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BLOW-UP OF POSITIVE-INITIAL-ENERGY SOLUTIONS FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH NONLINEAR DAMPING

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Abstract. The initial boundary value problem for an integro-differential equation with nonlinear damping in a bounded domain is considered. The local existence and blow-up of solutions with positive initial energy are discussed under some conditions. Estimates of the lifespan of solutions are also given.

1. INTRODUCTION

This paper is concerned with the initial boundary value problem for the following nonlinear integro-differential equation:

$$
\begin{aligned}
&u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + h(u_t) = f(u),\\
&u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega,\\
&u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0,
\end{aligned}
\tag{1.1}
$$

where Ω is a bounded domain in R^N, $N \geq 1$, with a smooth boundary $\partial\Omega$ so that the divergence theorem can be applied, ∇ denotes the gradient operator and Δ is the Laplacian operator. Here, g is a nonincreasing positive function, h is a nonlinear function, f is a nonlinear source term and $M(s)$ is a positive locally Lipschitz function with $M(s) \geq m_0 > 0$ for $s \geq 0$ like $M(s) = m_0 + bs^\gamma$, $m_0 > 0$, $b \geq 0$, $\gamma > 0$ and $s \geq 0$. The initial value functions $u_0(x)$, $u_1(x)$ are given and subscript t indicate the partial derivative with respect to t, and we denote $\|\cdot\|_p$ to be the norm of $L^p(\Omega)$.

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When $g \equiv 0$, for the case that $M \equiv 1$, it is a nonlinear wave equation which has been extensively studied and several results concerning existence and nonexistence have been established $([2, 5, 6, 7, 18])$. When M is not a constant function, the equation (1.1) without damping and the source terms is often called the Kirchhoff type equation; it was first introduced by Kirchhoff $([4])$ in order to describe the nonlinear vibrations of an elastic string. In this regard, the existence and nonexistence of solutions have been discussed by many authors and the references cited therein $([11, 12, 13, 19])$.

On the contrary, when g is not trivial on R and $M \equiv 1$, (1.1) becomes a semilinear viscoelastic wave equation. Messaoudi $([8, 9])$ studied (1.1) for $h(u_t) = |u_t|^{m-2} u_t$, $m > 2$ and $f(u) = |u|^{p-2} u$, $p > 2$. Under suitable conditions, he proved that any solution blow-up in finite time if $p > m$ and he also showed the global existence for arbitrary initial data if $m \geq p$. Later, Wu and Tsai $([20])$ extended Messaoudi's result to more general h and f. In that paper, we obtained the blowup result with small positive initial energy if $p > m$ and we also discussed the global existence and energy decay without the relation between m and p. In the event that M is not a constant function, the equation (1.1) is the model to describe the motion of deformable solids as hereditary effect is incorporated. The equation (1.1) was first studied by Torrejon and Yong $([17])$ who proved the existence of weakly asymptotic stable solution for large analytical datum. Later, Rivera $([10])$ showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions. Recently, Wu and Tsai $([21])$ studied (1.1) for $h(u_t) = -\Delta u_t$ and f is a power like function. The global existence, decay result and blowup properties had been proved.

In this paper we shall establish the result for blow-up properties of local solution for problem (1.1) with nonpositive as well as small positive initial energy by modifying the method in [18]. In this way, we can extend the result of $([14])$ in which he considered (1.1) with $g \equiv 0$ and the result of $([8, 20])$ to nonconstant $M(s)$. The content of this paper is organized as follows. In section 2, we present a lemma and some preliminaries, and state the local existence result. In section 3, we study the blow-up problem in cases of the initial energy being nonpositive and positive. Estimates of the blowup time are also given.

2. Preliminary and Local Existence Results

In this section, we shall discuss the local existence of solutions for problem (1.1). We first state a well-known lemma which will be used throughout this work.

Lemma 2.1. (Adam [1]). *If* $1 \leq p \leq \frac{2N}{[N-2m]^+}$ $(1 \leq p < \infty$ *if* $N = 2m)$, *then*

$$\|u\|_p \leq B_1 \left\| (-\Delta)^{\frac{m}{2}} u \right\|_2, \quad \textit{for } u \in D((-\Delta)^{\frac{m}{2}})$$

holds with some positive constant B_1, *where we put* $[a]^+ = \max\{0, a\}$ *and* $\frac{1}{[a]^+} = \infty$ *if* $[a]^+ = 0$.

Assume that

(A1) $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$g(0) > 0,\ g'(s) \le 0,\ m_0 - \int_0^\infty g(s)ds = l > 0,$$

here l is any arbitrary number larger than 0, less than m_0.

(A2) h is a C^1 function defined on R and there exist positive constants k_1 and k_2 such that

$$(h(u) - h(v))\,(u - v) \ge k_1\,|u - v|^m\,,$$

and

$$h(u)u \le k_2\,(|u|^\nu + |u|^m)\,,$$

for $u, v \in R$ and $2 < \nu \le m \le p^*$, here $p^* = \frac{2N}{N-2}(2 < \nu \le m < \infty$, if $N \le 2)$.

(A3) $f(0) = 0$ and there is a positive constant k_3 such that

$$|f(u) - f(v)| \le k_3\,|u - v|\left(|u|^{p-2} + |v|^{p-2}\right),$$

for $u, v \in R$ and $2 < p \le p_1^*$, here $p_1^* = \frac{2(N-3)}{N-4}(2 < p < \infty$, if $N \le 4)$.

Now, we are in a position to state the local existence result. For this purpose, we first take a related simpler problem into account. Then, we prove the existence of solutions to problem (1.1) by contraction mapping principle. Consider the following simpler problem:

$$\begin{aligned}
&u_{tt} - \mu(t)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + h(u_t) = f_1(x,t) \text{ on } \Omega \times (0,T),\\
&u\,(x,0) = u_0\,(x)\,,\ u_t\,(x,0) = u_1\,(x)\,,\ x \in \Omega,\\
&u\,(x,t) = 0,\ x \in \partial\Omega,\ t > 0.
\end{aligned} \tag{2.1}$$

Here, $T > 0$, f_1 is a fixed forcing term on $\Omega \times (0,T)$ and μ is a positive locally Lipschitz function on $[0,\infty)$ with $\mu(t) \ge m_0 > 0$ for $t \ge 0$.

By means of Galerkin method as in [21], we have the following lemma.

Lemma 2.2. *Suppose that* $(A1)$ *and* $(A2)$ *hold, and that* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ *and* $f_1 \in L^2([0,T), H_0^1(\Omega))$. *Then the problem* (2.1) *admits a unique* *solution* u *such that* $u \in H1$ *and* $u_t \in L^m(\Omega \times (0,T))$*, where*

$$H1 = C_w([0,T), H_0^1(\Omega) \cap H^2(\Omega)) \cap C([0,T), H_0^1(\Omega)) \\ \cap C_w^1([0,T), H_0^1(\Omega)) \cap C^1([0,T), L^2(\Omega)),$$

here the subscript "w" *means the weak continuity with respect to* t ([16]).

Theorem 2.3. *Assume that* $(A1)$, $(A2)$ *and* $(A3)$ *hold, and that* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, *then there exists a unique solution* u *of* (1.1) *satisfying* $u \in H1$ *and* $u_t \in L^m(\Omega \times (0,T))$, *and at least one of the following statements is valid*:

$$(i) T = \infty, \\ (ii) \|\nabla u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \to \infty \text{ as } t \to T^-. \tag{2.2}$$

Proof. For $T > 0$, $R_0 > 0$, we define a class of functions X_{T,R_0} which consists of functions v in $H1$ satisfying the initial conditions of (1.1) and $e(v(t)) \leq R_0^2$, $t \in [0,T)$, where

$$e(v(t)) = \|v_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla v\|_2^2 + \|\Delta v\|_2^2. \tag{2.3}$$

Then X_{T,R_0} is a complete metric space with the distance

$$d(y,z) = \sup_{0 \leq t \leq T} \left[\|(y-z)_t(t)\|_2^2 + \|\nabla(y-z)(t)\|_2^2 \right]^{\frac{1}{2}}, \tag{2.4}$$

where $y,\ z \in X_{T,R_0}$. Given $v \in X_{T,R_0}$, we consider the following problem

$$u_{tt} - M(\|\nabla v\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + h(u_t) = f(v), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x),\ x \in \Omega, \\ u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0. \tag{2.5}$$

By (A3) and $\|\nabla f\|_2 \leq k_4 \|v\|_{N(p-2)}^{p-2} \|\nabla v\|_{\frac{2N}{N-2}} \leq k_4 B_1^{p-1} \|\Delta v\|_2^{p-1}$, we see that $f \in L^2([0,T), H_0^1(\Omega))$, where $k_4 = k_3(p-1)vol(\Omega)^{\frac{1}{2}}$. Thus, by lemma 2.2, we derive that problem (2.5) admits a unique solution $u \in H1$ and $u_t \in L^m(\Omega \times (0,T))$. Then, we define the nonlinear mapping $Sv = u$, and we would like to show that there exist $T > 0$ and $R_0 > 0$ such that S is a contraction mapping from X_{T,R_0} into itself. For this, we multiply the first equation of (2.5) by $2u_t$ and integrate it over Ω to get

$$\begin{aligned}&\frac{d}{dt}\left[\|u_t\|_2^2+\left(M(\|\nabla v\|_2^2)-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2+(g\diamond\nabla u)(t)\right]\\ &+2\int_\Omega h(u_t)u_t dx\\ \le&\left(\frac{d}{dt}M(\|\nabla v\|_2^2)\right)\|\nabla u(t)\|_2^2+2\int_\Omega f(v)u_t dx.\end{aligned}\tag{2.6}$$

The equality in (2.6) is obtained, because

$$\begin{aligned}-2\int_0^t\int_\Omega g(t-\tau)\nabla u(\tau)\cdot\nabla u_t(t)dxd\tau =& \frac{d}{dt}\left[(g\diamond\nabla u)(t)-\int_0^t g(\tau)\|\nabla u(\tau)\|_2^2 d\tau\right]\\ &-(g'\diamond\nabla u)(t)+g(t)\|\nabla u(t)\|_2^2,\end{aligned}$$

where

$$(g\diamond\nabla u)(t)=\int_0^t g(t-\tau)\int_\Omega|\nabla u(\tau)-\nabla u(t)|^2 dxd\tau.$$

Next, multiplying the first equation of (2.5) by $-2\Delta u_t$ and integrating it over Ω, we have

$$\begin{aligned}&\frac{d}{dt}\left[\|\nabla u_t\|_2^2+\left(M(\|\nabla v\|_2^2)-\int_0^t g(s)ds\right)\|\Delta u(t)\|_2^2+(g\diamond\Delta u)(t)\right]\\ &+2\int_\Omega h'(u_t)|\nabla u_t|^2 dx\\ \le&\left(\frac{d}{dt}M(\|\nabla v\|_2^2)\right)\|\Delta u(t)\|_2^2-2\int_\Omega f(v)\Delta u_t dx.\end{aligned}\tag{2.7}$$

Combining (2.6) and (2.7) together, we obtain

$$\frac{d}{dt}e_1(t)+2\int_\Omega h(u_t)u_t dx+2\int_\Omega h'(u_t)|\nabla u_t|^2 dx\le I_1+I_2,\tag{2.8}$$

where

$$\begin{aligned}e_1(t) =& \|u_t\|_2^2+\left(M(\|\nabla v\|_2^2)-\int_0^t g(s)ds\right)\left(\|\nabla u(t)\|_2^2+\|\Delta u(t)\|_2^2\right)\\ &+(g\diamond\nabla u)(t)+\|\nabla u_t\|_2^2+(g\diamond\Delta u)(t),\\ I_1 =&\left(\frac{d}{dt}M(\|\nabla v\|_2^2)\right)\left(\|\nabla u(t)\|_2^2+\|\Delta u(t)\|_2^2\right),\quad I_2=2\int_\Omega f(v)\left(u_t-\Delta u_t\right)dx.\end{aligned}$$

To proceed further estimations, we note by (A1) that

$$|I_1| \leq \frac{2M_1}{l} \|\nabla v\|_2 \|\nabla v_t\|_2 e_1(t) \leq \frac{2M_1}{l} R_0^2 e_1(t), \tag{2.9}$$

and by (A3) and lemma 2.1 that

$$\begin{aligned} |I_2| &\leq 2k_3 \left(\|v\|_{2(p-1)}^{p-1} \|u_t\|_2 + (p-1) \|v\|_{N(p-2)}^{p-2} \|\nabla v\|_{\frac{2N}{N-2}} \|\nabla u_t\|_2 \right) \\ &\leq 2k_3 B_1^{p-1} \|\Delta v\|_2^{p-1} \left(\|u_t\|_2 + (p-1) \|\nabla u_t\|_2 \right) \\ &\leq c_1 R_0^{p-1} e_1(t)^{\frac{1}{2}}, \end{aligned} \tag{2.10}$$

where $M_1 = \sup\{|M'(s)|\,; 0 \leq s \leq R_0^2\}$ and $c_1 = 2k_3 p B_1^{p-1}$. Thus, integrating (2.8) over $(0,t)$ and using (2.9)-(2.10), we deduce that

$$\begin{aligned} & e_1(t) + 2\int_0^t \int_\Omega h(u_t) u_t dx dt + 2\int_0^t \int_\Omega h'(u_t) |\nabla u_t|^2 dx dt \\ \leq\ & e_1(0) + \int_0^t [\frac{2M_1}{l} R_0^2 e_1(t) + c_1 R_0^{p-1} e_1(t)^{\frac{1}{2}}] dt. \end{aligned} \tag{2.11}$$

Hence, by Gronwall's lemma and noting that $e_1(t) \geq c_*^{-1} e(u(t))$, here $c_*^{-1} = \min(1,l)$, we arrive at

$$e(u(t)) \leq \chi(u_0, u_1, R_0, T)^2 \mathrm{e}^{\frac{2M_1 R_0^2 T}{l}}, \text{ for any } t \in (0,T], \tag{2.12}$$

where $\chi(u_0, u_1, R_0, T) = \left(\sqrt{e_1(0)} + \frac{c_1}{2} R_0^{p-1} T \right) c_*^{\frac{1}{2}}$. Therefore, we see that for parameters T and R_0 satisfy

$$\chi(u_0, u_1, R_0, T)^2 \mathrm{e}^{\frac{2M_1 R_0^2 T}{l}} \leq R_0^2, \tag{2.13}$$

then S maps X_{T,R_0} into itself. On the other hand, by lemma 2.2, $u \in H1$. Moreover, it follows from (2.11) and (2.12) that $u_t \in L^m(\Omega \times (0,T))$.

Next, we will verify that S is a contraction mapping. Let $v_i \in X_{T,R_0}$ and $u^{(i)} \in X_{T,R_0}$, $i = 1,2$ be the corresponding solution to problem (2.5). Setting $w(t) = (u^{(1)} - u^{(2)})(t)$, then w satisfy the following system:

$$\begin{aligned} & w_{tt} - M\left(\|\nabla v_1\|_2^2\right) \Delta w + \textstyle\int_0^t g(t-\tau) \Delta w(\tau) d\tau + h(u_t^{(1)}) - h(u_t^{(2)}) \\ & = f(v_1) - f(v_2) + \left[M\left(\|\nabla v_1\|_2^2\right) - M\left(\|\nabla v_2\|_2^2\right) \right] \Delta u^{(2)}, \\ & w(0) = 0,\ w_t(0) = 0, \\ & w(x,t) = 0,\ x \in \partial\Omega, \text{ and } t \geq 0. \end{aligned} \tag{2.14}$$

We multiply the first equation of (2.14) by $2w_t$ and integrate it over Ω to get

$$\begin{aligned}(2.15)\qquad &\frac{d}{dt}e_2^*(w(t))+2\int_\Omega\left(h(u_t^{(1)})-h(u_t^{(2)})\right)w_t dx\\ &-(g'\diamond\nabla w)(t)+g(t)\left\|\nabla w(t)\right\|_2^2\\ =\ &I_3+I_4+I_5,\end{aligned}$$

where

$$\begin{aligned}(2.16)\qquad &e_2^*(w(t))\\ &=\|w_t\|_2^2+\left(M(\|\nabla v_1\|_2^2)-\int_0^t g(s)ds\right)\|\nabla w(t)\|_2^2+(g\diamond\nabla w)(t),\\ &I_3=2\left[M\left(\|\nabla v_1\|_2^2\right)-M\left(\|\nabla v_2\|_2^2\right)\right]\int_\Omega \Delta u^{(2)}w_t dx,\\ &I_4=2\int_\Omega\left(f(v_1)-f(v_2)\right)w_t dx \text{ and } I_5=\left(\frac{d}{dt}M(\|\nabla v_1\|_2^2)\right)\|\nabla w(t)\|_2^2.\end{aligned}$$

Applying the similar arguments as in estimating $I_i, i=1,2$, we observe that

$$\begin{aligned}|I_3|\ &\le\ 2L\left(\|\nabla v_1\|_2+\|\nabla v_2\|_2\right)\|\nabla v_1-\nabla v_2\|_2\left\|\Delta u^{(2)}\right\|_2\|w_t\|_2\\ &\le\ 4LR_0^2 e_2(v_1-v_2)^{\frac12}e_2^*(w(t))^{\frac12},\end{aligned}$$

$$|I_4|\le 4k_3B_1^{2(p-1)}R_0^{p-2}e_2(v_1-v_2)^{\frac12}e_2^*(w(t))^{\frac12},$$

and

$$|I_5|\le\frac{2M_1R_0^2}{l}e_2^*(w(t)),$$

where $e_2(v)=\|v_t\|_2^2+\|\nabla v\|_2^2$, and $L=L(R_0)$ is the Lipschitz constant of $M(r)$ in $[0,R_0]$. Exploiting these inequalities in (2.15) and integrating it over $(0,t)$, we obtain

$$e_2^*(w(t))\le e_2^*(w(0))+\int_0^t\left[\frac{2M_1R_0^2}{l}e_2^*(w(s))+c_2e_2(v_1-v_2)^{\frac12}e_2^*(w(s))^{\frac12}\right]ds,$$

where $c_2\ =\ 4\left(LR_0^2+k_3B_1^{2(p-1)}R_0^{p-2}\right)$. Thus, applying Gronwall's lemma and noting that $e_2^*(w(0))=0$, we have

$$e_2^*(w(t))\le\frac{c_2^2T^2}{4}\mathrm{e}^{\frac{2M_1R_0^2T}{l}}\sup_{0\le t\le T}e_2(v_1-v_2).$$

On the other hand, by (2.16), we note that $e_2^*(w(t)) \geq c_*^{-1} e_2(w)$. Hence, by (2.4), we deduce that

$$d(u^{(1)}, u^{(2)}) \leq C(T, R_0)^{\frac{1}{2}} d(v_1, v_2), \tag{2.17}$$

where $C(T, R_0)^2 = \frac{c_* c_2^2 T^2}{4} \mathrm{e}^{\frac{2M_1 R_0^2 T}{l}}$. Therefore, under inequality (2.13), S is a contraction mapping if $C(T, R_0) < 1$. We choose R_0 sufficiently large and T sufficiently small so that (2.13) and (2.17) are satisfied at the same time. By applying Banach fixed point theorem, we obtain the local existence result.

The second statement of the theorem is proved by a standard continuation argument $([15])$. Indeed, let $[0, T)$ be a maximal existence interval on which the solution of (1.1) exists. Suppose that $T < \infty$ and $\lim_{t \to T^-} \left(\|\nabla u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \right) < \infty$.Then, there are a sequence $\{t_n\}$ and a constant $K > 0$ such that $t_n \to T^-$ as $n \to \infty$ and $\|\nabla u_t(t_n)\|_2^2 + \|\Delta u(t_n)\|_2^2 \leq K$, $n = 1, 2, \ldots$. Since for all $n \in N$, there exists a unique solution of (1.1) with initial data $(u(t_n), u_t(t_n))$ on $[t_n, t_n + \tau]$, $\tau > 0$ depending on K and independent of $n \in N$. Thus, we can get $T < t_n + \tau$ for $n \in N$ large enough. It contradicts to the maximality of T. The proof of theorem 2.3 is now completed.

3. Blow-up Property

In this section, we shall discuss the blow up phenomena for a kind of problem (1.1):

$$\begin{gathered} u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + a\,|u_t|^{\nu-2}\,u_t \\ +a\,|u_t|^{m-2}\,u_t = |u|^{p-2}\,u, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \\ u(x,t) = 0, x \in \partial\Omega, t \geq 0, \end{gathered} \tag{3.1}$$

where $M(s) = 1 + bs^\gamma$, $b \geq 0$, $\gamma > 0$ and $s \geq 0$, $a > 0$, $2 < \nu \leq m \leq p^*$ and $2 < p \leq p_1^*$. In order to state our results, we make an extra assumption on g:

$$\int_0^\infty g(s)ds < \min\left(\frac{2(p-2)}{2p-3},\ \frac{p(E_1 - E(0))}{2\lambda_1^2} \right), \tag{3.2}$$

where E_1 and λ_1 are some positive constants given later. We first define the energy function associated with a solution u of (3.1) by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t) \text{ for } t \geq 0, \tag{3.3}$$

where

$$J(t) = \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\diamond\nabla u)(t) + \frac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \frac{1}{p}\|u\|_p^p.$$

We observe, from (A1) and lemma 2.1, that

$$\begin{aligned} &E(t) \\ &\geq \frac{1}{2}\left(l\,\|\nabla u(t)\|_2^2+\frac{b}{(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)}+(g\diamond\nabla u)(t)\right)-\frac{B_1^p l^{\frac{p}{2}}}{p}\|\nabla u\|_2^p \\ &\geq G\left[\left(l\,\|\nabla u(t)\|_2^2+\frac{b}{(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)}+(g\diamond\nabla u)(t)\right)^{\frac{1}{2}}\right], \end{aligned} \tag{3.4}$$

for $t \geq 0$, where

$$G(\lambda) = \frac{1}{2}\lambda^2 - \frac{B_1^p}{p}\lambda^p,\ B_1 = \frac{B}{\sqrt{l}},\ l = 1-\int_0^\infty g(s)ds > 0.$$

It is easy to verify that $G(\lambda)$ has a maximum at $\lambda_1 = B_1^{-\frac{p}{p-2}}$ and the maximum value is $E_1 = \frac{p-2}{2p}\lambda_1^2$. Before we prove our main result, we need the following lemmas.

Lemma 3.1. *Suppose that $(A1)$ holds, and that $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ and let u be a solution of (3.1). Then $E(t)$ is a nonincreasing function on $[0,T]$ and*

$$E'(t) = -a\int_\Omega (|u_t|^\nu + |u_t|^m)dx + \frac{1}{2}(g'\diamond\nabla u)(t) - \frac{1}{2}g(t)\,\|\nabla u\|_2^2. \tag{3.5}$$

Proof. Multiplying (3.1) by u_t and integrating it over Ω, and using integrating by parts, we obtain (3.5).

Lemma 3.2. [22]. *Assume that $(A1)$ holds, $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$* and $u_1 \in H_0^1(\Omega)$. *Let u be a solution of (3.1) with initial data satisfying $E(0) < E_1$ and $\left(l\,\|\nabla u_0\|_2^2 + \frac{b}{\gamma+1}\|\nabla u_0\|_2^{2(\gamma+1)}\right)^{\frac{1}{2}} > \lambda_1$. Then there exists $\lambda_2 > \lambda_1$ such that*

$$l\,\|\nabla u(t)\|_2^2 + \frac{b}{\gamma+1}\|\nabla u(t)\|_2^{2(\gamma+1)} + (g\diamond\nabla u)(t) \geq \lambda_2^2,\ \text{for } t>0. \tag{3.6}$$

Theorem 3.3. (Nonexistence of global solutions). *Let $p > m$ and $\gamma < \max(\frac{1-l}{4l}, \frac{p-2}{2})$. Assume that $(A1)$ and (3.2) hold, and that $u_0 \in H_0^1(\Omega)\cap H^2(\Omega)$, $u_1 \in$*

$H_0^1(\Omega)$. *Then any solution of* (3.1) *with initial data satisfying* $0 \le E(0) < E_1$ *and* $\left(l\|\nabla u_0\|_2^2 + \frac{b}{\gamma+1}\|\nabla u_0\|_2^{2(\gamma+1)}\right)^{\frac{1}{2}} > \lambda_1$ *blows up at finite time in the sense of* (2.2). *We remark that the lifespan* T *is estimated by* $0 < T \le \frac{L(0)^{1-\theta_1}}{c_{12}(\theta_1-1)}$, *where* $L(t)$ *and* c_{12} *are given in* (3.18) *and* (3.24) *respectively, and* θ_1 *is some positive constant given in the following proof.*

Proof. We set

$$H(t) = E_2 - E(t),\ t \ge 0, \tag{3.7}$$

where $E_2 = \frac{E_1+E(0)}{2}$. By (3.4), we see that $H'(t) \ge 0$. Thus we obtain

$$H(t) \ge H(0) = E_2 - E(0) > 0,\ t \ge 0. \tag{3.8}$$

Let

$$A(t) = \int_\Omega u u_t dx. \tag{3.9}$$

By differentiating (3.9) and using (3.1), we have

$$\begin{aligned} A'(t) \,=\, & \|u_t\|_2^2 - \|\nabla u\|_2^2 - b\|\nabla u\|_2^{2(\gamma+1)} \\ & + \int_\Omega \int_0^t g(t-s)\nabla u(s)\cdot\nabla u(t)dsdx \\ & - a\int_\Omega (|u_t|^{\nu-2} + |u_t|^{m-2})u_t u dx + \|u\|_p^p. \end{aligned} \tag{3.10}$$

Exploiting Hölder inequality and Young's inequality, we observe that

$$\begin{aligned} & \int_\Omega \int_0^t g(t-s)\nabla u(s)\cdot\nabla u(t)dsdx \\ = & \int_\Omega \int_0^t g(t-s)\nabla u(t)\cdot(\nabla u(s) - \nabla u(t))\,dsdx \\ & + \int_0^t g(t-s)ds\,\|\nabla u(t)\|_2^2 \\ \ge & -(g\diamond\nabla u)(t) + \frac{3}{4}\int_0^t g(s)ds\,\|\nabla u(t)\|_2^2, \end{aligned} \tag{3.11}$$

Then, by (3.11) and using (3.3) to substitute for $\|u\|_p^p$, (3.10) becomes

$$\begin{aligned} A'(t) \,\ge\, & a_1\|u_t\|_2^2 + a_2(g\diamond\nabla u)(t) + a_3\frac{b}{\gamma+1}\|\nabla u(t)\|_2^{2(\gamma+1)} + a_4 l\|\nabla u(t)\|_2^2 \\ & - a\int_\Omega (|u_t|^{\mu-2} + |u_t|^{m-2})u_t u dx + pH(t) - pE_2. \end{aligned}$$

where $a_1 = \frac{p+2}{2}$, $a_2 = \frac{p-2}{2}$, $a_3 = \frac{p-2(\gamma+1)}{2}$ and $a_4 = \frac{1}{l}\left(\frac{p-2}{2} - \frac{2p-3}{4}\int_0^\infty g(s)ds\right)$. By (3.2), we observe that $a_4 > 0$ and by the restriction on γ, we deduce that

$$A'(t) \geq a_1 \|u_t\|_2^2 + a_4\left[(g\diamond\nabla u)(t) + \frac{b}{\gamma+1}\|\nabla u(t)\|_2^{2(\gamma+1)} + l\|\nabla u(t)\|_2^2\right] - a\int_\Omega (|u_t|^{\mu-2} + |u_t|^{m-2})u_t u dx + pH(t) - pE_2. \tag{3.12}$$

Moreover,

$$\begin{aligned} & a_4\left[l\|\nabla u(t)\|_2^2 + \frac{b}{(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + (g\diamond\nabla u)(t)\right] - pE_2 \\ &= a_4\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}\left[l\|\nabla u(t)\|_2^2 + \frac{b}{(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + (g\diamond\nabla u)(t)\right] \\ &\quad + a_4\lambda_1^2\frac{\left[l\|\nabla u(t)\|_2^2 + \frac{b}{(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + (g\diamond\nabla u)(t)\right]}{\lambda_2^2} - pE_2 \\ &\geq c_1\left[l\|\nabla u(t)\|_2^2 + \frac{b}{(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} + (g\diamond\nabla u)(t)\right] + c_2, \end{aligned}$$

where the last inequality is obtained by lemma 3.2, λ_2 is given in lemma 3.2, $c_1 = a_4\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}$ and $c_2 = a_4\lambda_1^2 - pE_2$. By lemma 3.2, we have $c_1 > 0$ and by (3.2), we see that

$$\begin{aligned} c_2 &= \frac{1}{l}\left(\frac{p-2}{2} - \frac{2p-3}{4}\int_0^\infty g(s)ds\right)\lambda_1^2 - pE_2 \\ &> \left(\frac{p-2}{2} - \frac{2p-3}{4}\int_0^\infty g(s)ds\right)\lambda_1^2 - pE_2 \\ &= \frac{p(E_1 - E(0))}{2} - \frac{2p-3}{4}\int_0^\infty g(s)ds\lambda_1^2 > 0. \end{aligned}$$

Thus, (3.12) yields

$$\begin{aligned} A'(t) \geq\; & a_1\|u_t\|_2^2 + c_1(g\diamond\nabla u)(t) \\ & + \frac{bc_1}{\gamma+1}\|\nabla u(t)\|_2^{2(\gamma+1)} + c_1 l\|\nabla u(t)\|_2^2 \\ & - a\int_\Omega (|u_t|^{\mu-2} + |u_t|^{m-2})u_t u dx + pH(t). \end{aligned} \tag{3.13}$$

On the other hand, by Hölder inequality, we have

(3.14) $$\left|\int_\Omega |u_t|^{m-2}\, u_t u dx\right| \le \|u\|_m \|u_t\|_m^{m-1} \le c_3 \|u\|_p^{1-\frac{p}{m}} \|u\|_p^{\frac{p}{m}} \|u_t\|_m^{m-1},$$

where $c_3 = (\mathrm{vol}(\Omega))^{\frac{p-m}{mp}}$. Noting that, from (3.3) and lemma 3.2, we get

$$\begin{aligned} H(t) &\le E_1 - \frac{1}{2}\left(l\|\nabla u(t)\|_2^2 + \frac{b}{\gamma+1}\|\nabla u\|_2^{2(\gamma+1)} + (g\diamond\nabla u)(t)\right) + \frac{1}{p}\|u\|_p^p \\ &\le E_1 - \frac{1}{2}\lambda_1^2 + \frac{1}{p}\|u\|_p^p < \frac{1}{p}\|u\|_p^p. \end{aligned}$$

Hence,

(3.15) $$0 < H(0) \le H(t) \le \frac{1}{p}\|u\|_p^p \text{ for } t \ge 0.$$

Then, by Young's inequality and (3.5), (3.14) becomes

(3.16) $$\begin{aligned} &\left|\int_\Omega |u_t|^{m-2}\, u_t u dx\right| \\ &\le c_5\left(\varepsilon_1^m H(0)^{-\alpha_1}\|u\|_p^p + \varepsilon_1^{-m'} H(0)^{\alpha-\alpha_1} H(t)^{-\alpha} H'(t)\right), \end{aligned}$$

where $\alpha_1 = \frac{1}{m} - \frac{1}{p} > 0$, $0 < \alpha < \alpha_1$, $\varepsilon_1 > 0$, $m' = \frac{m}{m-1}$, $c_4 = c_3(p)^{\frac{1}{p}-\frac{1}{m}}$ and $c_5 = c_4 \max(1, \frac{1}{a})$. Similarly, we also have the following inequality

(3.17) $$\begin{aligned} &\left|\int_\Omega |u_t|^{\nu-2}\, u_t u dx\right| \\ &\le c_6\left(\varepsilon_2^\nu H(0)^{-\alpha_2}\|u\|_p^p + \varepsilon_2^{-\nu'} H(0)^{\alpha-\alpha_2} H(t)^{-\alpha} H'(t)\right), \end{aligned}$$

where $0 < \alpha < \alpha_2$, $\alpha_2 = \frac{1}{\nu} - \frac{1}{p} > 0$, $\varepsilon_2 > 0$, $\nu' = \frac{\nu}{\nu-1}$ and $c_6 = c_3(p)^{\frac{1}{p}-\frac{1}{\nu}} \max(1, \frac{1}{a})$. In order to satisfy both (3.16) and (3.17), we choose $0 < \alpha < \min\{\alpha_1, \alpha_2\}$.

Now, we define

(3.18) $$L(t) = H(t)^{1-\alpha} + \delta_1 A(t),\ t \ge 0,$$

where δ_1 is a positive constant to be specified later. By differentiating (3.18), and then using (3.16), (3.17) and (3.13), we derive that

(3.19) $$\begin{aligned} &L'(t) \\ &\ge \left(1-\alpha-\delta_1 a c_5 \varepsilon_1^{-m'} H(0)^{\alpha-\alpha_1} - \delta_1 a c_6 \varepsilon_2^{-\nu'} H(0)^{\alpha-\alpha_2}\right) H(t)^{-\alpha} H'(t) \\ &\quad + \delta_1\left[a_1\|u_t\|_2^2 + c_1(g\diamond\nabla u)(t) + \frac{bc_1}{(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} + c_1 l\|\nabla u(t)\|_2^2\right] \\ &\quad + \delta_1 p H(t) - \delta_1 a\left(c_5\varepsilon_1^m H(0)^{-\alpha_1} + c_6 \varepsilon_2^\nu H(0)^{-\alpha_2}\right)\|u\|_p^p. \end{aligned}$$

Letting $a_5 = \min\{a_1, c_1 l, \frac{(\gamma+1)c_1}{b}, \frac{p}{2}\}$ and decomposing $\delta_1 pH(t)$ in (3.19) by $\delta_1 pH(t) = 2a_5\delta_1 H(t) + (p-2a_5)\delta_1 H(t)$. Thus, by (3.7) and (3.3), we obtain

$$\begin{aligned} L'(t) \geq & \left(1-\alpha-\delta_1 ac_5\varepsilon_1^{-m'} H(0)^{\alpha-\alpha_1}\right. \\ & \left.-\delta_1 ac_6\varepsilon_2^{-\nu'} H(0)^{\alpha-\alpha_2}\right) H(t)^{-\alpha}H'(t) \\ & +\delta_1\left[\frac{2a_5}{p} - a\left(c_5\varepsilon_1^m H(0)^{-\alpha_1} + c_6\varepsilon_2^{\nu} H(0)^{-\alpha_2}\right)\right]\|u\|_p^p \\ & +\delta_1(a_1-a_5)\|u_t\|_2^2+\delta_1(c_1 l-a_5)\|\nabla u(t)\|_2^2+(p-2a_5)\delta_1 H(t) \\ & +\delta_1(c_1-a_5)\left(\frac{b}{(\gamma+1)}\|\nabla u\|_2^{2(\gamma+1)} + (g\diamond\nabla u)(t)\right). \end{aligned} \tag{3.20}$$

Now, we choose $\varepsilon_1, \varepsilon_2 > 0$ small enough such that $\frac{2a_5}{p} - a(c_5\varepsilon_1^m H(0)^{-\alpha_1} + c_6\varepsilon_2^{\nu} H(0)^{-\alpha_2}) \geq \frac{a_5}{2p}$, and $0 < \delta_1 < \frac{(1-\alpha)}{2}(c_5 a\varepsilon_1^{-m'} H(0)^{\alpha-\alpha_1} + c_6 a\varepsilon_2^{-\nu'} H(0)^{\alpha-\alpha_2})^{-1}$. Then (3.20) becomes

$$L'(t) \geq c_7\delta_1\left(\|u\|_p^p + \|u_t\|_2^2 + H(t) + (g\diamond\nabla u)(t) + \|\nabla u\|_2^{2(\gamma+1)}\right), \tag{3.21}$$

here $c_7 = \min\left\{\frac{a_5}{2p},\ a_1-a_5,\ c_1 l - a_5,\ \frac{b(c_1-a_5)}{\gamma+1},\ p-2a_5\right\}$. Thus $L(t)$ is a non-decreasing function on $t \geq 0$. Letting δ_1 be small enough in (3.18), then we have $L(0) > 0$. Hence $L(t) > 0$, for $t \geq 0$. Now set $\theta_1 = \frac{1}{1-\alpha}$. Since $\alpha < \min\{\alpha_1, \alpha_2\} < 1$, it is evident that $1 < \theta_1 < \frac{1}{1-\min\{\alpha_1,\alpha_2\}}$. By Young's inequality and Hölder inequality, it follows that

$$L(t)^{\theta_1} \leq 2^{\theta_1-1}\left[H(t) + \left(\delta_1\int_\Omega u_t u dx\right)^{\theta_1}\right]. \tag{3.22}$$

On the other hand, for $p > 2$ and using Hölder inequality and Young's inequality, we have

$$\left(\int_\Omega u_t u dx\right)^{\theta_1} \leq c_8\|u_t\|_2^{\theta_1}\|u\|_p^{\theta_1} \leq c_9\left(\|u\|_p^{\theta_1\beta_1} + \|u_t\|_2^{\theta_1\beta_2}\right),$$

where $c_8 = (\mathrm{vol}(\Omega))^{\frac{\theta_1(p-2)}{2p}}$, $\frac{1}{\beta_1}+\frac{1}{\beta_2} = 1$, and $c_9 = c_9(c_8,\beta_1,\beta_2) > 0$. Now choose $\alpha \in \left(0,\ \min(\alpha_1,\ \alpha_2,\ \frac{1}{2}-\frac{1}{p})\right)$ and take $\theta_1\beta_2 = 2$ to get $\theta_1\beta_1 = \frac{2}{1-2\alpha} < p$. Noting that from (3.15), we have

$$\|u\|_p^{\theta_1\beta_1} = \left[\left(\frac{1}{pH(0)}\right)^{\frac{1}{p}}\|u\|_p\right]^{\theta_1\beta_1}\left(\frac{1}{pH(0)}\right)^{-\frac{\theta_1\beta_1}{p}} \leq c_{10}\|u\|_p^p,$$

where $c_{10}=\left(\frac{1}{pH(0)}\right)^{1-\frac{\theta_1\beta_1}{p}}$. Consequently, (3.22) becomes

(3.23) $$L\left(t\right)^{\theta_1}\leq c_{11}\left[H\left(t\right)+\left\|u\right\|_p^p+\left\|u_t\right\|_2^2\right],$$

here c_{11} is some positive constant. Combining (3.22) and (3.23) together, we get

(3.24) $$L'\left(t\right)\geq c_{12}L\left(t\right)^{\theta_1},\ t\geq 0,$$

here $c_{12}=\frac{c_7\delta_1}{c_{12}}$. An integration of (3.24) over $(0,t)$ then yields

$$L\left(t\right)\geq\left(L\left(0\right)^{1-\theta_1}-c_{12}\left(\theta_1-1\right)t\right)^{-\frac{1}{\theta_1-1}}.$$

Since $L\left(0\right)>0$, (3.24) shows that L becomes infinite in a finite time $T\leq T^*=\frac{L(0)^{1-\theta_1}}{c_{12}(\theta_1-1)}$. From (3.7) and (3.3), we have

$$H(t)\leq E_2+\frac{1}{p}\left\|u\right\|_p^p.$$

Thus, by (3.23) and lemma 2.1, we deduce that

$$L\left(t\right)^{\theta_1}\leq c_{14}\left[c_{13}+\left\|\Delta u\right\|_2^2+\left\|\nabla u_t\right\|_2^2\right]^{\frac{p}{2}},$$

here c_{13}, c_{14} are some positive constants. Therefore, we complete the proof.

Remark. If $E(0)<0$, we replace the conditions of theorem 3.3 to be $p>\max\left\{2(\gamma+1),\ m\right\}$, and $\int_0^\infty g(s)ds<\frac{2(p-2)}{2p-3}$. Then we set $H(t)=-E(t)$, instead of (3.6). Applying the same arguments as in theorem 3.3, we have our result.

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