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RINGS WITH INDECOMPOSABLE RIGHT MODULES LOCAL

Surjeet Singh

Abstract. Every indecomposable module over a generalized uniserial ring is uniserial, hence local. This motivates one to study rings R satisfying the condition (*): R is a right artinian ring such that every finitely generated, indecomposable right R-module is local. The rings R satisfying (*) have been recently studied by Singh and Al-Bleahed (2004), they have proved some results giving the structure of local right R-modules. In this paper some more structure theorems for local right R-modules are proved. Examples given in this paper show that a rich class of rings satisfying condition (*) can be constructed. Using these results, it is proved that any ring R satisfying (*) is such that mod-R is of finite representation type. It follows from a theorem by Ringel and Tachikawa that any right R-module is a direct sum of local modules. If M is right module over a right artinian ring such that any finitely generated submodule of any homomorphic image of M is a direct sum of local modules, it is proved that it is a direct sum of local modules. This provides an alternative proof for that any right module over a right artinian ring R satisfying (*) is a direct sum of local modules.

0. INTRODUCTION

It is well known that an artinian ring R is generalized uniserial if and only if every finitely generated indecomposable right R-module is uniserial. Every uniserial module is local. This motivated Tachikawa [10] to study a ring R satisfying the condition (*): R is a right artinian ring such that every finitely generated indecomposable right R-module is local. Consider the dual condition (**): R is left artinian such that every finitely generated indecomposable left R-module is uniform. If a ring R satisfies (*), it is proved by Tachikawa that R admits a finitely generated

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injective cogenerator Q_R, then $B = End(Q_R)$ satisfies (**). Tachikawa had studied a ring R satisfying (*) through the corresponding ring B, but he did not give structure of local right R-modules. Singh and Al-Bleahed [8] have studied rings R satisfying (*) without using the duality, and they have proved some structure theorems on local right R-modules. In section 2, structure of a local right R-modules is further investigated. By using these results it is proved in Theorem 2.14 that R is of finite representation type. In section 3, general right R-modules are investigated. It is well known that exceptional rings as defined by Dlab and Ringel (see [2] or [3]) are balanced ring, and any right module over an exceptional $(1,2)$-ring is a direct sum of local modules. It follows from [2, Proposition 3] and also from [8, Theorem 2.13] that any exceptional $(1,2)$-ring also satisfies (*). It follows from [9, Corollary 4.4]., that any right R-module is a direct sum of local modules. A direct proof of this result is given, by proving the following: If M is a right module over a right artinian ring, such that any finitely generated submodule of any homomorphic image of M is a direct sum of local modules, then M is a direct sum of local modules (Theorem 3.4). As there is no known duality that can tell that a ring R satisfies (*) if and only if it satisfies (**), it would be interesting to examine condition (**) by itself. In section 4, some examples illustrating various results are given.

1. Preliminaries

All rings considered here are with identity $1 \neq 0$ and all modules are unital right modules unless otherwise stated. Let R be a ring and M be an R-module. $J(M)$, $E(M)$ and $socle(M)$ denote *radical*, *injecitve hull* and *socle* of M respectively, however $J(R)$ will be denoted by J. If R is right artinian, then $J(M) = MJ$. Further, $N \leqslant M$ denotes that N is a submodule of M. A ring R is called a *local ring*, if R/J is a division ring. Given two positive integers n, m, a ring R is called an $(n.m)$-ring if it is a local ring, $J^2 = 0$ and for $D = R/J$, $dim\ _DJ = n$ and dim $J_D = m$. Any $(1,2)$ (or $(2,1)$) ring R is called an *exceptional ring* if $E(_RR)$ (respectively $E(R_R)$) is of composition length 3 [4, p 446]. A module in which the lattice of submodules is linearly ordered under inclusion, is called a *uniserial module*, and module that is a direct sum of uniserial modules is called a *serial module* [5, Chapter V]. If for a ring R, $_RR$ (R_R) is serial, then R is called a *left (right) serial ring*. A ring that is local, both serial and artinian, is called a *chain* $ring$. A ring R is said to be of *finite right representation type*, if it admits only finitely many non-isomorphic indecomposable right R-modules [5, p 109]. If a module M has finite composition length, then $d(M)$ denotes the composition length of M. For definitions of M-injective and M-projective modules one may refer to [1, p 184].

2. Local Modules

Consider the following condition on a ring R: (*) R is a right artinian ring such that any finitely generated indecomposable right R-module is local.

The following is proved in [8, Proposition 2.2]

Proposition 2.1. *Let R be a right artinian ring. Then R satisfies (*) if and only if for any two non-simple local right R-modules A, B, simple submodules S, T of A, B respectively, and any R-isomorphism $\sigma : S \to T$, either σ or σ^{-1} extends to an R-homomorphism from A to B or from B to A respectively.*

Proposition 2.2. ([8]). *Let R be a ring satisfying $(*)$.*

(*i*) *Any uniform right R-module is uniserial.*

(*ii*) *R is left serial.*

(*iii*) *Let A, B be two uniserial right R-modules each of composition length at least three. Then $M = A \oplus B$ does not contain any local, non-uniserial submodule of composition length 3.*

(*iv*) *Let C_1, C_2 be two uniserial R-modules such that for some $k \geqslant 2$, $C_1/C_1J^k \cong C_2/C_2J^k$, and C_1J^k, C_2J^k are non-zero, then $C_1/C_1J^{k+1} \cong C_2/C_2J^{k+1}$.*

(*v*) *Let A_R, B_R be two local modules such that $d(A) = d(B)$, $AJ^2 = 0 = BJ^2$. For any simple submodule S of A, any R-monomorphism $\sigma : S \to B$ extends to an R-isomorphism from A onto B.*

For any local module A_R, AJ is a direct is a direct sum of uniserial modules [8, Lemma 2.7].

Theorem 2.3. ([8, Theorem 2.10]). *Let R be a ring satisfying (*) and A_R be a local module such that $AJ = C_1 \oplus C_2 \oplus \cdots \oplus C_t$ for some uniserial modules C_i. Then the following hold.*

(i) Either all C_i/C_iJ are isomorphic or $t \leq 2$.

(ii) Any local submodule of AJ is uniserial.

(iii) If $d(C_1) \geq 2$, then either $t \leq 2$ or any C_i is simple for $i \geq 2$.

Proposition 2.4. *Let R be a ring satisfying (*).*

(*i*) *Let A_1 and A_2 be any two uniserial right R-modules. Then $A_1J \oplus A_2J$ does not contain a submodule that is local but not uniserial.*

(*ii*) *If a non-zero homomorphic image of a uniserial right R-module L is injective, then L is injective.*

(*iii*) *Let A_R be a local module, and $AJ = C_1 \oplus D$, where C_1 is uniserial. Let σ be an R-endomorphism of A such that $ker\sigma \cap C_1 = 0$, and σ is not an automorphism. Then $\sigma(A)$ is a uniserial module of composition length more than $d(C_1)$, A/D embeds in $\sigma(A)$ and no homomorphic image of A/D is injective. If a module B_R embeds in C_1, then no non-zero homomorphic image of B is injective.*

(*iv*) *Let A_R be a local module, and $AJ = C_1 \oplus C_2 \oplus .... \oplus C_t$ for some uniserial submodules C_i. Let $s = max\{d(C_i) : 1 \leq i \leq t\}$. Then for any simple submodule S of A, and any uniserial submodule B of A of composition length s, any R-homomorphism $\sigma : S \to B$ extends to an R-endomorphism of A; if in addition S is contained in a uniserial submodule of composition length s, then σ is an automorphism.*

Proof.

(i) On the contrary suppose that $A_1J \oplus A_2J$ contains a non-uniserial local submodule uR. Then $u = u_1 + u_2$, $0 \neq u_i \in A_iJ$, and uJ is a direct sum of two non-zero uniserial submodules. As A_i are uniserial, without loss of generality we take $u_iR = A_iJ$. Then $uJ^2 = A_1J^3 \oplus A_2J^3$. This gives that $(A_1 \oplus A_2)/uJ^2 = B_1 \oplus B_2$ for some uniserial modules with $d(B_i) \geq 3$. But uR/uJ^2 is local, non-uniserial of composition length 3, and it embeds in $B_1 \oplus B_2$. This contradicts (2.2)(iii). Hence $A_1J \oplus A_2J$ does not contain a non-uniserial, local submodule

(ii) It is immediate from the fact that any uniform right R-module is uniserial.

(iii) By (2.3)(ii), $\sigma(A)$ is uniserial. As $B = ker\sigma$ embeds in D, it is immediate that $d(\sigma(A)) \geq d(A/D) = d(C_1) + 1$. As $B \cap C_1 = 0$, C_1 embeds in $\sigma(A)$. Also, C_1 embeds in A/D, it also follows that A/D embeds in $\sigma(A)$. As $\sigma(A)$ is not injective, by (ii) no homomorphic image of A/D is injective. The last part also follows from (ii)

(iv) Let $C = socle(B)$ and $\sigma : S \to B$ be an R-homomorphism. Suppose the contrary. As every uniserial R-module is quasi-injective, $t \geq 2$, $d(A) \geq s+2$ and AJ contains no uniserial submodule of composition length more than s. By (2.1), $\sigma^{-1} : C \to S$ extends to an R-endomorphism λ of A. Then λ is not automorphism, and $\lambda(A) \subseteq AJ$. As λ is one-to-one on B, we get $d(\lambda(A)) \geq s + 1$. But by (2.3)(ii), $\lambda(A)$ is uniserial, so we have a contradiction. The last part again follows from (2.3)(ii). ■

(2.1) gives the following.

Proposition 2.5. *Let a ring R satisfy (*). Let A_R, B_R be two local, modules such that A is B-projective and B is A-projective. Let $A_1 < A_2 < A$, $B_1 < B_2 < B$ be such that A_2/A_1 is simple and there exists an R-isomorphism $\sigma : A_2/A_1 \to B_2/B_1$. Then either there exists an R-homomorphism λ of $A \to B$ inducing σ or there exists an R-homomorphism $\lambda : B \to A$ inducing σ^{-1}.*

Henceforth, throughout this section R is a ring satisfying (*).

Lemma 2.6. *Let A_R be a local module.*

(*i*) *If $AJ = C_1 \oplus C_2$, where C_i are minimal submodules, then either A/C_1 or A/C_2 is injective.*

(*ii*) *If $AJ = C_1 \oplus C_2$, where C_i are uniserial, then either A/C_1 or A/C_2 is such that its every non-simple homomorphic image is injective. .*

(*iii*) *Suppose $AJ = C_1 \oplus C_2 \oplus .... \oplus C_t$ such that each C_i is uniserial and $t \geq 3$. For each $1 \leq i \leq t$, let L_i be the direct sum of all C_j with $j \neq i$. Then every non-simple homomorphic image of any A/L_i is injective.*

(*iv*) *Let $AJ = C_1 \oplus C_2 \oplus D$ with C_1 and C_2 both uniserial. Suppose for some k, l, C_1J^k/C_1J^{k+1} and C_2J^l/C_2J^{l+1} are isomorphic, and for some $u \geq 1$, $C_1J^{k+u} \neq 0 \neq C_2J^{l+u}$, then C_1J^{k+u}/C_1J^{k+u+1} and C_2J^{l+u}/C_2J^{l+u+1} are isomorphic.*

Proof.

(i) If none of A/C_i is injective, then A embeds in $E(A/C_1)J \oplus E(A/C_2)J$, which contradicts (2.4)(i). This proves (i).

(ii) By applying (i) to A/AJ^2 and by using Proposition $(2.4)(\text{ii})$, it follows.

(iii) For $t \geq 3$, as all C_i/C_iJ are isomorphic by (2.3), the result follows from (i).

(iv) It is enough to prove the result for $u = 1$. Suppose that C_1J^{k+1}/C_1J^{k+2} and C_2J^{l+1}/C_2J^{l+2} are not isomorphic. For some indecomposable idempotent $e \in R$, C_1J^k/C_1J^{k+2} and C_2J^l/C_1J^{l+2} both are homomorphic images of eR. This gives a local, non-uniserial module B_R of composition length 3 with $BJ = L_1 \oplus L_2$ such that $B/L_1 \cong C_1J^k/C_1J^{k+2}$ and $B/L_2 \cong C_2J^l/C_1J^{l+2}$. Let $\overline{A} = A/(C_1J^{k+2} \oplus C_2J^{l+2})$. Then B embeds in the radical of the direct sum of $\overline{A}/\overline{C_1 \oplus D}$ and $\overline{A}/\overline{C_2 \oplus D}$, which is a contradiction to (2.4)(i). This proves the result. ■

Lemma 2.7. *Let A_R be a local module and B_R any module. For some $C \leqslant B$, let $\sigma : A \to B/C$ be an R-homomorphism.*

(*i*) *There exists a local submodule D of $A \times B$ such that $D = (a,b)R$ with $aR = A$ and $\sigma(a) = b + C$. If D is uniserial and $d(B) \leqslant d(A)$,then σ can be lifted to some R-homomorphism $\eta : A \to B$*

(*ii*) *If $A \times B$ does not contain a local submodule D_1 with $d(D_1) > d(A)$, then A is B-projective.*

(*iii*) *If $A \times B$ has no non-uniserial local submodule and $d(B) \leqslant d(A)$, then A is B-projective.*

Proof. (i) Let $N = \{(x,y) \in A \times B : \sigma(x) = y + C\}$. Let $\pi : A \times B \to A$ be the natural projection. Then $\pi(N) = A$. There exists a local submodule D of N such that $\pi(D) = A$. Clearly $D = (a,b)R$ with $A = aR$ and $\sigma(a) = b + C$. Now

$d(D) \geqslant d(A)$. Suppose $d(D) = d(A)$. Then $D \cong A$ and $\eta : A \to B$ given by $\eta(ar) = br$ lifts σ. In case D is uniserial and $d(B) \leqslant d(A)$, then $D \cong A$, so once again σ can be lifted. After this (ii) is immediate. Under the hypothesis in (iii), the hypothsis in (ii) holds, so A is B-projective. ■

Lemma 2.8. *Let A_R and B_R be two uniserial modules and $\sigma : A \to B/C$ be an R-epimorphism for some $C < B$.*

(*i*) *If A is not injective and $d(B) \leqslant d(A)$, then either B is injective or σ can be lifted to some R-homomorphism $\eta : A \to B$.*

(*ii*) *If $d(B) \leqslant d(A)$ and neither A nor B is injective, then A is B-projective.*

(*iii*) *Any uniserial right R-module is either injective or quasi-projective.*

(*iv*) *Let $C = socle(A)$, and $C < D \leq A$ with D/C a simple module. If $C \cong D/C$, then all the composition factors of A are isomorphic*

Proof. By Lemma 2.7, there exists a local submodule $D = (a, b)R \subseteq A \times B$ such that $A = aR$ and $\sigma(a) = b + C$. Suppose $d(B) \leq d(A)$. If D is uniserial, it follows from (2.7)(i) that σ lifts to an R-homomorphism $\eta : A \to B$. Suppose D is not uniserial. Then $DJ = C_1 \oplus C_2$ for some non-zero uniserial submodules C_i. Let π_A and π_B be the natural projections of $A \times B$ onto A and B respectively. Then for one of the C_i say C_1, $\pi_A(C_1) = AJ$. But $\pi_B(C_1) \subseteq BJ$ and $d(B) \leqslant d(A)$, it follows that C_1 is isomorphic to AJ under π_A. Therefore $AJ \times BJ = C_1 \oplus (0 \times BJ)$, $DJ = C_1 \oplus (DJ \cap (0 \times BJ))$ and $C_2 \cong DJ \cap (0 \times BJ)$. Suppose that neither A nor B is injective, then $D \subseteq E(A)J \oplus E(B)J$, therefore by (2.4), D is uniserial. Then, by using (2.7)(i), we get A is B-projective. From this (i), (ii) and (iii) follow. (iv) is immediate from the fact that the injective hull of A is uniserial. ■

Lemma 2.9. *Let A_R be a local module such that $AJ = A_1 \oplus A_2$ for some uniserial submodules A_i and there exists an R-isomorphism $\sigma : soc(A_1) \to soc(A_2)$. Let there exists an R-endomorphism μ of A that extends σ. Let M_i be the maximal submodule of A_i. Then:*

(*i*) *$d(A_1) \leq d(A_2)$, A/A_1 is injective and $A/M_2 \oplus A_1$ is injective.*

(*ii*) *If $d(A_1) < d(A_2)$, then A/A_2 is quasi-projective, $A/(M_1 \oplus A_2)$ is not injective and $A/(M_2 \oplus A_1)$ is injective.*

(*iii*) *If $d(A_1) = d(A_2)$, then $A/M_2 \oplus A_1 \cong A/M_1 \oplus A_2$ and both are injective.*

(*iv*) *If $A_1/A_1J \cong A_2/A_2J$, then $A_1 \cong A_2$.*

Proof. Suppose $f : eR \to A$ is the projective cover of A. We take $A = eR/B$ and $A_i = C_i/B$ for some right ideals $B < C_i < eR$. Suppose there exists an R-endomorphism μ of A that extends σ. We can find an R-endomorphism λ of eR that lifts μ. Then $\lambda(B) \subseteq B$, $\lambda(socle(C_1)) + B = socle(C_2) + B \nsubseteq C_1 + B$. Hence C_1 is not invariant under the endomorphisms of eR, eR/C_1 is not quasi-projective, therefore A/A_1 being isomorphic to eR/C_1,is not quasi-projective. By (2.8)(iii), A/A_1 is injective. As $\mu(A_1) \cong A_1$ and $\mu(A_1) \cap A_1 = 0$, it follows that $d(A_1) \leq d(A_2)$ and A_1 embeds in A_2. Let $M_i \underset{\max}{<} A_i$. Suppose $d(A_1) < d(A_2)$, it follows that A/A_2 is isomorphic to a submodule A_2, and hence $A/(A_2 \oplus M_1)$ is not injective. Therefore by (2.6)(i), $A/(A_1 \oplus M_2)$ is injective. As A/A_2 is not injective, by (2.8)(iii), it is quasi-projective. If $d(A_1) = d(A_2)$, then the isomorphism σ gives that $A/A_1 \oplus M_2$ and $A/A_2 \oplus M_1$ are isomorphic, so once again, by (2.6)(i), both are injective. The hypothesis in (iv) gives that $A/(M_2 \oplus A_1) \cong A/(M_1 \oplus A_2)$, so they are injective by (i). By (ii), $d(A_1) = d(A_2)$.. Hence $A_1 \cong A_2$.

Theorem 2.10. *Let R be a local ring satisfying (*). If $J^2 \neq 0$, then R is a chain ring.*

Proof. By (2.2), R is a left serial ring. If R is not right serial, we get a local, right R-module A such that $AJ = C_1 \oplus C_2$ with each C_i uniserial, $d(C_1) = 2$, $d(C_2) = 1$. As every composition factor of A is isomorphic to R/J, it contradicts (2.9)(iv). Hence R is a chain ring.

Lemma 2.11. *Let A_R be a local module such that $AJ = A_1 \oplus A_2 \oplus L$ for some uniserial modules A_i, with $d(A_1) > 1$, and $L \neq 0$. Then no two composition factors of A_1 are isomorphic.*

Proof. By (2.3), AJ/AJ^2 is homogeneous. Suppose, A_1 has two isomorphic composition factors. Then for some $s \geq 1$, $A_1/A_1J \cong A_1J^s/A_1J^{s+1}$. Let $B = A/(A_1J^{s+1} + L)$. Then B contradicts (2.9)(ii).

Theorem 2.12. *Let A_R be a local module over a ring R satisfying (*) such that $AJ = C_1 \oplus C_2 \oplus .... \oplus C_t$ for some uniserial modules C_i such that $t \geq 2$, and $d(C_1) \geq 2$. Let $C_1/C_1J \cong C_i/C_iJ$ for some $i > 1$, then $t = 2$. If A is projective, then $C_1 \cong C_2$*

Proof. To start with, we take $A = eR$ for some indecomposable idempotent e. Suppose $C_1/C_1J \cong C_2/C_2J$. So there exists an indecomposable idempotent $f \in R$, such that for some $u, v \in eJf$, $C_1 = uR$, $C_2 = vR$. Then $u, v \in eJf \backslash J^2$. As R is left serial, $Rf = Ru = Rv$. We get $v = bu$ for some unit b in eRe, $C_2 = bC_1$, $d(C_1) = d(C_2)$. This contradicts (2.3)(iii) unless $t = 2$. By (2.6)(iv), $soc(C_1) \cong soc(C_2)$, hence $C_1 \cong C_2$. In general, as A is a homomorphic image of an eR, where $e = e^2$ is indecomposable, the result follows.

Theorem 2.13. *Let A_R be a local module such that $AJ = C_1 \oplus C_2$, where C_i are uniserial, and $C_1J^k/C_1J^{k+1} \cong C_1J^l/C_1J^{l+1} \neq 0$, for some $k < l$.*

(i) *A/C_1 has all its non-simple homomorphic images injective.*

(ii) *No two composition factors of C_2 are isomorphic.*

(iii) *No composition factor of C_2 is isomorphic to a composition factor of C_1.*

(iv) *A, A/C_1 and A/C_2 are all quasi-projective.*

Proof. Let $\lambda : eR \to A$ give the projective cover of A. Then $eJ = D_1 \oplus D_2 \oplus L$, where D_1, D_2 are uniserial and $C_1 = \lambda(D_1)$. If $L \neq 0$, by (2.11), D_1has no two composition factors isomorphic, which is a contradiction. Hence $L = 0$, and $eJ = D_1 \oplus D_2$. For some $s \geq 1$, $D_1/D_1J \cong D_1J^s/D_1J^{s+1}$. Thus $eR/(D_2 \oplus D_1J)$ embeds in D_1/D_1J^{s+1}, therefore it is not injective. Consequently, by (2.6)(i), $eR/(D_1 \oplus D_2J)$ is injective. Then, by (2.4)(ii), every non-simple homomorphic image of eR/D_1 is injective. If D_2 has two isomorphic composition factors, the interchange of the roles of D_1, D_2 will give that every non-simple homomorphic image of eR/D_2 is injective, in particular, $eR/(D_2 \oplus D_1J)$ is injective, which is a contradiction. Hence D_2 has no two composition factors isomorphic.

Suppose eR/D_2 is not quasi-projective. Then D_2 is not invariant under the R-endomorphisms of eR, consequently, there exists a non-zero homomorphism of D_2 into D_1. Therefore $D_2/D_2J \cong D_1J^v/D_1J^{v+1}$ for some $v \geq 0$. If $v > 0$, we get $eR/D_1 \oplus D_2J$ is not injective, which is a contradiction to (i) for eR. Hence $v = 0$. Then $eR/D_2 \oplus D_1J$ is isomorphic to $eR/D_1 \oplus D_2J$, so once again it is injective, which is a contradiction. Hence eR/D_2 is quasi-projective.

Suppose there exists an R-isomorphism $\sigma : D_1J^i/D_1J^{i+1} \to D_2J^j/D_2J^{j+1}$ for some i and j, with $D_1J^i \neq 0$. If $j \leq i$, then $D_2/D_2J \cong D_1J^u/D_1J^{u+1}$ for some u, and as in the above paragraph, we get a contradiction. Hence $i < j$. Then $D_1J^s/D_1J^{s+1} \cong D_1/D_1J \cong D_2J^u/D_2J^{u+1}$ for some $u \geq 1$. Then $eR/eJ \cong D_2J^{u-1}/D_2J^u \cong D_1J^{s-1}/D_1J^s$. It follows that eR/eJ is isomorphic to the top and bottom composition factors of $eR/D_2 \oplus D_1J^s$, and to the top and bottom composition factors of $eR/D_1 \oplus D_2J^u$. At the same time D_2/D_2J is isomorphic to a composition factor of $eR/D_1 \oplus D_2J^u$. The periodicity of the composition factors gives that D_2/D_2J is also isomorphic to a composition factor of $eR/D_2 \oplus D_1J^s$. Thus D_2/D_2J is either isomorphic to a composition factor of D_1/D_1J^s or it is isomorphic to eR/eJ. In the former case, we get a contradiction to $i < j$, and in the later case, every composition factor of $eR/D_1 \oplus D_2J^u$ and of $eR/D_2 \oplus D_1J^s$ is isomorphic to eR/eJ, and therefore $D_1/D_1J \cong D_2/D_2J$, which is a contradiction. Hence D_1 has no composition factor isomorphic to a composition factor of D_2. Hence $C_2 = \lambda(D_2)$. It follows that any submodule of $D_1 \oplus D_2$ is invariant under any R-endomorphism of eR. Consequently, A, A/C_1 and A/C_2 are all quasi-projective. ■

Theorem 2.14. *If a ring R satisfies $(*)$, then there exist only finitely many non-isomorphic, local right R-modules.*

Proof. All indcomposabe finitely generated right R-modules are local. As R is right artinian, there exists a bound on the composition lengths of the local modules and on the number of possible semi-simple modules that occur as socles of the local right R-modules. To prove the result it is enough to prove that given a triple (S_R, n, T_R), where S_R is simple, T_R is semi-simple and n is a positive integer, there do not exist more than two local modules A_R such that $S \cong A/AJ$, $d(A)$ = n and $socle(A) \cong T$.

Fix a local module A_R. Let B_R be another local module such that $A/AJ \cong B/BJ$, $d(A)$ = $d(B)$ and $socle(A) \cong socle(B)$. If A is uniserial, then so is B, and obviously $A_R \cong B_R$. So we shall suppose that A is not uniserial. Now A, B admit same projective cover, say eR.

Suppose AJ is semi-simple. Then BJ is also semi-simple. By (2.2)(v), A and B are isomorphic.

Henceforth we shall suppose that AJ is not semi-simple. Then AJ = $D_1 \oplus D_2 \oplus \ldots. \oplus D_u$, BJ = $H_1 \oplus H_2 \oplus \ldots. \oplus H_u$ and eJ = $C_1 \oplus C_2 \oplus \ldots. \oplus C_t$ for some uniserial modules D_i, H_j, C_k, with $u \leq t$. We take $d(D_1) \geq 2$, $d(H_1) \geq 2$ and D_1 a homomorphic image of C_1.

Suppose. $t \geq 3$. Then all other C_j for $j \geq 2$ are simple. As D_1 and H_1 have same composition length, and by (2.11), no two composition factors of C_1 are isomorphic, we get an isomorphicm $\sigma : socle(D_1) \to socle(H_1)$. Because of (2.1), we can take σ such that it extends to an R-homomorphism $\lambda : A \to B$. As in (2.4)(iv), λ is an isomorphism. Hence $A_R \cong B_R$.

Henceforth, we take t = 2. Then u = 2. It follows that $A/(D_1 \oplus D_2J)$ is either isomorphic to $eR/C_1 \oplus C_2J$ or to $eR/C_2 \oplus C_1J$. As $socle(A) \cong socle(B)$, we take $socle(D_i) \cong socle(H_i)$ for i = 1, 2. Suppose $d(D_1)$ = $d(H_1)$. By using (2.1), we can suppose that there exists an R-homomorphsm $\lambda : A \to B$ such that $\lambda(socle(D_1))$ = $socle(H_1)$. If λ is not an isomorphism, then $\lambda(A)$ is a uniserial module contained in BJ such tha $\lambda(A) \cap H_2$ = 0, and $d(\lambda(A)) > d(H_1)$. Therefore $d(\lambda(A) + H_2) > d(BJ)$, which is a contradiction. Hence $A_R \cong B_R$.

Suppose $d(D_1) \neq d(H_1)$. Because of (2.6)(ii), we take D_1 such that every non-simple homomorphic image of A/D_1 is injective. If $d(D_2) < d(H_2)$, then as $socle(D_2) \cong socle(H_2)$, A/D_1 embeds in H_2, so A/D_1 is not injective, which is a contradiction. Hence $d(H_2) < d(D_2)$. Then B/H_1 embeds in D_2, therefore B/H_1 has no non-zero homomorphic image injective. Hence every non-simple homomorphic image of B/H_2 is injective. Therefore, $A/D_1 \oplus D_2J$ and $B/H_2 \oplus H_1J$ are isomorphic, that gives $D_2/D_2J \cong H_1/H_1J$ and $D_1/D_1J \cong H_2/H_2J$. Now $d(D_1) < d(H_1)$, so D_1 embeds in H_1. Therefore D_1/D_1J is isomorphic to a composition factor of H_1. Thus D_1/D_1J is isomorphic to a composition factor of H_1

as well as of H_2. Then by (2.13), no two composition factors of H_1 are isomorphic and no two composition factors of H_2 are isomorphic. So there exists unique positive integer t such that $D_1/D_1J \cong H_1J^t/H_1J^{t+1}$. That gives $D_1 \cong H_1J^t$. Thus $d(D_1) = d(H_1) - t$ and $d(D_2) = d(H_2) + t$. Hence by the cases discussed above, the result follows. ∎

3. Decomposition Theorem

Lemma 3.1. *Let M be any right module over a ring R.*

(i) *Let L be a finitely generated submodule of M such that L is a summand of any finitely generated submodule of M containing L. Let $S < M$ be such that S is finitely generated and in $\overline{M} = M/L$, $\overline{S}$ is a summand of every finitely generated submodule of $\overline{M}$. Then $L+S$ is a summand of any finitely generated submodule of M containing $L+S$.*

(ii) *Let $N \leqslant M$ such that N is finitely generated and is summand of any finitely generated submodule of M containing N. Then $NJ = MJ \cap N$.*

(iii) *If L is a finitely generated submodule of M such that it is a summand of every finitely generated submodule of M containing L, then any summand K of L is also a summand of any finitely generated submodule of M containing K.*

Proof.

(i) Let $L + S \leqslant T$, where T is a finitely generated submodule of M. Then $T = L \oplus C$, $L + S = L \oplus W$ for some $C \leqslant M, W \leqslant M$. Therefore $\overline{S} = \overline{W}$ and $\overline{S} \leqslant \overline{C}$ in $\overline{M} = M/L$. By the hypothesis, $\overline{C} = \overline{S} \oplus \overline{K}$ for some $K \leqslant M$ containing L. Thus $T = S + K = W + K$ and $W \cap K \subseteq L$. As K is finitely generated, $K = L \oplus V$ for some $V \leqslant K$, $T = (W + L) + V$. Suppose for some $w \in W$, $x \in L$, and $v \in V$, $w + x = v$. Then $w \in W \cap K \subseteq L$, $v \in L \cap V = 0$. Hence $(W + L) \oplus V = T = (S + L) \oplus V$.

(ii) Let $x \in MJ \cap N$. Then $x = \sum_i x_i a_i$ for some finitely many $x_i \in M$, $a_i \in J$. Set $K = \sum_i x_i R + N$. Then K is finitely generated, $x \in KJ$, $K = N \oplus P$ for some $P \leqslant K$, and $KJ = NJ \oplus PJ$. Hence $x \in NJ$.

(iii) Now $L = K \oplus S$ for some $S \leqslant L$. Suppose $K \leqslant T$, a finitely generated submodule of M. Then $T + S = L \oplus V = K \oplus (S \oplus V)$. This gives $T = K \oplus W$, where $W = T \cap (S \oplus V)$. ∎

Definition 3.2. A module M is said to satisfy $(\diamond)$ if any finitely generated submodule of any homomorphic image of M is a direct sum of local modules having finite composition lengths.

Lemma 3.3. *Let M_R be a module satisfying $(\diamond)$ and R be right artinian. Let $A = \oplus \sum_{\alpha \in \Lambda} A_\alpha \leqslant M$ such that, each A_α is finitely generated and for any finite subset X of Λ, $A_X = \sum_{\alpha \in X} A_\alpha$ is a summand of any finitely generated submodule of M containing it. Let S be a local submodule of M such that $\overline{S}$ in $\overline{M} = M/A$ is non-zero and is a summand of any finitely generated submodule of $\overline{M}$ containing $\overline{S}$.*

- (a) *Let Γ be any finite subset of Λ such that $S \cap A = S \cap C$, where $C = A_\Gamma$. Then $\overline{S}$ in M/C is also a summand of any finitely generated submodule of M/C containing $\overline{S}$.*
- (b) *There exists a local submodule S_1 of M such that $A \cap S_1 = 0$, $\overline{S_1} = \overline{S}$ in M/A, and for any finite subset Γ of Λ, $A_\Gamma \oplus S_1$ is a summand of any finitely generated submodule of M containing it.*

Proof.

- (a) It follows from (3.1)(ii) that $AJ = MJ \cap A$. Now $S \cap A = SJ \cap A = SJ \cap (MJ \cap A) = S \cap AJ$. As S is finitely generated, we get a finite subset $\Gamma$of Λ such that $S \cap A = S \cap CJ$, where $C = A_\Gamma$. In $\overline{M} = M/C$, let $\overline{S}$ be contained in a finitely generated submodule $\overline{T}$, with $C \leqslant T$. Then T is finitely generated. Now $A = C \oplus D$ for some $D \leqslant A$. Consider $T_1 = T + D$. In M/A, $\overline{T_1} = \overline{T}$ and $\overline{S} \leqslant \overline{T_1}$. Therefore $\overline{T_1} = \overline{S} \oplus \overline{L}$ for some $A \leqslant L$, $S \cap L = S \cap A = S \cap CJ$. We get $T = S + (T \cap L)$ with $S \cap (L \cap T) \subseteq CJ$. This gives $(S + C) \cap [(L \cap T) + C] = C + [(S + C) \cap (L \cap T)] = C$, as $C \subseteq L \cap T$. Hence, $\overline{S}$ in M/C is a summand of $\overline{T}$.
- (b) Let Γ be a finite subset of Λ such that $S \cap A = S \cap CJ$, where $C = A_\Gamma$. We choose S to be of smallest composition length among those local submodules S' for which $\overline{S} = \overline{S'}$. By the hypothesis, $C + S = C \oplus S_1$ for some local submodule S_1 of M. Then in M/A, $\overline{S} = \overline{S_1}$ and $d(S_1) \leqslant d(S)$. That gives $d(S) = d(S_1)$ and $C + S = C \oplus S$. Hence $A \cap S = 0$. Let X be any finite subset of Λ. Now $A \cap S = A_X \cap S = 0$. Let T be any finitely generated submodule of M containing A_X such that in M/A_X, $\overline{S} \subseteq \overline{T}$, then by (a), $\overline{S}$ is a summand of $\overline{T}$. Now $T = A_X \oplus P$ for some $P \leqslant T$. In M/A_X, $\overline{S} \subseteq \overline{P}$, $\overline{P} = \overline{S} \oplus \overline{Q}$ for some $Q \leqslant M$ containing A_X. Therefore, $T = S \oplus Q$, as $S \cap Q \subseteq A_X \cap S = 0$. But A_X is also a summand of Q. Hence $A_X \oplus S$ is a summand of T. This proves the result. ∎

Theorem 3.4. *If a module M_R satisfies satisfies $(\diamond)$, where R is right artinian, then M is a direct sum of local modules. Any module over a ring R saisfying (*) is a direct sum of local modules.*

Proof. Let xR be a local submodule of M of smallest composition length such that $xR \nsubseteq MJ$. Let T be a finitely generated submodule of M containing xR. Now $T = \oplus \sum_{i=1}^{n} A_i$ for some local submodules A_i. Let $\pi_i : T \rightarrow A_i$ be the projections giving this decomposition of T. If for every i, either $\pi_i(xR) \subseteq A_iJ$ or $A_i \subseteq MJ$, then $xR \subseteq MJ$, which is a contradiction. Thus for some i, $\pi_i(xR) \nsubseteq A_iJ$ and $A_i \nsubseteq MJ$. Then $\pi_i(xR) = A_i$, $d(x_iR) = A_i$. Therefore π_i maps xR isomorphically onto A_i. Hence xR is a summand of T. Let F be the family of all those local submodules of M that are summand of any finitely generated submodule that contains them. Thus F is non-empty. A subfamily $F^{'}$ of F is said to satisfy condition (S), if the sum of the members of $F^{'}$ is direct and the sum of any finite subfamily of $F^{'}$ is a summand of any finitely generated submodule of M containing that sum. The set of all such subfamilies is non-empty. Union of any chain of subfamilies of F satisfying (S) satisfies (S). So, there exists a maximal subfamily $\{A_\alpha\}_{\alpha\in\Lambda}$ of F satisfying (S). Thus $\{A_\alpha\}_{\alpha\in\Lambda}$ satisfies the hypothesis in (3.3). Now $N = \sum_{\alpha\in\Lambda} A_\alpha = \oplus \sum_{\alpha\in\Lambda} A_\alpha$. Suppose $M \neq N$. Then as for M, M/N has a local submodule $\overline{B}$ that is a summand of any finitely generated submodule of M/N containing $\overline{B}$. As seen in the proof of (3.3)(b), we can choose B such that it is local, $N \cap B = 0$ and the family $\{A_\alpha\}_{\alpha\in\Lambda} \cup \{B\}$ satisfies (S), which is a contradiction to the maximality of $\{A_\alpha\}_{\alpha\in\Lambda}$. Hence $M = N$, a direct sum of local submodules. As any module over a ring satisfying (*), satisfies ($\diamond$), the second part follows. ■

Theorem 3.5. *Let R be a ring satisfying (*), and M be any right R-module. Then any local submodule of MJ is uniserial and MJ is a direct sum of uniserial submodules. $R/r.ann(J)$ is a generalized uniserial ring.*

Proof. Let T be a finitely generated submodule of MJ. Suppose T is not a direct sum of uniserial submodules. So there exists a local submodule uR of T that is not uniserial. There exists a finitely generated submodule K of M such that $T \subseteq KJ$. Now $K = \oplus \sum_{i=1}^{n} A_i$ for some local submodules A_i. Let $\pi_i : K \rightarrow A_i$ be the corresponding projections and $L_i = ker(\pi_i \mid uR)$. As uR/L_i embeds in A_iJ, by (2.2), each uR/L_i is uniserial. Therefore $L_i \neq 0$ for any i. However, $\bigcap_i L_i = 0$, so we get, say L_1, L_2 such that $L_1 \nsubseteq L_2$ and $L_2 \nsubseteq L_1$. Let $v = \pi_1(u) + \pi_2(u)$. Then $vR \cong uR/(L_1 \cap L_2)$, it is local but not uniserial. As $\pi_i(u)R \subseteq A_iJ$, by [8, Lemma 2.7], $\pi_i(u)R$ is uniserial. For any local module A_R, as AJ is a direct sum of uniserial modules, any uniserial submodule wR of AJ embeds in a uniserial summand of AJ. From this it follows that there exist two uniserial R-modules B_i such that vR embeds in $B_1J \oplus B_2J$, which contradicts (2.4)(i). Hence any submodule of MJ is a direct sum of uniserial modules.

Now $R' = R/r.ann(J)$ embeds in a finite direct sum K of copies of J_R. As any local submodule of K is uniserial, R' is right serial. As R' is also left serial, is a generalized uniserial ring. ■

4. Some Examples

The following is easy to prove.

Lemma 4.1. *Let A be a uniserial module over a generalized uniserial ring R, such that no two composition factors of A are isomorphic. Then the module $M = A \oplus A$ has the following properties.*

- (i) *If L is any submodule of M, then $L = L_1 \oplus L_2$ and $M = M_1 \oplus M_2$ for some uniserial modules L_i, M_i such that $L_i \subseteq M_i$.*
- (ii) *If $K < L \subseteq M$ such that K is maximal in L, then $L = L_1 \oplus L_2$, $K = K_1 \oplus L_2$ for some uniserial modules L_i, $K_1 \underset{\max}{<} L_1$.*
- (iii) *Let $L = L_1 \oplus L_2$ be a submodule of M such that L_i are uniserial and $d(L_1) = d(L_2)$. Then $K = L_1 \oplus L_1'$ is fully invariant in M.*

Example A. Let F be a field admitting an endomorphism σ such that $[F : \sigma(F)] = 2$. Consider matrix units $\{e_{ij},\ 1 \leq i \leq j \leq n\}$ such that for $i > 1$, $ae_{ij} = e_{ij}a$, $ae_{11} = e_{11}a$, $e_{1k}a = \sigma(a)e_{1k}$ for any $k > 1$ and any $a \in F$. Let R be the set of all upper triangular matrices over F. We write its members as $\sum_{i\leq j} a_{ij}e_{ij}$. Two member of R are added componentwise, and multiplication is defined by using the above specified laws for the matrix units. We also look at R as $T_n(F)$ the ring of $n \times n$ upper triangular matrices over F. Using the fact that $T_n(F)$ is generalized uniserial, we get that R is left serial. We see that for any $1 < k < n$, $a \in F$, $ae_{1k} = e_{11}(ae_{1k})$. Hence the right ideal $e_{11}R$ is the set of all matrices in R, whose last $n-1$ rows are zero rows. Now $F = \sigma(F) + u\sigma(F)$, where $u \in F\backslash\sigma(F)$. $e_{11}J = A \oplus B$, where A, B are right ideals such that any member of A is of the form of $\sum_{k>1} \sigma(a_{1k})e_{1k}$, and any member of B is of the form $\sum_{k>1} u\sigma(a_{1k})e_{1k}$. By comparing with the right ideal $\sum_{j>1} e_{1j}F$ in $T_n(F)$, we see that A and B are isomorphic uniserial right ideals of R, such that they are quasi-injective and quasi-projective. They can be regarded as modules over $T_n(F)$. No two composition factors of A are isomorphic. For some submodules K, K' of $e_{11}J$, consider $M = e_{11}R/K$ and $N = e_{11}R/K'$. Let L/K, L'/K' be simple submodules of M, N respectively and $\mu : L/K \to L'/K'$ be an R-isomorphism. By (4.1), $L = L_1 \oplus L_2$, $K = K_1 \oplus L_2$, $L' = L_1' \oplus L_2'$, $K' = K_1' \oplus L_2'$ for some unisrial modules L_i, L_i', $K_1 \underset{\max}{<} L_1$and $K_1' \underset{\max}{<} L_1'$. Let

$\eta : L_1/K_1 \to L_1'/K_1'$ be the R-isomorphism induced by μ. Write $e_{11}R = M_1 \oplus M_2 = M_1' \oplus M_2'$where each M_i , M_i' is uniserial, $L_i \subseteq M_i$. and $L_i' \subseteq M_i'$. Then there exists unique R- isomorphism $\lambda : M_1 \to M_1'$ which induces η. Now $soc(L_1) = x_1e_{1n}F$, $soc(L_1') = x_1'e_{1n}F$, for some $x_1,\ x_1' \in F$ such that $\lambda(x_1e_{1n}) = x_1'e_{1n}$. Further $d(L_1) = d(L_2)$. Let $soc(L_2) = x_2e_{1n}F$, $soc(L_2') = x_2'e_{1n}F, x_2, x_2' \in F$ We can find $w \in F$ such that $wx_2 = x_2'$. Let λ_w be the R-automorphism of $e_{11}R$ given by left multiplication by w. If λ_w extend λ, then λ_w lifts η. Otherwise, let $\lambda_w(x_1e_{1n}) = x_1'e_{1n}a + x_2'e_{1n}b$ for some $a,\ b \in F$. If $a = 0$, then $\lambda_w(soc(e_{11}R)) = x_2'e_{1n}F$ which is a contradiction. Hence $a \neq 0$. Then ϕ the R-automorphism of $e_{11}R$ given by left multiplication by $w\sigma(a)^{-1}$ is such that $\phi(x_1e_{1n}) = x_1'e_{1n} + x_2'e_{1n}c$ for some $c \in F$. Then ϕ lifts σ.

We verify the condition in (2.1) to prove that R satisfies (*). Let $M,\ N$ be any two local R-modules, and S be a simple submodule of M. Let $\phi : S \to N$ be an R-monomorphism. We can take $M = e_{rr}R/K$, and $N = e_{ss}R/L$ for some $1 \leq r, s \leq n,\ K < e_{rr}R$, and $L < e_{ss}R$. Now the case for $r = s = 1$, has been discussed above. Notice that the last $n-1$ rows of R constitute the ring $R'$of $n-1 \times n-1$ upper triangular matrices over F, $e_{11}J$ being a direct sum of two copies of the first row of R', is injective as a right R'-module. Using this it can be verified that R satisfies the condition given in (2.1). Hence R satisfies (*) on the right. ∎

Example B. Let F be a field, $R = \begin{bmatrix} F & F+Fx \\ 0 & F+Fx \end{bmatrix}$, where $x^2 = 0$. As a left ideal, $Je_{22} = Fxe_{22} + Fe_{12} + Fxe_{12} = C_1 \oplus C_2$, where $C_1 = Fe_{12}$, $C_2 = Fxe_{22} + Fxe_{12} = Rxe_{22}$, $J^2xe_{22} = \begin{bmatrix} 0 & F+Fx \\ 0 & Fx \end{bmatrix}\begin{bmatrix} 0 & F+Fx \\ 0 & Fx \end{bmatrix} = \begin{bmatrix} 0 & Fx \\ 0 & 0 \end{bmatrix}$ $\cong Re_{11} \cong C_1$. Observe that $socle(Re_{22}) = Fe_{12} \oplus Fxe_{12}$. As C_2 is invariant under all endomorphisms of Re_{22}, Re_{22}/C_2 is quasi-projective. Also Re_{22}/Fxe_{22} is quasi-projective. Let $M = Re_{22}/C_1 = \overline{Fxe_{12} + Fe_{22} + Fxe_{22}}$. It is uniserial and its proper submodules are $\overline{C_2} > B = \overline{Fxe_{12}}$. Let σ be an endomorphism of B. Suppose $\sigma(\overline{xe_{12}}) = \overline{zxe_{12}}$, $z \in F$. Then the R-endomorphism of M given by multiplication by z extends σ. Similarly for $\overline{C_2}$, as any endomorphism of $\overline{C_2}$ is given by multiplication by an elemeent of F.This gives M is quasi-injective. As M contains a copy of Re_{11}, M is Re_{11}-injective. Let L be a left ideal properly contained in Re_{22}. If $L = Fxe_{22} + Fxe_{12}$, then $\sigma(xe_{22}) = \overline{\alpha xe_{22}}$ for some $\alpha \in F$ and σ is given by right multiplication by $\overline{\alpha e_{22}}$ in M. If $L = C_1 \oplus C_2$, then $\sigma(xe_{22}) = \overline{\alpha xe_{22}}$, $\sigma(e_{12}) = \overline{\beta xe_{12}}$ for some $\alpha,\ \beta \in F$, and σ is given by right multiplication by $\overline{(\alpha + \beta x)e_{22}}$. If L is any of Fxe_{12}, Fe_{12}, then $L \cong Re_{11}$, as M is Re_{11}-injective, σ is given by right multiplication by a member of M. if $L = Fxe_{12} \oplus Fe_{12}$, then $\sigma(e_{12}) = \overline{\alpha xe_{12}}$ for some $\alpha \in F$, and σ is given by right multiplication by $\overline{\alpha xe_{22}}$. Hence M is Re_{22}-injective. This proves that M is injective. Similarly, one can prove that any non-simple, uniserial, homomorphic image of Re_{22} is injective. After this one can easily verify that R satisfies (*) on the left. Then the ring R' anti-isomorphic to R satisfies (*) on the right. Observe that in $Je_{22} = C_1 \oplus C_2$, $C_1 \cong JC_2$, but $C_1 \ncong C_2/JC_2$.

We are yet not aware of an example of a local module over a ring R satisfying (*), for which $t \geq 3$ as in (2.6).

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Surjeet Singh
House No. 424, Sector No. 35 A,
Chandigarh-160036,
India
E-mail: ossinghpal@yahoo.co.in