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IDENTIFICATION PROBLEMS FOR ISOTROPIC VISCOELASTIC MATERIALS WITH LONG NONLINEAR MEMORY

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Abstract. This paper is concerned with the identification problems of unknown parameters in viscoelastic materials with long nonlinear memory. The unknown parameters are diffusion constants and kernels in nonlinear memory terms, and the identification of such parameters is studied by means of quadratic optimal control theory due to Lions [10]. The existence of optimal parameters is proved, and the necessary condition is established for distributive and terminal values observation by using the transposition method.

1. Introduction

In this paper, we study the identification problems of unknown parameters in viscoelastic materials with long nonlinear memory. The term of long memory is represented by Volterra integro-differential equation. Let Ω be a domain in $\mathbf{R}^n$ with smooth boundary Γ and let us denote $Q = (0,T)\times\Omega,\ \Sigma = (0,T)\times\Gamma$ for $T > 0$. The general nonlinear vibrating equation of isotropic viscoelastic materials occupying a domain Ω with long memory is given by

$$\text{(1)}\quad \frac{\partial^2 y}{\partial t^2} - \mathrm{div}(g(\nabla y)) + \int_0^t k(t-s)\mathrm{div}(h(\nabla y(s,x)))ds = f(x, y, \nabla y, \frac{\partial y}{\partial t}) \text{ in } Q,$$

where f, g and h are sufficiently smooth nonlinear functions. In the case of vibrating membrane the nonlinear functions g and h are given by $g(x) = h(x) = \frac{x}{\sqrt{1+|x|^2}},\ x \in \mathbf{R}^n$. This type of nonlinear equations with memory is studied in Dafermos and Nohel [3], Engler [5], Renardy, Hrusa and Nohel [14], Cavalcanti

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and Oquendo [2], Qin and Ni [13] and others mainly on the existence of global solutions and their asymptotic behaviors. Various types of partially linearized equations for (1) such as

$$\frac{\partial^2 y}{\partial t^2} - \alpha\Delta y - \int_0^t k(t-s)\Delta y(s,x)ds = f(y) + g(\frac{\partial y}{\partial t}) \text{ in } Q, \tag{2}$$

are proposed and studied in many references cited in [14].

In this paper we consider the following partially linearized Volterra integro-differential equation with nonlinear kernel

$$\frac{\partial^2 y}{\partial t^2} - \alpha\Delta y - \int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla y(s,x)}{\sqrt{1+|\nabla y(s,x)|^2}}\Big)ds = f \text{ in } Q, \tag{3}$$

under the Dirichlet boundary condition and the initial conditions

$$y = 0 \quad \text{on } \Sigma, \tag{4}$$

$$y(0,x) = y_0(x), \qquad \frac{\partial y}{\partial t}(0,x) = y_1(x) \qquad \text{in } \Omega. \tag{5}$$

The partially linearized equation (3) can be considered as vibrating models where the derivation of y is sufficiently small in the instantaneous vibrations compared with the memory effects.

Physically, the constant $\alpha > 0$ and the integral kernel $k(\cdot)$ in (3) represent the velocity of vibration and the fading rate of memory effects, respectively.

For the problem (3)-(5), it is proved in Hwang and Nakagiri [9] that the fundamental results on existence, uniqueness and regularity of weak and strong solutions corresponding to the various data conditions on y_0, y_1 and f in [9].

Based on the fundamental results in [9], in this paper we study our problem with strong solutions satisfying the problem (3)-(5) under stronger data conditions to obtain our main results.

To formulate the identification problem, we replace α in (3) by $\alpha_0 + \alpha^2$ for fixed $\alpha_0 > 0$, and we introduce a new parameter $q = (\alpha, k(\cdot))$ of a pair of α and $k(t)$. We define the Banach space $\mathcal{P}$ of parameters $q = (\alpha, k(\cdot))$ by $\mathcal{P} = \mathbf{R} \times C^1[0,1]$. In our situation α and $k(\cdot)$ are unknown parameters to be identified for the determination of realistic dynamics of viscoelastic materials. For the related identification problems for constant parameters, we refer to Ha and Nakagiri [8].

We emphasis that the identification problems of relaxation kernels depending only on time are still little studied, in spite of their physical importance (cf. A. Favaron and A. Lorenzi [6], A. Favini and A. Lorenzi [7] and A. Lorenzi and F. Messina [12]). In the above references the uniqueness of kernels in identification process was studied. However, in this paper we study the quadratic cost identification problems for $q = (\alpha, k(\cdot)) \in \mathcal{P}$ in the framework of Lions [10] (cf. Ahmed [1]).

Now we introduce a cost functionals subject to (3)-(5) by

$$J(q) = \|Cy(q) - z_d\|_M^2 \quad \text{for} \quad q \in \mathcal{P}, \tag{6}$$

where $z_d \in M$ is the desired value, C is an observation operator from the solution space to M and M is a Hilbert space of observation variables. Assume that an admissible subset $\mathcal{P}_{ad}$ of $\mathcal{P}$ is convex and closed. The parameter $q^* \in \mathcal{P}_{ad}$ satisfying

$$\inf_{q \in \mathcal{P}_{ad}} J(q) = J(q^*) \tag{7}$$

is called an optimal parameter and $y(q^*)$ is called the corresponding optimal state. We study the existence and characterization problem of optimal parameters q^* for the cost (6). With the energy inequality for weak solutions we prove the continuity of solutions $y(q)$ in the parameter q, and the existence of an optimal parameter q^* is proved provided that the admissible set $\mathcal{P}_{ad}$ is bounded and closed in $\mathcal{P} = \mathbf{R} \times C^1[0,1]$. Also we prove the Gâteaux differentiability of solution map which is a map from the space of parameters to the space of weak solutions. Finally by using the Gâteaux differentiability we establish the necessary condition of optimality for the case of distributive and terminal values observations. In the description of adjoint systems in the necessary condition, we utilized the transposition method to define the adjoint state appropriately because of the lack of solvability of the formal adjoint system.

2. Viscoelastic Equations with Long Memory

We consider the following Dirichlet boundary value problem for the viscoelastic equations with long nonlinear memory:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \alpha \Delta y - \displaystyle\int_0^t k(t-s) \operatorname{div}\Big(\frac{\nabla y(s,x)}{\sqrt{1 + |\nabla y(s,x)|^2}} \Big) ds = f \quad \text{in} \ \ Q, \\ y = 0 \quad \text{on} \ \ \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in} \ \ \Omega, \end{cases} \tag{2.1}$$

where $\alpha > 0$, k is a scalar kernel function, f is a external forcing term and y_0, y_1 are given initial functions. We shall give the notations used throughout this paper. The scalar product and norm on $[L^2(\Omega)]^n$ are also denoted by (ϕ, ψ) and $|\phi|$. Then the scalar product $(\phi, \psi)_{H_0^1(\Omega)}$ and the norm $\|\phi\|$ of $H_0^1(\Omega)$ are given by $(\nabla \phi, \nabla \psi)$ and $\|\phi\| = (\nabla \phi, \nabla \phi)^{\frac{1}{2}}$, respectively. Let $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. The scalar product and norm on $D(\Delta)$ are denoted by $(\phi, \psi)_{D(\Delta)} = (\Delta \phi, \Delta \psi)$ and $\|\phi\|_{D(\Delta)} = |\Delta \phi|$, respectively. The duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle \phi, \psi \rangle$.

Related to the nonlinear term in (2.1), we define the function $G : \mathbf{R}^n \to \mathbf{R}^n$ by $G(x) = \dfrac{x}{\sqrt{1+|x|^2}}$, $x \in \mathbf{R}^n$. Then it is easily verified that

$$|G(x) - G(y)| \le 2|x-y|, \quad \forall x, y \in \mathbf{R}^n. \tag{2.2}$$

The nonlinear operator $G(\nabla\cdot) : H_0^1(\Omega) \to [L^2(\Omega)]^n$ is defined by

$$G(\nabla\phi)(x) = \frac{\nabla\phi(x)}{\sqrt{1+|\nabla\phi(x)|^2}}, \quad \text{a.e. } x \in \Omega, \quad \forall \phi \in H_0^1(\Omega). \tag{2.3}$$

By the definition of $G(\nabla\cdot)$ in (2.3), we have the following useful property on $G(\nabla\cdot)$:

$$|G(\nabla\phi)| \le |\nabla\phi|, \quad |G(\nabla\phi) - G(\nabla\psi)| \le 2|\nabla\phi - \nabla\psi|, \quad \forall \phi, \psi \in H_0^1(\Omega). \tag{2.4}$$

The solution space $W(0,T)$ for weak solutions of (2.1) is defined by

$$W(0,T) = \{g | g \in L^2(0,T; H_0^1(\Omega)), g' \in L^2(0,T; L^2(\Omega)), g'' \in L^2(0,T; H^{-1}(\Omega))\}$$

endowed with the norm

$$\|g\|_{W(0,T)} = \left(\|g\|^2_{L^2(0,T;H_0^1(\Omega))} + \|g'\|^2_{L^2(0,T;L^2(\Omega))} + \|g''\|^2_{L^2(0,T;H^{-1}(\Omega))} \right)^{\frac{1}{2}},$$

where g' and g'' denote the first and second order distributive derivatives of g.

Definition 2.1. A function y is said to be a weak solution of (2.1) if $y \in W(0,T)$ and y satisfies

$$\begin{cases} \langle y''(\cdot), \phi\rangle + \alpha(\nabla y(\cdot), \nabla\phi) + \displaystyle\int_0^{\cdot} k(\cdot - s)(G(\nabla y(\cdot)), \nabla\phi) ds = (f(\cdot), \phi) \\ \qquad \textit{for all } \ \phi \in H_0^1(\Omega) \textit{ in the sense of } \mathcal{D}'(0,T), \\ y(0) = y_0 \in H_0^1(\Omega), \qquad y'(0) = y_1 \in L^2(\Omega). \end{cases} \tag{2.5}$$

The following theorem proved in Hwang and Nakagiri [9] gives the fundamental results on existence, uniqueness and regularity of weak solutions of (2.1).

Theorem 2.1. *Assume that $k(\cdot) \in C^1[0,T]$ and*

$$y_0 \in H_0^1(\Omega), \quad y_1 \in L^2(\Omega), \quad f \in L^2(0,T; L^2(\Omega)). \tag{2.6}$$

Then the problem (2.1) has a unique weak solution $y \in W(0,T) \cap C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. Moreover, y has the following estimate

$$|\nabla y(t)|^2 + |y'(t)|^2 \le C(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}), \quad \forall t \in [0,T],$$

where C is a constant depending only on α and $\|k\|_{C^1[0,T]}$.

Remark 2.1. If $G(\nabla\cdot)$ in (2.3) is replaced by a linear bounded operator $\tilde{G}(\nabla\cdot) : H_0^1(\Omega) \to [L^2(\Omega)]^n$, then Theorem 2.1 still holds for this $\tilde{G}$ (cf. Dautray and Lions [4, p.660]).

Next we introduce the solution space $\tilde{W}(0,T)$ for strong solutions of (2.1) defined by

$$\tilde{W}(0,T) = \{g | g \in L^2(0,T; D(\Delta)), g' \in L^2(0,T; H_0^1(\Omega)), g'' \in L^2(0,T; L^2(\Omega))\}.$$

Definition 2.2. *A function y is said to be a strong solution of (2.1) if $y \in \tilde{W}(0,T)$, $\mathrm{div}\ G(\nabla y) \in L^2(0,T; L^2(\Omega))$ and y satisfies*

$$\text{(2.7)} \quad \begin{cases} y''(t) - \alpha\Delta y(t) - \displaystyle\int_0^t k(t-s)\mathrm{div}\ G(\nabla y(s))ds = f(t), \ a.e.\ t \in [0,T], \\ y(0) = y_0 \in D(\Delta), \qquad y'(0) = y_1 \in H_0^1(\Omega). \end{cases}$$

The next theorem on the well-posedness for strong solutions of (2.1) is also proved in Hwang and Nakagiri [9].

Theorem 2.2. *Assume that $k(\cdot) \in C^1[0,T]$ and*

$$\text{(2.8)} \qquad y_0 \in D(\Delta), \quad y_1 \in H_0^1(\Omega), \quad f \in H^1(0,T; L^2(\Omega)).$$

Then the problem (2.1) has a unique strong solution $y \in \tilde{W}(0,T)$ which satisfies

$$y \in C([0,T]; D(\Delta)) \cap C^1([0,T]; H_0^1(\Omega)) \cap C^2([0,T]; L^2(\Omega))$$

and the estimates

$$\begin{aligned} & |\Delta y(t)|^2 + |\nabla y'(t)|^2 + |y''(t)|^2 \\ \text{(2.9)} \qquad \le\ & C(\|y_0\|^2_{D(\Delta)} + \|y_1\|^2 + \|f\|^2_{H^1(0,T;L^2(\Omega))}), \quad \forall t \in [0,T], \end{aligned}$$

where C is a constant depending only on α and $\|k\|_{C^1[0,T]}$.

3. Identification Problems

In this section we study the identification problems for the following viscoelastic system with nonlinear memory

$$\text{(3.1)} \quad \begin{cases} \dfrac{\partial^2 y}{\partial t^2} - (\alpha_0 + \alpha^2)\Delta y - \displaystyle\int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla y(s,x)}{\sqrt{1+|\nabla y(s,x)|^2}}\Big)ds = f \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \\ y(0,x) = y_0(x), \ \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \text{ in } \Omega, \end{cases}$$

where $\alpha_0 > 0, y_0 \in D(\Delta), y_1 \in H_0^1(\Omega)$ and $f \in H^1(0,T;L^2(\Omega))$ are fixed, and $\alpha \in \mathbf{R}$, $k(\cdot) \in C^1[0,T]$. In (3.1) the diffusion parameter α and the fading rate $k(\cdot)$ are unknown parameters which should be identified. To study the identification of parameters α and $k(\cdot)$ in (3.1), we introduce the Banach space $\mathcal{P}$ of parameters $q = (\alpha, k(\cdot))$ by

$$\mathcal{P} = \mathbf{R} \times C^1[0,T] \tag{3.2}$$

The norm of $\mathcal{P}$ is defined by

$$\|(\alpha, k(\cdot))\|_{\mathcal{P}} = |\alpha| + \|k(\cdot)\|_{C^1[0,T]} \quad \text{for } (\alpha, k(\cdot)) \in \mathcal{P}.$$

Since the data in (3.1) satisfy (2.8), we know by Theorem 2.2 that (3.1) admits a unique strong solution $y \in \tilde{W}(0,T)$. The reason why we study our problems with strong solution is summarized as follows.

As we will show later, our approach rely on the optimal control theory due to Lions [10]. Therefore, it is necessary to establish the existence and necessary conditions for optimal parameters. When we prove the necessary conditions for optimal parameters we need to take variation on the set $\mathcal{P}$ into the solution space. Especially, in taking variation for the unknown parameter α which appears as the coefficient of Δy term, we will have a forcing term involving Δy term which has to belong to the space $L^2(0,T;L^2(\Omega))$, i.e., y must belong to $L^2(0,T;D(\Delta))$. Since the space $L^2(0,T;L^2(\Omega))$ is the forcing function's space in Theorem 2.1, we can obtain our related results in $W(0,T)$.

Thus we can introduce the solution map $q \to y(q) : \mathcal{P} \to W(0,T)$. We note that the topology of $W(0,T)$ is natural to study the identification problems. We attach a cost functional subject to (3.1) by the quadratic cost

$$J(q) = \|Cy(q) - z_d\|_M^2 \quad \text{for} \quad q \in \mathcal{P}, \tag{3.3}$$

where $z_d \in M$ is the desired value, $C \in \mathcal{L}(W(0,T), M)$ is an observer and M is a Hilbert space of observation variables. Assume that an admissible set $\mathcal{P}_{ad} = \mathcal{P}_{ad}^1 \times \mathcal{P}_{ad}^2 \subset \mathcal{P}$ is nonempty, closed and convex. The identification problem is formulated as the minimization problem for (3.1) subject to the cost (3.3), and we shall solve the following two problems. That is

(i) Find an element $q^* \in \mathcal{P}_{ad}$ satisfying

$$\inf_{q \in \mathcal{P}_{ad}} J(q) = J(q^*). \tag{3.4}$$

(ii) Deduce necessary conditions on $q^* = (\alpha^*, k^*(\cdot))$.

We shall call such the q^* the optimal parameter and $y(q^*)$ the corresponding optimal state.

3.1. Existence of optimal parameters

The strong continuity of the map $q \to y(q) : \mathcal{P} \to W(0,T)$ is crucial to solve the problems i) and ii). In this subsection we shall solve i) under the conditions that $\mathcal{P}^1_{ad}$ is compact in $\mathbf{R}$ and $\mathcal{P}^2_{ad}$ is bounded and closed in $C^1[0,T]$.

Theorem 3.1. *The map $q \to y(q) : \mathcal{P} \to W(0,T)$ is continuous. That is, $y(q_m) \to y(q)$ in $W(0,T)$ as $q_m = (\alpha_m, k_m(\cdot)) \to q = (\alpha, k(\cdot))$ in $\mathbf{R} \times C^1[0,1]$.*

Proof. Suppose that $q_m = (\alpha_m, k_m(\cdot)) \to q = (\alpha, k(\cdot))$ in $\mathcal{P}$, i.e., $\alpha_m \to \alpha$ in $\mathbf{R}$ and $k_m(\cdot) \to k(\cdot)$ in $C^1[0,T]$. Since the imbedding $C^1[0,T] \hookrightarrow L^1(0,T)$ is compact, there exists a subsequence of $\{k_m\}$ written again by $\{k_m\}$ such that $k_m(\cdot) \to k(\cdot)$ in $L^1(0,T)$. Let $y_m = y(q_m)$ be the weak solutions of (3.1) with $\alpha = \alpha_m$ and $k(\cdot) = k_m(\cdot)$. We denote by $z = y(q)$ the weak solution y of (3.1) for notational convenience. We put $y_m - z = \phi_m$, then ϕ_m satisfies

$$(3.5)\quad \begin{cases} \dfrac{\partial^2 \phi_m}{\partial t^2} - (\alpha_0 + \alpha^2)\Delta\phi_m \\ \qquad - \displaystyle\int_0^t k(t-s)\mathrm{div}\Big(G(\nabla y_m(s,x)) - G(\nabla z(s,x))\Big)ds \\ = (\alpha_m^2 - \alpha^2)\Delta y_m + \displaystyle\int_0^t (k_m(t-s) - k(t-s))\mathrm{div}\, G(\nabla y_m(s,x))ds \text{ in } Q, \\ \phi_m = 0 \quad \text{on } \Sigma, \\ \phi_m(0,x) = 0, \quad \dfrac{\partial \phi_m}{\partial t}(0,x) = 0 \text{ in } \Omega \end{cases}$$

in the weak sense. In what follows, we will omit the variables s and/or x in the representation $y(s,x)$ such as $y(s)$ or more simply as y whenever no confusion. Also we use the convolution operation $*$ defined by $f * g(t) = \int_0^t f(t-s)g(s)ds$. Now we put

$$(3.6)\quad (\alpha_m^2 - \alpha^2)\Delta y_m + (k_m - k) * \mathrm{div}\, G(\nabla y_m) = F_m.$$

By Theorem 2.2 and Remark 2.2 the functions $F_m \in L^2(0,T; L^2(\Omega))$ and are uniformly bounded in $L^2(0,T; L^2(\Omega))$. We shall derive the estimations on ϕ_m. By multiplying $\phi_m'(t)$ to the weak form of the equation in (3.5) we have

$$(3.7)\quad \begin{aligned} &(\phi_m''(t), \phi_m'(t)) + (\alpha_0 + \alpha^2)(\nabla\phi_m(t), \nabla\phi_m'(t)) \\ &+(k * (G(\nabla\phi_m)(t) - G(\nabla z))(t), \nabla\phi_m'(t)) = (F_m(t), \phi_m'(t)). \end{aligned}$$

We set $\beta = \alpha_0 + \alpha^2$ and $\mathcal{G}_m = G(\nabla y_m) - G(\nabla z)$ for simplicity. Since $k \in C^1[0,T]$, we have

$$(k * \mathcal{G}_m(t), \nabla\phi'_m(t)) = \frac{d}{dt}(k * \mathcal{G}_m(t), \nabla\phi_m(t)) \\ -k(0)(\mathcal{G}_m(t), \nabla\phi_m(t)) - (k' * \mathcal{G}_m(t), \nabla\phi_m(t)).$$

Then wee see that (3.7) is rewritten as

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big[\beta(\nabla\phi_m(t), \nabla\phi_m(t)) + |\phi'_m(t)|^2 + 2(k * \mathcal{G}_m(t), \nabla\phi_m(t))\Big] \\ &= (F_m(t), \phi'_m(t)) + k(0)(\mathcal{G}_m(t), \nabla\phi_m(t)) + (k' * \mathcal{G}_m(t), \nabla\phi_m(t)). \end{aligned} \tag{3.8}$$

Let $\epsilon > 0$ be a positive real number. We set $k_0 = \|k\|_{C[0,T]}$ and $k_1 = \|k'\|_{C[0,T]}$. By (2.4) and Schwarz inequality, we have $|\mathcal{G}_m(t)| \le 2|\nabla y_m - \nabla z| = 2|\nabla\phi_m|$ and

$$\begin{aligned} \Big|(2k * \mathcal{G}_m(t), \nabla\phi_m(t))\Big| &\le 2k_0|\nabla\phi_m(t)| \int_0^t |\nabla\phi_m(s)|ds \\ &\le \epsilon|\nabla\phi_m(t)|^2 + c(\epsilon) \int_0^t |\nabla\phi_m(s)|^2 ds \end{aligned} \tag{3.9}$$

for some $c(\epsilon) > 0$. Also we have by Schwarz inequality that

$$\begin{cases} \Big|\displaystyle\int_0^t 2k(0)(\mathcal{G}_m, \nabla\phi_m)ds\Big| \le 2k_0 \int_0^t |\nabla\phi_m|^2 ds, \\ \Big|\displaystyle\int_0^t (2k' * \mathcal{G}_m, \nabla\phi_m)ds\Big| \le 2k_1\Big(\int_0^t |\nabla\phi_m|ds\Big)^2 \le 2k_1 T \int_0^t |\nabla\phi_m|^2 ds, \\ \Big|\displaystyle\int_0^t 2(F_m, \phi'_m)ds\Big| \le \|F_m\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^t |\phi'_m|^2 ds. \end{cases} \tag{3.10}$$

By integrating (3.8) on $[0, t]$ and using the estimates (3.9), (3.10) and 0 initial conditions on ϕ_m, we can obtain the following inequality

$$\begin{aligned} |\phi'_m(t)|^2 + \beta|\nabla\phi_m(t)|^2 &\le \epsilon|\nabla\phi_m(t)|^2 + \|F_m\|^2_{L^2(0,T;L^2(\Omega))} \\ &+ (2k_0 + c(\epsilon) + 2k_1 T) \int_0^t |\nabla\phi_m|^2 ds + \int_0^t |\phi'_m|^2 ds. \end{aligned} \tag{3.11}$$

By choosing $\epsilon = \frac{\beta}{2}$ in (3.11), we can find a $K > 0$ such that

$$|\phi'_m(t)|^2 + |\nabla\phi_m(t)|^2 \le K\|F_m\|^2_{L^2(0,T;L^2(\Omega))} + K \int_0^t (|\phi'_m|^2 + |\nabla\phi_m|^2)ds. \tag{3.12}$$

Thus by applying the Bellman-Gronwall's inequality, we have from (3.12) that

$$|\nabla\phi_m(t)|^2 + |\phi'_m(t)|^2 \le C\|F_m\|^2_{L^2(0,T;L^2(\Omega))}, \quad \forall t \in [0, T] \tag{3.13}$$

for some $C > 0$. Since

$$\|F_m\|_{L^2(0,T;L^2(\Omega))} \leq |\alpha_m^2 - \alpha^2| \|\Delta y_m\|_{L^2(0,T;L^2(\Omega))} + \|k_m - k\|_{L^1(0,T)} \|\operatorname{div} G(\nabla y_m)\|_{L^2(0,T;L^2(\Omega))} \tag{3.14}$$

and $\{\Delta y_m\}$ and $\{\operatorname{div} G(\nabla y_m)\}$ are in $L^2(0,T;L^2(\Omega))$ by Theorem 2.2, we see that $\|F_m\|_{L^2(0,T;L^2(\Omega))} \to 0$ as $m \to \infty$ by $q_m \to q$ in $\mathcal{P}$. Therefore, it follows from (3.13) that we can assert that

$$(y_m(t), y_m'(t)) \to (z(t), z'(t)) \quad \text{strongly in} \quad H_0^1(\Omega) \times L^2(\Omega), \quad \forall t \in [0,T] \tag{3.15}$$

Since y_m and z are strong solutions, we can verify by (3.15) via (2.7) that

$$y_m'' \to z'' \quad \text{strongly in} \quad L^2(0,T;H^{-1}(\Omega)).$$

Therefore

$$y_m \to z \quad \text{strongly in} \quad W(0,T).$$

This implies that $y(q_m) \to y(q)$ strongly in $W(0,T)$ as $q_m \to q$ in $\mathcal{P}$. This proves Theorem 3.1.

The following existence theorem of an optimal parameter $q^* = (\alpha^*, k^*(\cdot))$ over the admissible set $\mathcal{P}_{ad}$ follows from Theorem 3.2.

Theorem 3.2. *Let $\mathcal{P}_{ad} = \mathcal{P}_{ad}^1 \times \mathcal{P}_{ad}^2$. If $\mathcal{P}_{ad}^1$ is compact in $\mathbf{R}$ and $\mathcal{P}_{ad}^2$ is bounded and closed in $C^1[0,T]$, then there exists at least one optimal parameter $q^* = (\alpha^*, k^*(\cdot)) \in \mathcal{P}_{ad}$ for the cost (3.3).*

Proof. Let $\{q_n = (\alpha_n, k_n(\cdot))\}$ be the minimizing sequence such that $\lim_{n\to\infty} J(q_n) = \inf_{q\in\mathcal{P}_{ad}} J(q)$. Since $\mathcal{P}_{ad}^1$ is compact and $\mathcal{P}_{ad}^2$ is bounded and closed in $C^1[0,T]$, we can find a subsequence $\{q_{n_l}\} = \{(\alpha_{n_l}, k_{n_l}(\cdot))\}$ of $\{q_n\}$ and $q^* = (\alpha^*, k^*(\cdot)) \in \mathcal{P}_{ad}$ such that $\alpha_{n_l} \to \alpha^* \in \mathcal{P}_{ad}^1$ in $\mathbf{R}$ and $k(\cdot)_{n_l} \to k^*(\cdot) \in \mathcal{P}_{ad}^2$ in $L^1(0,T)$ as $l \to \infty$. Then by Theorem 3.1 and the lower semi-continuity of costs, the limit $q^* = (\alpha^*, k^*(\cdot))$ is shown to be an optimal parameter.

3.2. Gâteaux differentiability of solution map

In the proof of necessary conditions to solve the problem ii) we utilize the Gâteaux differential of $y(q)$ with respect to $q \in \mathcal{P}$. Thus it needs to estimate quotients $z_\lambda = (y(q_\lambda) - y(q^*))/\lambda$ in the space $W(0,T)$, where $q_\lambda = q^* + \lambda(q - q^*)$, $\lambda \in [-1,1], \lambda \neq 0$ and $q, q^* \in \mathcal{P}$. We set $y_\lambda = y(q_\lambda)$ and $y^* = y(q^*)$ for simplicity.

Let us begin to prove the Gâteaux differentiability of the solution map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$.

Theorem 3.3. *The map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is Gâteaux differentiable. That is, for fixed $q = (\alpha, k(\cdot))$ and $q^* = (\alpha^*, k^*(\cdot))$ the Gâteaux derivative $z = Dy(q^*)(q - q^*)$ of $y(q)$ at $q = q^*$ in the direction $q - q^*$ exists in $W(0,T)$ and it is a unique weak solution of the following problem*

$$
(3.16)\qquad \begin{cases} \dfrac{\partial^2 z}{\partial t^2} - (\alpha_0 + \alpha^{*2})\Delta z \\ \qquad - \displaystyle\int_0^t k^*(t-s)\mathrm{div}\Big\{\frac{\nabla z(s,x)}{\sqrt{1+|\nabla y^*(s,x)|^2}} \\ \qquad - \dfrac{\nabla y^*(s,x)\cdot\nabla z(s,x)}{(1+|\nabla y^*(s,x)|^2)^{\frac{3}{2}}}\nabla y^*(s,x)\Big\}ds \\ = 2\alpha^*(\alpha - \alpha^*)\Delta y^* + (k - k^*) * \textit{div}\, G(\nabla y^*) \quad \textit{in} \quad Q, \\ z = 0 \quad \textit{on} \quad \Sigma, \\ z(0,x) = 0, \quad \dfrac{\partial z}{\partial t}(0,x) = 0 \quad \textit{in} \quad \Omega. \end{cases}
$$

Proof. For fixed q and q^* we set $q_\lambda = (\alpha_\lambda, k_\lambda(\cdot)) = q^* + \lambda(q - q^*)$. We recall the simplified notations $y_\lambda = y(q_\lambda)$ and $y^* = y(q^*)$ corresponding to given parameters q_λ and q^*, respectively. Then $q_\lambda \in \mathcal{P}$ and $|q_\lambda - q^*| = |\lambda||q - q^*| \to 0$ as $\lambda \to 0$. Hence by Theorem 3.1 we have

$$
(3.17)\qquad \lim_{\lambda\to 0} y_\lambda = y^* \quad \text{in} \quad C([0,T]; H_0^1(\Omega)).
$$

We set $z_\lambda = \lambda^{-1}(y_\lambda - y^*)$, $\lambda \neq 0$. Then z_λ solves the following problem in the weak sense:

$$
(3.18)\qquad \begin{cases} \dfrac{\partial^2 z_\lambda}{\partial t^2} - (\alpha_0 + \alpha^{*2})\Delta z_\lambda - \displaystyle\int_0^t k_\lambda(t-s)\frac{1}{\lambda}\mathrm{div}\Big(G(\nabla y_\lambda) - G(\nabla y^*)\Big)ds \\ = (2\alpha^*(\alpha - \alpha^*) + \lambda(\alpha - \alpha^*)^2)\Delta y_\lambda + (k - k^*) * \mathrm{div}\, G(\nabla y^*) \quad \text{in} \quad Q, \\ z_\lambda = 0 \quad \text{on} \quad \Sigma, \\ z_\lambda(0,x) = 0, \quad \dfrac{\partial z_\lambda}{\partial t}(0,x) = 0 \quad \text{in} \quad \Omega. \end{cases}
$$

We put

$$
F_\lambda = (2\alpha^*(\alpha - \alpha^*) + \lambda(\alpha - \alpha^*)^2)\Delta y_\lambda + (k - k^*) * \mathrm{div}\, G(\nabla y^*).
$$

Then by Theorem 2.2, we see that $F_\lambda \in L^2(0,T; L^2(\Omega))$ and $\{F_\lambda\}_{\lambda\in[-1,1]}$ is in $L^2(0,T; L^2(\Omega))$. We set $\beta^* = \alpha_0 + \alpha^{*2}$ and

$$
\mathcal{G}_\lambda = \frac{1}{\lambda}(G(\nabla y_\lambda) - G(\nabla y^*))
$$

for simplicity. Then as in the proof of Theorem 3.1, we can deduce that for each $t \in [0,T]$, $z_\lambda(t)$ satisfies

$$\begin{aligned} &\beta^*|\nabla z_\lambda(t)|^2 + |z_\lambda'(t)|^2 \\ &= -2\int_0^t (k_\lambda' * \mathcal{G}_\lambda, \nabla z_\lambda)ds + 2(k_\lambda * \mathcal{G}_\lambda(t), \nabla z_\lambda(t)) \\ &\quad - 2\int_0^t (k_\lambda(0)\mathcal{G}_\lambda, \nabla z_\lambda)ds + 2\int_0^t (F_\lambda, z_\lambda')ds. \end{aligned} \tag{3.19}$$

The kernel terms of (3.19) can be estimated as

$$|\mathcal{G}_\lambda(t)| = \Big|\frac{G(\nabla y_\lambda(t)) - G(\nabla y^*(t))}{\lambda}\Big| \le 2|\nabla z_\lambda(t)|. \tag{3.20}$$

Then we can see that (3.19), (3.20) imply

$$|\nabla z_\lambda(t)|^2 + |z_\lambda'(t)|^2 \le K\int_0^t (|\nabla z_\lambda|^2 + |z_\lambda'|^2)ds + K\|F_\lambda\|^2_{L^2(0,T;L^2(\Omega))} \tag{3.21}$$

for some $K > 0$. Hence by applying the Gronwall's inequality to (3.21), we have

$$|\nabla z_\lambda(t)|^2 + |z_\lambda'(t)|^2 \le K_1\|F_\lambda\|^2_{L^2(0,T;L^2(\Omega))} \le K_2^2 < \infty, \tag{3.22}$$

for some $K_1 > 0$ and some $K_2 > 0$ independent of λ. Therefore there exist a $z \in W(0,T)$ and a sequence $\{\lambda_k\} \subset (-1,1)$ tending to 0 such that

$$\begin{cases} z_{\lambda_k} \to z \ \text{ weakly star in } \ L^\infty(0,T;H_0^1(\Omega)) \ \text{ as } \ k \to \infty, \\ z_{\lambda_k}' \to z' \ \text{ weakly star in } \ L^\infty(0,T;L^2(\Omega)) \ \text{ as } \ k \to \infty, \\ z(0) = 0, \quad z'(0) = 0. \end{cases} \tag{3.23}$$

Let us prove that

$$k_{\lambda_k} * \mathcal{G}_{\lambda_k} \quad \to \quad k^* * \mathrm{div}\Big\{\frac{\nabla z}{\sqrt{1+|\nabla y^*|^2}} - \frac{\nabla y^* \cdot \nabla z}{\big(1+|\nabla y^*|^2\big)^{\frac{3}{2}}}\nabla y^*\Big\}$$

$$\text{weakly in } \ L^2(0,T;H^{-1}(\Omega)). \tag{3.24}$$

For all $\phi \in L^2(0,T;H_0^1(\Omega))$ we have

$$\int_0^T (k_{\lambda_k} * \mathcal{G}_{\lambda_k}, \nabla\phi)dt = \int_0^T ((k_{\lambda_k} - k^*) * \mathcal{G}_{\lambda_k}, \nabla\phi)dt + \int_0^T (k^* * \mathcal{G}_{\lambda_k}, \nabla\phi)dt. \tag{3.25}$$

Since $k_{\lambda_k} - k^* = \lambda_k(k - k^*)$, the first term of the right hand side of (3.25) can be written as

$$\int_0^T ((k - k^*) * \Big(G(\nabla y_{\lambda_k}) - G(\nabla y^*)\Big), \nabla\phi)dt. \tag{3.26}$$

We can easily verify that (3.26) can be bounded by

$$\|k-k^*\|_{C[0,T]}\|\nabla y_{\lambda_k}-\nabla y^*\|_{L^2(0,T;L^2(\Omega))}\|\nabla\phi\|_{L^2(0,T;L^2(\Omega))}$$

which tends to 0 as $\lambda_k \to 0$. For simplicity of calculations, we introduce $K(\nabla\cdot):H_0^1(\Omega)\to L^\infty(\Omega)$ by

$$[K(\nabla\phi)](x)=\frac{1}{\sqrt{1+|\nabla\phi(x)|^2}},\quad \text{a.e. } x\in\Omega,\quad \forall\phi\in H_0^1(\Omega).$$

Then clearly $\|K(\nabla\phi)\|_{L^\infty(\Omega)}\le 1$ for all $\phi\in H_0^1(\Omega)$ and

$$|K(\nabla\phi)-K(\nabla\varphi)|\le|\nabla\psi-\nabla\varphi|,\quad \forall\psi,\ \varphi\in H_0^1(\Omega). \tag{3.27}$$

The function $\mathcal{G}_{\lambda_k}$ is decomposed as

$$\begin{aligned}
&\mathcal{G}_{\lambda_k}\\
&= \frac{1}{\lambda_k}(K(\nabla y_{\lambda_k})\nabla y_{\lambda_k}-K(\nabla y^*)\nabla y^*))\\
&= K(\nabla y_{\lambda_k})\nabla z_{\lambda_k}+\frac{1}{\lambda_k}(K(\nabla y_{\lambda_k})-K(\nabla y^*))\nabla y^*\\
&= K(\nabla y^*)\nabla z+K(\nabla y^*)(\nabla z_{\lambda_k}-\nabla z)+(K(\nabla y_{\lambda_k})-K(\nabla y^*))\nabla z_{\lambda_k}\\
&\quad+\frac{1}{\lambda_k}(K(\nabla y_{\lambda_k})-K(\nabla y^*))\nabla y^*\\
&\equiv K(\nabla y^*)\nabla z+\mathcal{G}^1_{\lambda_k}+\mathcal{G}^2_{\lambda_k}+\mathcal{G}^3_{\lambda_k}.
\end{aligned} \tag{3.28}$$

By (3.23), we have

$$\int_0^T(k^* * \mathcal{G}^1_{\lambda_k},\nabla\phi)dt\to 0 \tag{3.29}$$

as $\lambda_k\to 0$. By (3.27) and

$$\left|\left(\Big(K(\nabla y_{\lambda_k})-K(\nabla y^*)\Big)\nabla z_{\lambda_k},\nabla\phi)\right|\le 2K_2|\nabla\phi|\ \in\ L^1(0,T) \tag{3.30}$$

by (3.22), it follows by applying Lebesgue dominated convergence theorem that

$$\int_0^T(k^* * \mathcal{G}^2_{\lambda_k},\nabla\phi)dt\to 0 \tag{3.31}$$

as $\lambda_k\to 0$. We shall prove

$$\int_0^T(k^* * \mathcal{G}^3_{\lambda_k},\nabla\phi)dt\to-\int_0^T(k^* * (K(\nabla y^*)^3\nabla y^*\cdot\nabla z)\nabla y^*,\nabla\phi)dt \tag{3.32}$$

as $\lambda_k \to 0$. To prove (3.32) we will employ the following simplified notation

$$\mathcal{H}_i(\phi,\psi)=\frac{\phi_{x_i}+\psi_{x_i}}{\sqrt{1+|\nabla\phi|^2}\sqrt{1+|\nabla\psi|^2}(\sqrt{1+|\nabla\phi|^2}+\sqrt{1+|\nabla\psi|^2})}, \tag{3.33}$$

where $\phi, \psi \in H_0^1(\Omega)$. Here we remark that the symbol $|\cdot|$ in (3.33) denotes the absolute value of $\mathbf{R}^n$. Then we can verify by direct calculations that

$$|\mathcal{H}_i(\phi,\psi)-\mathcal{H}_i(\phi,\phi)| \le C_1|\phi_{x_i}-\psi_{x_i}|, \tag{3.34}$$

where C_1 is some positive constant. Using the above notation, we have the following representation

$$\int_0^T (k^* * \mathcal{G}^3_{\lambda_k}, \nabla\phi)dt = -\sum_{i=1}^n \int_0^T \Big(k^* * (z_{\lambda_k x_i}\mathcal{H}_i(y_{\lambda_k}, y^*)\nabla y^*), \nabla\phi\Big)dt. \tag{3.35}$$

The i-the term of the right hand side of (3.35) can be rewritten as

$$\begin{aligned}
&-\int_0^T \Big(k^* * ((z_{\lambda_k x_i} - z_{x_i})\mathcal{H}_i(y_{\lambda_k}, y^*)\nabla y^*), \nabla\phi\Big)dt\\
&-\int_0^T \Big(k^* * (z_{x_i}(\mathcal{H}_i(y_{\lambda_k}, y^*) - \mathcal{H}_i(y^*, y^*))\nabla y^*), \nabla\phi\Big)dt\\
&-\int_0^T \Big(k^* * (z_{x_i}\mathcal{H}_i(y^*, y^*)\nabla y^*), \nabla\phi\Big)dt.
\end{aligned} \tag{3.36}$$

By (3.23) the first term of (3.36) tends to 0. The inequality (3.34) together with (3.17) implies that

$$|\mathcal{H}_i(y_{\lambda_k}, y^*) - \mathcal{H}_i(y^*, y^*)| \;\to\; 0 \quad \text{a.e. in } \; Q \tag{3.37}$$

as $k \to \infty$. Moreover

$$|z_{x_i}(\mathcal{H}_i(y_{\lambda_k}, y^*) - \mathcal{H}_i(y^*, y^*))\nabla y^* \cdot \nabla\phi| \;\le c|\nabla z||\nabla\phi| \;\in\; L^1(Q). \tag{3.38}$$

Hence from (3.37) and (3.38) the second term of (3.36) tends to 0. Thus (3.32) is proved. From (3.25) and (3.32) the convergence (3.24) follows. Therefore it is verified that by routine treatments as in Dautray and Lions [4], z is a unique weak solution satisfying (3.16). Next, it remains to show the strong convergence of $\{z_\lambda\}$ in $W(0,T)$. The integral terms in (3.18) and (3.16) are rewritten as

$$\int_0^t k^*(t-s)\mathrm{div}\Big(K(\nabla y_\lambda)\nabla z_\lambda\Big)ds - \sum_{i=1}^n \Big(k^* * \mathrm{div}\Big(z_{\lambda x_i}\mathcal{H}_i(y_\lambda, y^*)\nabla y^*\Big)\Big)(t), \tag{3.39}$$

$$\int_0^t k^*(t-s)\mathrm{div}\Big(K(\nabla y^*)\nabla z_\lambda\Big)ds - \sum_{i=1}^n \Big(k^* * \mathrm{div}\Big(z_{x_i}\mathcal{H}_i(y^*,y^*)\nabla y^*\Big)\Big)(t), \tag{3.40}$$

respectively. Now we set the forcing term in (3.16) by

$$F = 2\alpha^*(\alpha-\alpha^*)\Delta y^* + (k-k^*) * \mathrm{div}\, G(\nabla y^*). \tag{3.41}$$

If we put

$$\delta_\lambda = -\sum_{i=1}^n k^* * \mathrm{div}\Big(z_{x_i}(\mathcal{H}_i(y_\lambda,y^*) - \mathcal{H}_i(y^*,y^*))\nabla y^*\Big), \tag{3.42}$$

$$\epsilon_\lambda = F_\lambda - F, \qquad \eta_\lambda = \delta_\lambda + \epsilon_\lambda, \tag{3.43}$$

then we can assert that $\delta_\lambda,\ \epsilon_\lambda \to 0$ in $H^1(0,T;H^{-1}(\Omega))$ as $\lambda \to 0$. Subtracting (3.18) from (3.16) and using the above notations (3.42) and (3.43), we see that the difference and we put $\phi_\lambda = z_\lambda - z$ satisfies

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\phi_\lambda - (\alpha_0 + {\alpha^*}^2)\Delta\phi_\lambda \\ \qquad - \displaystyle\int_0^t k^*(t-s)\mathrm{div}\Big(\nabla z_\lambda K(\nabla y_\lambda) - \nabla z K(\nabla y^*)\Big)ds \\ \qquad + \displaystyle\sum_{i=1}^n k^* * \mathrm{div}(\phi_{\lambda x_i}\mathcal{H}_i(y_\lambda,y^*)\nabla y^*) = \eta_\lambda \ \text{ in } \ Q, \\ \phi_\lambda = 0 \quad \text{ on } \ \Sigma, \\ \phi_\lambda(0,x) = 0, \quad \dfrac{\partial}{\partial t}\phi_\lambda(0,x) = 0 \ \text{ in } \ \Omega, \end{cases} \tag{3.44}$$

in the weak sense. To estimate ϕ_λ, we utilize the energy equality for (3.44) as in the proof of (3.13). Then ϕ_λ satisfies

$$\begin{aligned} &|\phi_\lambda'(t)|^2 + \beta^*|\nabla\phi_\lambda(t)|^2 = -2\int_0^t \langle \eta_\lambda', \phi_\lambda\rangle ds + 2\langle \eta_\lambda(t), \phi_\lambda(t)\rangle \\ &+ 2(k^* * \Big(\nabla z_\lambda K(\nabla y_\lambda) - \nabla z K(\nabla y^*)\Big)(t), \nabla\phi_\lambda(t)) \\ &- 2\int_0^t k(0)(\nabla z_\lambda K(\nabla y_\lambda) - \nabla z K(\nabla y^*), \nabla\phi_\lambda)ds \\ &- 2\int_0^t (k^{*\prime} * \Big(\nabla z_\lambda K(\nabla y_\lambda) - \nabla z K(\nabla y^*)\Big), \nabla\phi_\lambda)ds \\ &- 2\textstyle\sum_{i=1}^n (k^* * \Big(\phi_{\lambda x_i}\mathcal{H}_i(y_\lambda,y^*)\nabla y^*\Big)(t), \nabla\phi_\lambda(t)) \\ &+ 2\textstyle\sum_{i=1}^n \displaystyle\int_0^t k^*(0)(\phi_{\lambda x_i}\mathcal{H}_i(y_\lambda,y^*)\nabla y^*), \nabla\phi_\lambda)ds \\ &+ 2\textstyle\sum_{i=1}^n \displaystyle\int_0^t (k^{*\prime} * \Big(\phi_{\lambda x_i}\mathcal{H}_i(y_\lambda,y^*)\nabla y^*\Big), \nabla\phi_\lambda)ds. \end{aligned} \tag{3.45}$$

The left hand side of (3.45) is rewritten as

$$
\begin{aligned}
&\sum_{i=1}^{4}\Psi_\lambda^i(t) + 2(k^* * (\nabla\phi_\lambda K(\nabla y_\lambda))(t), \nabla\phi_\lambda(t)) \\
&- 2\int_0^t k(0)(\nabla\phi_\lambda K(\nabla y_\lambda), \nabla\phi_\lambda)ds \\
&- 2\int_0^t (k^{*\prime} * \Big(\nabla\phi_\lambda K(\nabla y_\lambda)\Big), \nabla\phi_\lambda)ds \\
&- 2\sum_{i=1}^{n}(k^* * (\phi_{\lambda x_i}\mathcal{H}_i(y_\lambda, y^*)\nabla y^*)(t), \nabla\phi_\lambda(t)) \\
&+ 2\sum_{i=1}^{n}\int_0^t k^*(0)(\phi_{\lambda x_i}\mathcal{H}_i(y_\lambda, y^*)\nabla y^*, \nabla\phi_\lambda)ds \\
&+ 2\sum_{i=1}^{n}\int_0^t (k^{*\prime} * \Big(\phi_{\lambda x_i}\mathcal{H}_i(y_\lambda, y^*)\nabla y^*\Big), \nabla\phi_\lambda)ds,
\end{aligned}
\tag{3.46}
$$

where

$$
\begin{aligned}
\Psi_\lambda^1(t) &= -2\int_0^t \langle \eta_\lambda', \phi_\lambda\rangle ds + 2\langle \eta_\lambda(t), \phi_\lambda(t)\rangle, \\
\Psi_\lambda^2(t) &= 2(k^* * \Big(\nabla z K(\nabla y_\lambda) - \nabla z K(\nabla y^*)\Big)(t), \nabla\phi_\lambda(t)), \\
\Psi_\lambda^3(t) &= -2\int_0^t (k^*(0)(\nabla z K(\nabla y_\lambda) - \nabla z K(\nabla y^*), \nabla\phi_\lambda)ds, \\
\Psi_\lambda^4(t) &= -2\int_0^t (k^{*\prime} * \Big(\nabla z K(\nabla y_\lambda) - \nabla z K(\nabla y^*)\Big), \nabla\phi_\lambda)ds.
\end{aligned}
$$

We put

$$S_\lambda(t) = \sum_{i=1}^{4}|\Psi_\lambda^i(t)|.$$

Now we use the same symbol $\mathcal{H}_i(\psi, \phi)$ for $\psi, \phi \in H_0^1(\Omega)$ by the $L^\infty(\Omega)$ function given by

$$\Big[\mathcal{H}_i(\psi, \phi)\Big](x) = \mathcal{H}_i(\psi(x), \phi(x)) \quad \text{a.e. } x \in \Omega.$$

Then, in the above sense we have

$$\Big|(\phi_{\lambda x_i}(t)\mathcal{H}_i(y_\lambda, y^*)\nabla y^*(t), \nabla\phi_\lambda(t))\Big| \le c|\nabla\phi_\lambda(t)|^2, \quad \forall t \in [0, T], \tag{3.47}$$

where $c > 0$. Therefore, by (3.46) and (3.47), we can deduce from (3.45) the following inequality

$$|\phi_\lambda'(t)|^2 + |\nabla\phi_\lambda(t)|^2 \le C_1 S_\lambda(t) + C_2\int_0^t |\nabla\phi_\lambda|^2 ds, \tag{3.48}$$

where C_1 and C_2 are some positive constants. Hence by the Bellman-Gronwall's inequality, it follows that

$$|\phi'_\lambda(t)|^2 + |\nabla\phi_\lambda(t)|^2 \le C_1 S_\lambda(t) + C_1 C_2 \exp(C_2 T) \int_0^t S_\lambda(s)ds. \tag{3.49}$$

By (3.23), we can find a sequence $\{\lambda_k\} \subset (-1, 1)$ tending to 0 such that

$$S_{\lambda_k}(t) \;\to\; 0 \quad \text{as} \;\; \lambda_k \;\to\; 0, \tag{3.50}$$

$$|S_{\lambda_k}(t)| \le C < \infty \quad \forall t \in [0, T]. \tag{3.51}$$

Therefore the inequality (3.49) together with (3.50) and (3.51) implies

$$(\phi_{\lambda_k}(t), \phi'_{\lambda_k}(t)) \;\to\; 0 \;\; \text{strongly in} \;\; H_0^1(\Omega) \times L^2(\Omega). \tag{3.52}$$

With (3.44) and (3.52) we have $z_{\lambda_k} \to z$ in $W(0, T)$, so that by the uniqueness of weak solutions

$$z_\lambda \to z \qquad \text{strongly in} \quad W(0, T) \tag{3.53}$$

as $\lambda \to 0$. This completes the proof.

Theorem 3.3 implies that the cost $J(q)$ is Gâteaux differentiable at q^* in the direction $q - q^*$ and the optimality condition is rewritten by

$$(Cy(q^*) - z_d, Cz)_M \ge 0, \quad \forall q \in \mathcal{P}_{ad}, \tag{3.54}$$

where $z = Dy(q^*)(q - q^*)$ and $z_d \in M$.

We shall consider the following case of distributive and terminal values observations. That is, we set $M = L^2(0, T; L^2(\Omega)) \times L^2(\Omega)$, $Cy(q) = (y(q), y(q; T)) \in L^2(0, T; L^2(\Omega)) \times L^2(\Omega)$ for $q \in \mathcal{P}$, and the cost is given by

$$J(q) = \|y(q) - z_d^1\|^2_{L^2(0,T;L^2(\Omega))} + |y(q; T) - z_d^2|^2, \quad q \in \mathcal{P}_{ad}, \tag{3.55}$$

where $z_d^1 \in L^2(0, T; L^2(\Omega))$ and $z_d^2 \in L^2(\Omega)$.

To give necessary conditions on the above observation, we need to construct a suitable adjoint system. However, in the representation of formal adjoint system the well-posedness cannot be verified by the integral kernel part of it and the presence of $L^2(0, T; H^{-1}(\Omega))$-valued forcing terms. Thus we will use transposition method due to Lions and Magenes [11] to avoid these difficulties.

3.3. Transposition and necessary conditions

Let $g \in L^2(0, T; L^2(\Omega))$. Then by Remark 2.1 via Theorem 2.1, we have a unique weak solution $\phi \in W(0, T) \cap C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ of

$$
(3.56) \qquad \begin{cases} \dfrac{\partial^2\psi}{\partial t^2} - (\alpha_0 + \alpha^{*2})\Delta\psi \\ \qquad - \displaystyle\int_0^t k^*(t-s)\mathrm{div}\Big\{ \frac{\nabla\psi(s,x)}{\sqrt{1+|\nabla y^*(s,x)|^2}} \\ \qquad - \dfrac{\nabla y^*(s,x)\cdot\nabla\psi(s,x)}{\big(1+|\nabla y^*(s,x)|^2\big)^{\frac{3}{2}}}\nabla y^*(s,x)\Big\}ds \ = g \quad \text{in} \quad Q, \\ \psi = 0 \quad \text{on} \quad \Sigma, \\ \psi(0,x) = 0, \quad \dfrac{\partial\psi}{\partial t}(0,x) = 0 \quad \text{in} \quad \Omega. \end{cases}
$$

Therefore we can define the space

$$
X \equiv \{\psi|\psi \text{ satisfies } (3.56) \text{ with } g \in L^2(0,T;L^2(\Omega))\}.
$$

It is seen in Theorem 2.1 that $X \subset W(0,T) \cap C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$. We give a inner product $(\cdot,\cdot)_X$ on X by $(\psi_1,\psi_2)_X = (g_1,g_2)_{L^2(0,T;L^2(\Omega))}$, where ψ_1,ψ_2 are the weak solutions of (3.56) for given $g = g_1, g_2 \in L^2(0,T;L^2(\Omega))$, respectively. We can see that $(X,(\cdot,\cdot)_X)$ is a Hilbert space. Also the map

$$
\begin{aligned} \Theta : \psi \quad \to \quad & \psi'' - (\alpha_0 + \alpha^{*2})\Delta\psi \\ & - \int_0^t k^*(t-s)\mathrm{div}\Big\{ \frac{\nabla\psi}{\sqrt{1+|\nabla y^*|^2}} - \frac{\nabla y^*\cdot\nabla\psi}{\big(1+|\nabla y^*|^2\big)^{\frac{3}{2}}}\nabla y^*\Big\}ds \end{aligned}
$$

of X onto $L^2(0,T;L^2(\Omega))$ is an isomorphism. Hence for each continuous linear functional $L : X \to \mathbf{R}$, there exists uniquely a $p = p_L \in L^2(0,T;L^2(\Omega))$ such that

$$
(3.57) \qquad \int_0^T (p(t), \Theta\psi(t))dt = L(\psi), \quad \forall\psi \in X.
$$

For $h \in L^1(0,T;H^{-1}(\Omega))$, $p_0 \in L^2(\Omega)$ and $p_1 \in H^{-1}(\Omega)$, let us define the functional $L = L(h,p_0,p_1)$ by

$$
(3.58) \qquad L(\psi) = \int_0^T \langle h(t), \psi(t)\rangle dt + \langle p_1, \psi(T)\rangle - (p_0, \psi'(T)).
$$

Then this L is linear on X. Next we shall show the boundedness of L. It is easily checked from the fact $\psi \in X \subset C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ that

$$
\begin{aligned} |L(\psi)| \leq & \ (\|h\|_{L^1(0,T;H^{-1}(\Omega))} + \|p_1\|_{H^{-1}(\Omega)} + |p_0|) \\ & \times(\|\psi\|_{C([0,T];H_0^1(\Omega))} + |\nabla\psi(T)| + |\psi'(T)|). \end{aligned}
$$

Proposition 3.1. *For* $h \in L^1(0,T;H^{-1}(\Omega))$, $p_0 \in L^2(\Omega)$ *and* $p_1 \in H^{-1}(\Omega)$, *there is a unique solution* $p \in L^2(0,T;L^2(\Omega))$ *such that*

$$\begin{cases} \displaystyle\int_0^T (p(t), \Theta\psi(t))dt \\ \displaystyle = \int_0^T \langle h(t), \psi(t)\rangle dt + \langle p_1, \psi(T)\rangle - (p_0, \psi'(T)), \quad \forall \psi \in X. \end{cases}$$

It is verified that the optimality condition for the cost $J(q)$ in (3.55) is written as

$$\int_0^T (y(q^*;t) - z_d^1(t), z(t))dt + (y(q^*;T) - z_d^2, z(T)) \geq 0, \quad \forall q \in \mathcal{P}_{ad}, \tag{3.59}$$

where q^* is the optimal parameter for (3.55) and z is the solution of (3.16). Hence by Proposition 3.1, there exist a unique $p \in L^2(0,T;L^2(\Omega))$ satisfying

$$\begin{cases} \displaystyle\int_0^T (p(t), \Theta\psi(t))dt \\ \displaystyle = \int_0^T (y(q^*;t) - z_d^1(t), \psi(t))dt - (y(q^*;T) - z_d^2, \psi(T)), \\ \forall\psi \quad \text{such that} \quad \Theta\psi \in L^2(0,T;L^2(\Omega)), \\ \psi \in W(0,T) \cap C([0,T];H_0^1(\Omega)), \quad \psi(0) = 0, \quad \psi'(0) = 0. \end{cases} \tag{3.60}$$

In fact the Gâteaux derivative $\psi = z = Dy(q^*)(q - q^*)$ belongs to $X \subset W(0,T) \cap C([0,T];H_0^1(\Omega))$ and

$$\Theta\psi = \mathcal{G}(q - q^*) \in L^2(0,T;L^2(\Omega)), \tag{3.61}$$

where

$$\mathcal{G}(q - q^*) = 2\alpha^*(\alpha - \alpha^*)\Delta y^* + (k - k^*) * \text{div}\, G(\nabla y^*). \tag{3.62}$$

If we take $\psi = z = Dy(q^*)(q - q^*)$ in (3.60), then we have

$$\begin{aligned} &(y(q^*) - z_d^1, z)_{L^2(0,T;L^2(\Omega))} + (y(q^*;T) - z_d^2, z(T)) \\ &= \int_0^T (y(q^*;t) - z_d^1(t), z(t))dt - (y(q^*;T) - z_d^2), z(T)) \\ &= \int_0^T (p(t), \Theta\psi(t))dt = \int_0^T (p(t), \mathcal{G}(q - q^*;t))\, dt. \end{aligned}$$

Therefore we conclude that the optimality condition (3.59) is equivalent to

$$\int_0^T (p(t), \mathcal{G}(q - q^*;t))\, dt \geq 0, \quad \forall q \in \mathcal{P}_{ad}.$$

Hence, we have the following theorem on necessary conditions.

Theorem 3.4. *The optimal parameter q^* for the cost (3.55) is characterized by the two states $y = y(q^*)$, $p = p(q^*)$ of the following system*

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - (\alpha_0 + \alpha^{*2})\Delta y - \displaystyle\int_0^t k^*(t-s)\mathit{div}\Big(\frac{\nabla y(s,x)}{\sqrt{1+|\nabla y(s,x)|^2}}\Big)ds = f \quad \text{in} \quad Q, \\ y = 0 \quad \text{on} \quad \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in} \quad \Omega, \end{cases}$$

$$\begin{cases} \displaystyle\int_Q p\left[\frac{\partial^2 \psi}{\partial t^2} - (\alpha_0 + \alpha^{*2})\Delta\psi \right. \\ \displaystyle\quad \left. - \int_0^t k^*(t-s)\mathrm{div}\Big\{\frac{\nabla\psi}{\sqrt{1+|\nabla y^*|^2}} - \frac{\nabla y^*\cdot\nabla\psi}{\big(1+|\nabla y^*|^2\big)^{\frac{3}{2}}}\nabla y^*\Big\}ds\right] dxdt \\ \displaystyle = \int_Q (y(q^*) - z_d^1)\psi dxdt + \int_\Omega (y(q^*;T) - z_d^2)\psi(T)dx, \\ \forall\psi \quad \text{such that} \\ \dfrac{\partial^2\psi}{\partial t^2} - (\alpha_0 + \alpha^{*2})\Delta\psi \\ \displaystyle\quad - \int_0^t k^*(t-s)\mathrm{div}\Big\{\frac{\nabla\psi(s,x)}{\sqrt{1+|\nabla y^*(s,x)|^2}} - \frac{\nabla y^*(s,x)\cdot\nabla\psi(s,x)}{\big(1+|\nabla y^*(s,x)|^2\big)^{\frac{3}{2}}}\nabla y^*\Big\}ds \in L^2(Q), \\ \psi = 0 \quad \text{on} \quad \Sigma, \\ \psi(0,x) = 0, \quad \dfrac{\partial\psi}{\partial t}(0,x) = 0 \quad \text{in} \quad \Omega, \end{cases}$$

and one inequality

$$\int_Q p\left[2\alpha^*(\alpha - \alpha^*)\Delta y + (k - k^*) * \mathit{div}\, G(\nabla y)\right] dxdt \geq 0,$$
$$\forall q = (\alpha, k(\cdot)) \in \mathcal{P}_{ad}.$$

Remark 3.1 The *adjoint* state p in Theorem 3.4 satisfies formally

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2} - (\alpha_0 + \alpha^{*2})\Delta p \\ \displaystyle\quad - \int_t^T k^*(s-t)\mathrm{div}\Big(\frac{\nabla p(s,x)}{\sqrt{1+|\nabla y(t,x)|^2}} - \nabla y(t,x)\frac{\nabla y(t,x)\cdot\nabla p(s,x)}{(1+|\nabla y(t,x)|^2)^{\frac{3}{2}}}\Big)ds \\ = y(t,x) - z_d^1(t,x) \quad \text{in} \quad Q, \\ p = 0 \quad \text{on} \quad \Sigma, \\ p(T,x) = 0, \quad \dfrac{\partial p}{\partial t}(T,x) = -(y(T,x) - z_d^2(x)) \quad \text{in} \quad \Omega. \end{cases}$$

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