n x - T^n x\| &\le \|PT^{n-1}x - T^n x\| \\ &\le \|PT^{n-1}x - T^{n-1}x\|.\end{aligned}$$

This implies that $\{\|PT^n x - T^n x\|\}$ is nonincreasing. We also know that for any $v \in C$ and $u \in F(T)$,

$$\langle v - Pv, Pv - u\rangle \ge 0$$

and hence

$$\|v - Pv\|^2 \le \langle v - Pv, v - u\rangle.$$

So, we get

$$\begin{aligned}\|Pv - u\|^2 &= \|Pv - v + v - u\|^2 \\ &= \|Pv - v\|^2 - 2\langle Pv - v, u - v\rangle + \|v - u\|^2 \\ &\le \|v - u\|^2 - \|Pv - v\|^2.\end{aligned}$$

Let $m, n \in \mathbb{N}$ with $m \ge n$. Putting $v = T^m x$ and $u = PT^n x$, we have

$$\begin{aligned}\|PT^m x - PT^n x\|^2 &\le \|T^m x - PT^n x\|^2 - \|PT^m x - T^m x\|^2 \\ &\le \|T^n x - PT^n x\|^2 - \|PT^m x - T^m x\|^2.\end{aligned}$$

So, $\{PT^n x\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{PT^n x\}$ converges strongly to an element p of $F(T)$. Take $u \in F(T)$. Then we obtain, for any $n \in \mathbb{N}$,

$$\|S_n x - u\| \le \frac{1}{n}\sum_{k=0}^{n-1}\|T^k x - u\| \le \|x - u\|.$$

So, $\{S_n x\}$ is bounded and hence there exists a weakly convergent subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$. If $S_{n_i} x \rightharpoonup v$, then we have $v \in F(T)$. In fact, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned}0 &\le \beta\|T^{k+1}x - y\|^2 + (1-\beta)\|T^k x - y\|^2 \\ &\quad - \alpha\|T^{k+1}x - Ty\|^2 - (1-\alpha)\|T^k x - Ty\|^2 \\ &= \beta\big\{\|T^{k+1}x - Ty\|^2 + 2\left\langle T^{k+1}x - Ty, Ty - y\right\rangle + \|Ty - y\|^2\big\} \\ &\quad + (1-\beta)\big\{\|T^k x - Ty\|^2 + 2\left\langle T^k x - Ty, Ty - y\right\rangle + \|Ty - y\|^2\big\} \\ &\quad - \alpha\|T^{k+1}x - Ty\|^2 - (1-\alpha)\|T^k x - Ty\|^2 \\ &= \|Ty - y\|^2 + 2\left\langle \beta T^{k+1}x + (1-\beta)T^k x - Ty, Ty - y\right\rangle \\ &\quad + (\beta - \alpha)\big\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\big\}.\end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \ldots, n-1$,

$$0 \le n\|Ty - y\|^2 + 2\left\langle \sum_{k=0}^{n-1} T^k x + \beta(T^n x - x) - nTy, Ty - y \right\rangle$$
$$+(\beta - \alpha)\big\{\|T^n x - Ty\|^2 - \|x - Ty\|^2\big\}.$$

Deviding this inequality by n, we have

$$0 \le \|Ty - y\|^2 + 2\left\langle S_n x + \frac{1}{n}\beta(T^n x - x) - Ty, Ty - y \right\rangle$$
$$+\frac{1}{n}(\beta - \alpha)\big\{\|T^n x - Ty\|^2 - \|x - Ty\|^2\big\},$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Replacing n by n_i and letting $n_i \to \infty$, we obtain from $S_{n_i} x \rightharpoonup v$ that

$$0 \le \|Ty - y\|^2 + 2\left\langle v - Ty, Ty - y \right\rangle.$$

Putting $y = v$, we have $0 \le -\|Tv - v\|^2$ and hence $Tv = v$. To complete the proof, it is sufficient to show that if $S_{n_i} x \rightharpoonup v$, then $v = p$. We have that

$$\langle T^k x - PT^k x, PT^k x - u \rangle \ge 0$$

for all $u \in F(T)$. Since $\{\|T^k x - PT^k x\|\}$ is nonincreasing, we have

$$\begin{aligned}\langle u - p, T^k x - PT^k x \rangle &\le \langle PT^k x - p, T^k x - PT^k x \rangle \\ &\le \|PT^k x - p\| \cdot \|T^k x - PT^k x\| \\ &\le \|PT^k x - p\| \cdot \|x - Px\|.\end{aligned}$$

Adding these inequalities from $k = 0$ to $k = n-1$ and dividing n, we have

$$\langle u - p, S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x \rangle \le \frac{\|x - Px\|}{n}\sum_{k=0}^{n-1}\|PT^k x - p\|.$$

Since $S_{n_i} x \rightharpoonup v$ and $PT^k x \to p$, we have

$$\langle u - p, v - p \rangle \le 0.$$

We know $v \in F(T)$. So, putting $u = v$, we have $\langle v - p, v - p \rangle \le 0$ and hence $\|v - p\|^2 \le 0$. So, we obtain $v = p$. This completes the proof. ■

Remark 1. From Theorem 4.1, we can prove Theorems 1.1 and 1.2. We do not know whether a nonlinear ergodic theorem of Baillon's type for super hybrid mappings in a Hilbert space holds or not.

5. Weak Convergence Theorem of Mann's Type

In this section, we prove a weak convergence theorem of Mann's type [12] for generalized hybrid mappings in a Hilbert space. Before proving the theorem, we need the following lemma.

Lemma 5.1. *Let H be a Hilbert space and let C be a closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping. Then, $I - T$ is demiclosed, i.e., $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$ imply $z \in F(T)$.*

Proof. Since $T : C \to C$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{5.1}$$

for all $x, y \in C$. Suppose $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Let us consider

$$\alpha\|Tx_n - Tz\|^2 + (1-\alpha)\|x_n - Tz\|^2 \le \beta\|Tx_n - z\|^2 + (1-\beta)\|x_n - z\|^2. \tag{5.2}$$

From this inequality, we have

$$\begin{aligned}\alpha\|Tx_n - x_n + x_n - Tz\|^2 &+ (1-\alpha)\|x_n - Tz\|^2 \\ &\le \beta\|Tx_n - x_n + x_n - z\|^2 + (1-\beta)\|x_n - z\|^2\end{aligned}$$

and hence

$$\begin{aligned}&\alpha(\|Tx_n - x_n\|^2 + \|x_n - Tz\|^2 + 2\langle Tx_n - x_n, x_n - Tz\rangle) + (1-\alpha)\|x_n - Tz\|^2 \\ &\le \beta(\|Tx_n - x_n\|^2 + \|x_n - z\|^2 + 2\langle Tx_n - x_n, x_n - Tz\rangle) + (1-\beta)\|x_n - z\|^2.\end{aligned}$$

We apply a Banach limit μ to both sides of the inequality. Then, we have

$$\begin{aligned}&\alpha\mu_n(\|Tx_n - x_n\|^2 + \|x_n - Tz\|^2 + 2\langle Tx_n - x_n, x_n - Tz\rangle) + (1-\alpha)\mu_n\|x_n - Tz\|^2 \\ &\le \beta\mu_n(\|Tx_n - x_n\|^2 + \|x_n - z\|^2 + 2\langle Tx_n - x_n, x_n - Tz\rangle) + (1-\beta)\mu_n\|x_n - z\|^2\end{aligned}$$

and hence

$$\begin{aligned}\alpha\mu_n\|x_n - Tz\|^2 &+ (1-\alpha)\mu_n\|x_n - Tz\|^2 \\ &\le \beta\mu_n\|x_n - z\|^2 + (1-\beta)\mu_n\|x_n - z\|^2.\end{aligned}$$

So, we have $\mu_n\|x_n - Tz\|^2 \le \mu_n\|x_n - z\|^2$. From $\mu_n\|x_n - z + z - Tz\|^2 \le \mu_n\|x_n - z\|^2$, we also have

$$\mu_n\|x_n - z\|^2 + \mu_n\|z - Tz\|^2 + 2\mu_n\langle x_n - z, z - Tz\rangle \le \mu_n\|x_n - z\|^2.$$

So, we obtain $\mu_n\|z - Tz\|^2 \le 0$ and hence $\|z - Tz\|^2 \le 0$. Then, $Tz = z$. This implies that $I - T$ is demiclosed. ∎

Using Lemma 5.1 and Ibaraki and Takahashi [6], we can prove the following theorem. The proof is due to the technique developed by Ibaraki and Takahashi [6] and [7].

Theorem 5.2. *Let H be a Hilbert space and let C be a closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$ and let P be the mertic projection of H onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n = 1, 2, \dots.$$

Then, the sequence $\{x_n\}$ converges weakly to an element v of $F(T)$, where $v = \lim_{n\to\infty} Px_n$.

Proof. Let $z \in F(T)$. Since T is quasi-nonexpansive, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)Tx_n - z\|^2\\ &\le \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|Tx_n - z\|^2\\ &\le \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2\\ &= \|x_n - z\|^2\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \|x_n - z\|^2$ exists. So, we have that $\{x_n\}$ is bounded. We also have from (2.1) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)Tx_n - z\|^2\\ &= \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|Tx_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2\\ &\le \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2\\ &= \|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2.\end{aligned}$$

So, we have

$$\alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2 \le \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Since $\lim_{n\to\infty} \|x_n - z\|^2$ exists and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, we have $\|Tx_n - x_n\|^2 \to 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. By Lemma 5.1, we obtain $v \in F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. To complete the proof, we show $v_1 = v_2$. We know $v_1, v_2 \in F(T)$ and hence $\lim_{n\to\infty} \|x_n - v_1\|^2$ and $\lim_{n\to\infty} \|x_n - v_2\|^2$ exist. Put

$$a = \lim_{n\to\infty}(\|x_n - v_1\|^2 - \|x_n - v_2\|^2).$$

Note that for $n = 1, 2, \ldots,$

$$\|x_n - v_1\|^2 - \|x_n - v_2\|^2 = 2\langle x_n, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2.$$

From $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$, we have

(5.3) $$a = 2\langle v_1, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2$$

and

(5.4) $$a = 2\langle v_2, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2.$$

Combining (5.3) and (5.4), we obtain $0 = 2\langle v_2-v_1, v_2-v_1\rangle$ and hence $\|v_2-v_1\|^2 = 0$. So, we obtain $v_2 = v_1$. This implies that $\{x_n\}$ converges weakly to an element v of $F(T)$. Since $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $z \in F(T)$ and $n \in \mathbb{N}$, we obtain from Takahashi and Toyoda [18] that $\{Px_n\}$ converges strongly to an element p of $F(T)$. On the other hand, we have from the property of P that

$$\langle x_n - Px_n, Px_n - u\rangle \geq 0$$

for all $u \in F(T)$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $Px_n \to p$, we obtain

$$\langle v - p, p - u\rangle \geq 0$$

for all $u \in F(T)$. Putting $u = v$, we obtain $p = v$. This means $v = \lim_{n\to\infty} Px_n$. This completes the proof. ■

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