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STABILITY OF A MIXED ADDITIVE AND QUADRATIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN BANACH MODULES

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Abstract. In this paper we establish the general solution of mixed additive and quadratic functional equation

$$f(x + 2y) + f(x - 2y) + 8f(y) = 2f(x) + 4f(2y)$$

and investigate the generalized Hyers-Ulam-Rassias stability of this equation in non-Archimedean Banach modules over a unital Banach algebra.

1. Introduction and Preliminaries

In 1940, S.M. Ulam [25] asked the first question on the stability problem. In 1941, D. H. Hyers [10] solved the problem of Ulam. This result was generalized by Aoki [4] for additive mappings and by Rassias [23] for linear mappings by considering an *unbounded Cauchy difference*. The paper of Rassias [23] has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. Since then, several stability problems for various functional equations have been investigated by numerous mathematicians; cf. e.g. [6, 11] and the bibliography quoted there. The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to a symmetric bi-additive function [15]. It is natural that such equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f

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between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ for all x (see [15]). The biadditive function B is given by

$$B(x, y) = \frac{1}{4}\Big(f(x+y) - f(x-y)\Big). \tag{1.2}$$

A Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space (see [24]). Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In [7], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). Grabiec in [9] has generalized the above mentioned results. Jun and Lee [13] proved the Hyers-Ulam-Rassias stability of the pexiderized quadratic equation (1.1). Moslehian [18] and Mirzavazir [18] have investigated the orthogonal stability of the pexiderized quadratic equation (1.1).

By a non-Archimedean field we mean a field K equipped with a function (valuation) $|\ .\ |$ from K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$, and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in N$.

Let X be a vector space over a scalar field K with a non-Archimedean nontrivial valuation $|\ .\ |$. A function $\|\ .\ \|: X \longrightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\| x \| = 0$ if and only if $x = 0$;

(ii) $\| rx \| = | r | \| x \|$ $(r \in K, x \in X)$;

(iii) the strong triangle inequality; namely,

$$\| x + y \| \leq \max\{\| x \|, \| y \|\} \quad (x, y \in X)$$

Then $(X, \| \quad \|)$ is called a non-Archimedean space. For nontrivial example of non-Archimedean space we refer to [17]. Due to the fact that

$$\| x_n - x_m \| \leq \max\{\| x_{j+1} - x_j \|: m \leq j \leq n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

In [17] Moslehian and Rassias have solved the stability problem for Cauchy and quadratic functional equations in non-Archimedean normed spaces. Stability of two types of cubic functional equations in non-Archimedean spaces have been investigated in [19]. In this context we refer to Arriola and Beyer [3] and Kaiser [14].

G. Z. Eskandani [8] has investigated the Hyers-Ulam-Rassias stability of the following functional equation

$$\sum_{i=1}^{m} f\left(mx_i + \sum_{j=1, j\neq i}^{m} x_j\right) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right)$$

$(m \in \mathbb{N}, m \geq 2)$ in quasi-Banach spaces.(See also [21, 22])

In this paper, using extensively the ideas and termonolgy of [17], we deal with the following functional equation deriving from additive and quadratic functions:

$$f(x+2y) + f(x-2y) + 8f(y) = 2f(x) + 4f(2y) \tag{1.3}$$

It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of the functional equation (1.3) . The main purpose of this paper is to establish the general solution of Eq. (1.3) and investigate the Hyers-Ulam-Rassias stability for Eq. (1.3) in non-Archimedean Banach modules over a unital Banach algebra.

2. Solutions of Eq. (1.3)

Throughout this section, X and Y will be real vector spaces. Before proceeding the proof of Theorem (2.3) which is the main result in this section, we shall need the following two lemmas.

Lemma 2.1. *If an odd function $f : X \to Y$ satisfies (1.3) for all $x, y \in X$, then the mapping $f : X \to Y$ is additive .*

Proof. Let an odd function $f : X \longrightarrow Y$ satisfy the functional equation (1.3) for all $x, y \in X$. Setting $x = 0$ in (1.3), we get

$$f(2y) = 2f(y) \tag{2.1}$$

for all $y \in X$. Therefore we have

$$f(x+2y) + f(x-2y) = 2f(x) \tag{2.2}$$

for all $x, y \in X$. Replace x by $2x$ in (2.2), we get

$$f(x+y) + f(x-y) = 2f(x) \tag{2.3}$$

for all $x, y \in X$. Replace x and y by y and x in (2.3), respectively, we get

$$f(x+y) - f(x-y) = 2f(y) \tag{2.4}$$

for all $x, y \in X$. Adding (2.3) to (2.4), we get

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. Therefore the function $f : X \to Y$ is additive. ∎

Lemma 2.2. *If an even function* $f : X \to Y$ *satisfies* (1.3) *for all* $x, y \in X$*, then the mapping* $f : X \to Y$ *is quadratic.*

Proof. Let an even function $f : X \longrightarrow Y$ satisfy the functional equation (1.3) for all $x, y \in X$. Putting $x = y = 0$ in (1.3), we get $f(0) = 0$. Setting $x = 0$ in (1.3), we get

$$f(2y) = 4f(y) \tag{2.5}$$

for all $y \in X$. Therefore we have

$$f(x + 2y) + f(x - 2y) = 2f(x) + 8f(y) \tag{2.6}$$

for all $x, y \in X$. Replace x by $2x$ in (2.6), we get

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.7}$$

for all $x, y \in X$. Therefore the function $f : X \to X$ is quadratic. ∎

Theorem 2.3. *A function* $f : X \to Y$ *satisfies* (1.3) *for all* $x, y \in X$ *if and only if there exists a symmetric bi-additive function* $B : X \times X \to Y$ *and an additive function* $A : X \to Y$ *such that* $f(x) = B(x, x) + A(x)$ *for all* $x \in X$.

Proof. If there exists a symmetric bi-additive function $B : X \times X \to Y$ and an additive function $A : X \to Y$ such that $f(x) = B(x, x) + A(x)$ for all $x \in X$, Then by a simple computation one can show that the functions B and A satisfy the functional equation (1.3). So the function f satisfies the equation (1.3).

Conversely, we decompose f into the even part and odd part by putting

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. It is easy to show that the functions f_e and f_o satisfy (1.3). Hence by Lemma 2.1 and Lemma 2.2 we achieve that the functions f_e and f_o are quadratic and additive, respectively. Therefore there exists a symmetric bi-additive function $B : X \times X \to Y$ such that $f_e(x) = B(x, x)$ for all $x \in X$ (see [1]). So

$$f(x) = B(x, x) + A(x)$$

for all $x \in X$, where $A(x) = f_o(x)$ for all $x \in X$. ∎

3. Hyers-Ulam-Rassias Stability of Eq. (1.3)

Throughout this section, let B be a unital Banach algebra with norm $|\ .\ |$ and let ${}_BX$ be left Banach B-module with norm $\|\ .\ \|$ and ${}_BY$ be non-Archimedean left Banach B-module with norm $\|\ .\ \|$. In this section, we prove the stability of Eq. (1.3) in the spirit of Hyers, Ulam and Rassias. For convenience, we use the following abbreviations for a given function $f : {}_BX \to {}_BY$:

$$\begin{aligned} D_af(x,y) :&= f(ax+2ay)+f(ax-2ay)+8f(ay)-2af(x)-4af(2y)\\ \Delta_af(x,y) :&= f(ax+2ay)+f(ax-2ay)+8f(ay)-2a^2f(x)-4a^2f(2y)\\ M_af(x,y) :&= f(ax+2ay)+f(ax-2ay)+8f(ay)\\ &\quad -(a^2+a)f(x)-(a^2-a)f(-x)-2(a^2+a)f(2y)-2(a^2-a)f(-2y) \end{aligned}$$

for all $x, y \in {}_BX$ and $a \in B$.

Theorem 3.1. *Let $\varphi : {}_BX \times {}_BX \to [0,\infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{1}{|2|^n}\varphi(2^nx, 2^ny) = 0 \tag{3.1}$$

for all $x, y \in {}_BX$ and

$$\tilde{\varphi}(x) := \sup\left\{\frac{\varphi(0, 2^jx)}{|2|^j} : j \in N \cup \{0\}\right\} \tag{3.2}$$

exists for all $x \in {}_BX$. Suppose that an odd mapping $f : {}_BX \to {}_BY$ satisfies the inequality

$$\|D_af(x,y)\| \le \varphi(x,y) \tag{3.3}$$

for all $x, y \in {}_BX$ and $a \in B$. Then there exists a unique additive mapping $A : {}_BX \to {}_BY$ satisfying $A(ax) = aA(x)$ for all $x \in {}_BX$ and $a \in B$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|8|}\tilde{\varphi}(x) \tag{3.4}$$

for all $x \in {}_BX$.

Proof. Putting $x = 0$ and $a = 1$ in (3.3), we get

$$\|f(2y) - 2f(y)\| \le \frac{1}{|4|}\varphi(0,y) \tag{3.5}$$

for all $y \in {}_BX$. Replacing y by $2^n x$ in (3.5) and dividing both sides of (3.5) by $|2|^{n+1}$, we get

$$\Big\|\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^n}f(2^n x)\Big\| \leq \frac{1}{|2|^{n+3}}\varphi(0, 2^n x) \tag{3.6}$$

for all $x \in {}_BX$ and all non-negative integers n. Therefore, we conclude from (3.1) and (3.6) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence in ${}_BY$ for all $x \in {}_BX$. Since ${}_BY$ is complete the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges in ${}_BY$ for all $x \in {}_BX$. So one can define the mapping $A : {}_BX \to {}_BY$ by

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}f(2^n x) \tag{3.7}$$

for all $x \in {}_BX$. Using induction one can show that

$$\Big\|\frac{f(2^n x)}{2^n} - f(x)\Big\| \leq \frac{1}{|8|}\max\Big\{\frac{\varphi(0, 2^j x)}{|2|^j} : 0 \leq j < n\Big\} \tag{3.8}$$

for all $n \in N$ and all $x \in {}_BX$. By taking n to approach infinity in (3.8) and using (3.2) and (3.7) one obtains (3.4). Now, we show that A is an additive mapping. It follows from (3.1), (3.3) and (3.7) that

$$\|D_1 A(x, y)\| = \lim_{n\to\infty} \frac{1}{|2|^n}\big\|D_1 f(2^n x, 2^n y)\big\| \leq \lim_{n\to\infty} \frac{1}{|2|^n}\varphi(2^n x, 2^n y) = 0$$

for all $x, y \in {}_BX$. Hence, the mapping A satisfies (1.3), So by Lemma (2.1) the mapping A is additive. Replacing y by 0 in (3.3), we get

$$\|f(ax) - af(x)\| \leq \frac{1}{|2|}\varphi(x, 0) \tag{3.9}$$

for all $x \in {}_BX$ and $a \in B$. Replacing x by $2^n x$ in (3.9) and dividing both sides of (3.9) by $|2|^n$, we get

$$\Big\|\frac{1}{2^n}f(2^n ax) - \frac{a}{2^n}f(2^n x)\Big\| \leq \frac{1}{|2|^{n+1}}\varphi(2^n x, 0)$$

for all $x \in {}_BX$, $a \in B_1$ and for all non-negative integers n. Therefor

$$\Big\|\frac{1}{2^n}f(2^n ax) - \frac{a}{2^n}f(2^n x)\Big\| \longrightarrow 0$$

as $n \longrightarrow \infty$ for all $x \in {}_BX$, $a \in B$. Hence

$$A(ax) = \lim_{n\to\infty} \frac{1}{2^n}f(2^n ax) = \lim_{n\to\infty} \frac{a}{2^n}f(2^n x) = aA(x)$$

for all $x \in {}_BX$ and $a \in B$.

To prove the uniqueness of A, let $T : {}_BX \to {}_BY$ be another additive mapping satisfying (3.4). Then

$$\begin{aligned}\|A(x) - T(x)\| &= \lim_{k\to\infty} |2|^{-k} \big\|A(2^k x) - T(2^k x)\big\| \\ &\le \lim_{k\to\infty} |2|^{-k} \max\big\{\|A(2^k x) - f(2^k x)\|, \|T(2^k x) - f(2^k x)\|\big\} \\ &\le \frac{1}{|8|} \lim_{k\to\infty} \lim_{n\to\infty} \max\Big\{\frac{\varphi(0, 2^j x)}{|2|^j} : k \le j < n+k\Big\} \\ &= \frac{1}{|8|} \lim_{k\to\infty} \sup\Big\{\frac{\varphi(0, 2^j x)}{|2|^j} : k \le j < \infty\Big\} = 0\end{aligned}$$

for all $x \in {}_BX$. So $A = T$. ∎

Theorem 3.2. *Let $\Phi : {}_BX \times {}_BX \to [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} |2|^n \Phi\big(\frac{x}{2^n}, \frac{y}{2^n}\big) = 0 \tag{3.10}$$

for all $x, y \in {}_BX$ and

$$\widetilde{\Phi}(x) := \sup\Big\{|2|^j \Phi(0, \frac{x}{2^{j+1}}) : j \in N \cup \{0\}\Big\} \tag{3.11}$$

exists for all $x \in {}_BX$. Suppose that an odd mapping $f : {}_BX \to {}_BY$ satisfies the inequality

$$\|D_a f(x, y)\| \le \Phi(x, y)$$

for all $x, y \in {}_BX$ and $a \in B$. Then there exists a unique additive mapping $A : {}_BX \to {}_BY$ satisfying $A(ax) = aA(x)$ for all $x \in {}_BX$ and $a \in B$ such that

$$\|f(x) - A(x)\| \le \frac{1}{|4|}\tilde{\Phi}(x) \tag{3.12}$$

for all $x \in {}_BX$.

Proof. Similar to the proof of Theorem 3.1, we have

$$\|f(2y) - 2f(y)\| \le \frac{1}{|4|}\Phi(0, y) \tag{3.13}$$

for all $y \in {}_BX$. Replacing y by $\dfrac{x}{2^{n+1}}$ in (3.13) and multiplying both sides of (3.13) to $|2|^n$, we get

$$\Big\|2^{n+1} f\big(\frac{x}{2^{n+1}}\big) - 2^n f\big(\frac{x}{2^n}\big)\Big\| \le |2|^{n-2}\Phi\big(0, \frac{x}{2^{n+1}}\big) \tag{3.14}$$

for all $x \in {}_BX$ and all non-negative integers n. Therefor, we conclude from (3.10) and (3.14) that the sequence $\{2^n f(x/2^n)\}$ is a Cauchy sequence in ${}_BY$ for all $x \in {}_BX$. Since ${}_BY$ is complete the sequence $\{2^n f(x/2^n)\}$ converges in ${}_BY$ for all $x \in {}_BX$. So one can define the mapping $A : {}_BX \to {}_BY$ by

$$A(x) := \lim_{n\to\infty} 2^n f\big(\frac{x}{2^n}\big) \tag{3.15}$$

for all $x \in {}_BX$. Using induction one can show that

$$\Big\|2^n f(\frac{x}{2^n}) - f(x)\Big\| \le \frac{1}{|4|} \max\big\{|2|^j \Phi(0, \frac{x}{2^{j+1}}) : 0 \le j < n\big\} \tag{3.16}$$

for all $n \in N$ and all $x \in H$. By taking n to approach infinity in (3.16) and using (3.11) and (3.15) one obtains (3.12). The rest of the proof is similar to the proof of Theorem 3.1. ∎

Corollary 3.3. *Let θ, r, s be positive real numbers such that $r, s \neq 1$ and $|2| < 1$. Suppose that an odd mapping $f : {}_BX \to {}_BY$ satisfies the inequality*

$$\|D_a f(x, y)\| \le \theta(\|x\|^r + \|y\|^s)$$

for all $x, y \in {}_BX$ and $a \in B$. Then there exists a unique additive mapping $A : {}_BX \to {}_BY$ satisfying $A(ax) = aA(x)$ for all $x \in {}_BX$ and $a \in B$ such that

$$\|f(x) - A(x)\| \le \begin{cases} \dfrac{\theta\|x\|^s}{|\ 8\ |} & (r, s > 1) \\ \dfrac{\theta\|x\|^s}{|\ 2\ |^{s+2}} & (r, s < 1) \end{cases}$$

for all $x \in {}_BX$.

Corollary 3.4. *Let θ, r, s be positive real numbers such that $r + s \neq 1$ and $|2| < 1$. Suppose that an odd function $f : {}_BX \to {}_BY$ satisfies the inequality*

$$\|D_a f(x, y)\| \le \theta\|x\|^r\|y\|^s$$

for all $a \in B$ and $x, y \in {}_BX$. Then $f : {}_BX \to {}_BY$ is an additive mapping satisfying $f(ax) = af(x)$ for all $x \in {}_BX$ and $a \in B$.

Theorem 3.5. *Let $\varphi : {}_BX \times {}_BX \to [0, \infty)$ be a mapping such that*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y)}{|4|^n} = 0 \tag{3.17}$$

for all $x, y \in {}_BX$ *and*

$$\widetilde{\varphi}(x) := \sup\Big\{\frac{\varphi(0, 2^j x)}{|4|^j} : j \in N \cup \{0\}\Big\} \tag{3.18}$$

exists for all $x \in {}_BX$*. Suppose that an even mapping* $f : {}_BX \to {}_BY$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|\Delta_a f(x, y)\| \le \varphi(x, y) \tag{3.19}$$

for all $x, y \in {}_BX$ *and* $a \in B$*, Then there exists a unique quadratic mapping* $Q : {}_BX \to {}_BY$ *satisfying* $Q(ax) = a^2 Q(x)$ *for all* $x \in {}_BX$ *and* $a \in B$ *such that*

$$\|f(x) - Q(x)\| \le \frac{1}{|8|}\widetilde{\varphi}(x) \tag{3.20}$$

for all $x \in {}_BX$*.*

Proof. Putting $x = 0, a = 1$ in (3.19) , we get

$$\|f(2y) - 4f(y)\| \le \frac{1}{|2|}\varphi(0, y) \tag{3.21}$$

for all $y \in {}_BX$. Replacing y by $2^n x$ in (3.21) and dividing both sides of (3.21) by $|4|^{n+1}$, we get

$$\Big\|\frac{1}{4^{n+1}} f(2^{n+1}x) - \frac{1}{4^n} f(2^n x)\Big\| \le \frac{1}{|2|^{2n+3}}\varphi(0, 2^n x) \tag{3.22}$$

for all $x \in {}_BX$ and all non-negative integers n. Therefore, we conclude from (3.17) and (3.22) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence in ${}_BY$ for all $x \in {}_BX$. Since ${}_BY$ is complete the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges for all $x \in {}_BX$. So one can define the mapping $Q : {}_BX \to {}_BY$ by:

$$Q(x) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x) \tag{3.23}$$

for all $x \in {}_BX$. Using induction one can show that

$$\Big\|\frac{f(2^n x)}{4^n} - f(x)\Big\| \le \frac{1}{|8|}\max\Big\{\frac{\varphi(0, 2^j x)}{|4|^j} : 0 \le j < n\Big\} \tag{3.24}$$

for all $n \in N$ and all $x \in {}_BX$. By taking n to approach infinity in (3.24) and using (3.18) and (3.23) one obtains (3.20).

Now, we show that the mapping Q is quadratic. It follows from (3.17), (3.19) and (3.23) that

$$\begin{aligned}\|\Delta_1 Q(x,y)\| &= \lim_{n\to\infty}\frac{1}{|4|^n}\big\|\Delta_1 f(2^n x, 2^n y)\big\| \\ &\le \lim_{n\to\infty}\frac{1}{|4|^n}\varphi(2^n x, 2^n y) = 0\end{aligned}$$

for all $x, y \in {}_BX$, So the mapping Q satisfies (1.3), hence by Lemma 2.2, the mapping Q is quadratic. Replacing $y = 0$ in (3.19), we have

$$\|f(ax) - a^2 f(x)\| \le \frac{1}{|2|}\varphi(x, 0) \tag{3.25}$$

for all $x \in {}_BX$ and $a \in B$. Replacing x by $2^n x$ in (3.25) and dividing both sides of (3.25) by $|4|^n$, we get

$$\Big\|\frac{1}{4^n}f(2^n ax) - \frac{a^2}{4^n}f(2^n x)\Big\| \le \frac{1}{|2|^{2n+1}}\varphi(2^n x, 0)$$

for all $x \in {}_BX$, $a \in B$ and for all non-negative integers n. Therefore

$$\Big\|\frac{1}{4^n}f(2^n ax) - \frac{a^2}{4^n}f(2^n x)\Big\| \longrightarrow 0$$

as $n \longrightarrow \infty$ for all $x \in {}_BX$, $a \in B_1$. Hence

$$Q(ax) = \lim_{n\to\infty}\frac{1}{4^n}f(2^n ax) = \lim_{n\to\infty}\frac{1}{4^n}a^2 f(2^n x) = a^2 Q(x)$$

for all $x \in {}_BX$, $a \in B$. To prove the uniqueness of Q, let $T : {}_BX \to {}_BX$ be another quadratic mapping satisfying (3.20). So it follows from (3.20) and (3.23) that

$$\begin{aligned}\|Q(x) - T(x)\| &= \lim_{k\to\infty}|4|^{-k}\big\|Q(2^k x) - T(2^k x)\big\| \\ &\le \lim_{k\to\infty}|4|^{-k}\max\big\{\|Q(2^k x) - f(2^k x)\|, \|T(2^k x) - f(2^k x)\|\big\} \\ &\le \frac{1}{|8|}\lim_{k\to\infty}\lim_{n\to\infty}\max\Big\{\frac{\varphi(0, 2^j x)}{|4|^j} : k \le j < n + k\Big\} \\ &\le \frac{1}{|8|}\lim_{k\to\infty}\sup\Big\{\frac{\varphi(0, 2^j x)}{|4|^j} : k \le j < \infty\Big\} = 0\end{aligned}$$

for all $x \in {}_BX$. So $Q = T$. ∎

Theorem 3.6. *Let* $\Phi : {}_BX \times {}_BX \to [0, \infty)$ *be a mapping such that*

$$\lim_{n\to\infty} |4|^n \Phi\big(\frac{x}{2^n}, \frac{y}{2^n}\big) = 0 \tag{3.26}$$

for all $x, y \in {}_BX$ *and*

$$\widetilde{\Phi}(x) := \sup\big\{|4|^j \Phi(0, \frac{x}{2^{j+1}}) : j \in N \cup \{0\}\big\} \tag{3.27}$$

exists for all $x \in {}_BX$. *Suppose that an even mapping* $f : {}_BX \to {}_BY$ *with* $f(0) = 0$ *satisfies the inequality*

$$\|\Delta_a f(x, y)\| \le \Phi(x, y)$$

for all $x, y \in {}_BX$ *and* $a \in B$. *Then there exists a unique quadratic mapping* $Q : {}_BX \to {}_BY$ *satisfying* $Q(ax) = a^2 Q(x)$ *for all* $x \in {}_BX$ *and* $a \in B$ *such that*

$$\|f(x) - Q(x)\| \le \frac{1}{|2|}\widetilde{\Phi}(x) \tag{3.28}$$

for all $x \in {}_BX$.

Proof. Similar to the proof of Theorem 3.5, we have

$$\|f(2y) - 4f(y)\| \le \frac{1}{|2|}\Phi(0, y) \tag{3.29}$$

for all $y \in {}_BX$. Replacing y by $\dfrac{x}{2^{n+1}}$ in (3.29) and multiplying both sides of (3.29) to $|4|^n$, we get

$$\Big\|4^{n+1} f\big(\frac{x}{2^{n+1}}\big) - 4^n f\big(\frac{x}{2^n}\big)\Big\| \le |2|^{2n-1}\Phi\big(0, \frac{x}{2^{n+1}}\big) \tag{3.30}$$

for all $x \in {}_BX$ and all non-negative integers n. Therefore we conclude from (3.26) and (3.30) that the sequence $\{4^n f(x/2^n)\}$ is a Cauchy sequence in ${}_BY$ for all $x \in {}_BX$. Since ${}_BY$ is complete the sequence $\{4^n f(x/2^n)\}$ converges in ${}_BY$ for all $x \in {}_BX$. So one can define the mapping $Q : {}_BX \to {}_BY$ by

$$Q(x) := \lim_{n\to\infty} 4^n f\big(\frac{x}{2^n}\big) \tag{3.31}$$

for all $x \in {}_BX$. Using induction one can show that

$$\Big\|4^n f(\frac{x}{2^n}) - f(x)\Big\| \le \frac{1}{|2|}\max\big\{|4|^j \Phi(0, \frac{x}{2^{j+1}}) : 0 \le j < n\big\} \tag{3.32}$$

for all $n \in N$ and all $x \in {}_BX$. By taking n to approach infinity in (3.32) and using (3.27) and (3.31) one obtains (3.28). The rest of the proof is similar to Theorem 3.5. ■

Corollary 3.7. *Let θ, r, s be positive real numbers such that $r, s \neq 2$ and $|2| < 1$. Suppose that an even mapping $f : {}_BX \to {}_BY$ satisfies the inequality*

$$\|\Delta_a f(x,y)\| \leq \theta(\|x\|^r + \|y\|^s)$$

for all $x, y \in {}_BX$ and $a \in B$. Then there exists a unique quadratic mapping $Q : {}_BX \to {}_BY$ satisfying $Q(ax) = a^2 Q(x)$ for all $x \in {}_BX$ and $a \in B$ such that

$$\|f(x) - Q(x)\| \leq \begin{cases} \dfrac{\theta\|x\|^s}{|\ 8\ |} & (r, s > 2) \\[2ex] \dfrac{\theta\|x\|^s}{|\ 2\ |^{s+1}} & (r, s < 2) \end{cases}$$

for all $x \in {}_BX$.

Corollary 3.8. *Let θ, r, s be positive real numbers such that $r + s \neq 2$ and $|2| < 1$. Suppose that an even mapping $f : {}_BX \to {}_BY$ satisfies the inequality*

$$\|\Delta_a f(x,y)\| \leq \theta\|x\|^r \|y\|^s$$

for all $x, y \in {}_BX$ and $a \in B$. Then $f : {}_BX \to {}_BY$ is a quadratic mapping satisfying $f(ax) = a^2 f(x)$ for all $x \in {}_BX$ and $a \in B$.

We now prove our main theorem in this section.

Theorem 3.9. *Let $\varphi : {}_BX \times {}_BX \to [0, \infty)$ be a mapping satisfying (3.17) for all $x, y \in {}_BX$, and let*

(3.33) $$\widetilde{\psi}_m(x) := \sup\Big\{\frac{\varphi(0, 2^j x)}{|m|^j} + \frac{\varphi(0, -2^j x)}{|m|^j} : j \in N \cup \{0\}\Big\}$$

exists for all $x \in {}_BX$ and $m \in \{2, 4\}$. Suppose that a mapping $f : {}_BX \to {}_BY$ with $f(0) = 0$ satisfies the inequality

(3.34) $$\|M_a f(x,y)\| \leq \varphi(x,y)$$

for all $x, y \in {}_BX$ and $a \in B$. Then there exists a unique additive mapping $A : {}_BX \to {}_BY$ and a unique quadratic mapping $Q : {}_BX \to {}_BY$ satisfying $A(ax) = aA(x)$, $Q(ax) = a^2 Q(x)$ for all $x \in {}_BX$ and $a \in B$ such that

(3.35) $$\|f(x) - A(x) - Q(x)\| \leq \frac{1}{|16|} \max\Big\{\widetilde{\psi}_2(x), \widetilde{\psi}_4(x)\Big\}$$

for all $x \in {}_BX$.

Proof. Let $f_e(x) = \dfrac{f(x)+f(-x)}{2}$ and $f_o(x) = \dfrac{f(x)-f(-x)}{2}$, then $f_e(-x) = f_e(x)$ and $f_o(-x) = -f_o(x)$ for all $x \in {}_BX$. Let

$$\psi(x,y) := \frac{1}{|2|}\big[\varphi(x,y)+\varphi(-x,-y)\big] \tag{3.36}$$

for all $x, y \in {}_BX$. By using (3.34) and (3.36), we have

$$\|\Delta_a f_e(x,y)\| = \|M_a f_e(x,y)\| \le \psi(x,y)$$

$$\|D_a f_o(x,y)\| = \|M_a f_o(x,y)\| \le \psi(x,y)$$

for all $x, y \in {}_BX$ and $a \in B$. Hence, the result follows by using Theorems 3.1 and 3.5 for ψ instead of φ . ■

Theorem 3.10. *Let $\Phi : {}_BX \times {}_BX \to [0,\infty)$ be a mapping satisfying (3.10) for all $x, y \in {}_BX$, and let*

$$\widetilde{\Psi}_m(x) := \sup\big\{|m|^j\Phi(0,\frac{x}{2^{j+1}}) + |m|^j\Phi(0,-\frac{x}{2^{j+1}}) : j \in N \cup \{0\}\big\} \tag{3.37}$$

exists for all $x \in {}_BX$. Suppose that a mapping $f : {}_BX \to {}_BY$ with $f(0) = 0$ satisfies the inequality

$$\|M_a f(x,y)\| \le \Phi(x,y)$$

for all $x, y \in {}_BX$ and $a \in B$. Then there exists a unique additive mapping $A : {}_BX \to {}_BY$ and a unique quadratic mapping $Q : {}_BX \to {}_BY$ satisfying $A(ax) = aA(x)$, $Q(ax) = a^2Q(x)$ for all $x \in {}_BX$ and $a \in B$ such that

$$\|f(x) - A(x) - Q(x)\| \le \frac{1}{|8|}\max\big\{\widetilde{\Psi}_2(x), \widetilde{\Psi}_4(x)\big\} \tag{3.38}$$

for all $x \in {}_BX$.

Proof. Similar to Theorem 3.9, the results can be obtained by using Theorems 3.2 and 3.6 . ■

Corollary 3.11. *Let θ, r, s be positive real numbers such that $(r, s < 1)$ or $(r, s > 2)$ and $|2| < 1$. Suppose that a mapping $f : {}_BX \to {}_BY$ satisfies the inequality*

$$\|M_a f(x,y)\| \le \theta(\|x\|^r + \|y\|^s)$$

for all $x, y \in {}_BX$ and $a \in B$. Then there exists a unique additive mapping $A : {}_BX \to {}_BY$ and a unique quadratic mapping $Q : {}_BX \to {}_BY$ satisfying

$A(ax) = aA(x)$, $Q(ax) = a^2Q(x)$ *for all* $x \in {}_BX$ *and* $a \in B$ *such that*

$$\|f(x) - A(x) - Q(x)\| \leq \begin{cases} \dfrac{2\theta\|x\|^s}{|16|} & (r, s > 2) \\ \dfrac{2\theta\|x\|^s}{|2|^{s+3}} & (r, s < 1) \end{cases}$$

for all $x \in {}_BX$.

Corollary 3.12. *Let* θ, r, s *be positive real numbers such that* $(r + s < 1)$ *or* $(r + s > 2)$ *and* $|2| < 1$. *Suppose that a mapping* $f : {}_BX \to {}_BY$ *satisfies the inequality*

$$\|M_af(x, y)\| \leq \theta\|x\|^r\|y\|^s$$

for all $x, y \in {}_BX$ *and* $a \in B$. *Then there exists a unique additive mapping* $A : {}_BX \to {}_BY$ *and a unique quadratic mapping* $Q : {}_BX \to {}_BY$ *satisfying* $A(ax) = aA(x)$, $Q(ax) = a^2Q(x)$ *for all* $x \in {}_BX$ *and* $a \in B$ *such that*

$$f(x) = A(x) + Q(x)$$

for all $x \in {}_BX$.

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REFERENCES

1. J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989.
2. D. Amir, *Characterizations of Inner Product Spaces*, Birkhäuser, Basel, 1986.
3. L. M. Arriola and W. A. Beyer, Stability of the Cauchy functional equation over p-adic fields, *Real Analysis Exchange*, **31** (2005/2006), 125-132.
4. T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, **2** (1950), 64-66.
5. P. W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.*, **27** (1984), 76-86.
6. S. Czerwik, *Functional equations and inequalities in several variables*, World Scientific, New Jersey-London-Singapore-Hong Kong, 2002.

7. S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg* **62** (1992), 59-64.

8. G. Z. Eskandani, On the Hyers-Ulam-Rassias stability of an additive functional equation in quasi-Banach spaces, *J. Math. Anal. Appl.*, **345** (2008), 405-409.

9. A. Grabiec, The generalized Hyers-Ulam stability of a class of functional equations, *Publ. Math. Debrecen*, **48** (1996), 217-235.

10. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.*, **27** (1941), 222-224.

11. D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of functional equations in several variables*, Birkhäuser, Basel, 1998.

12. P. Jordan and J. Von Neumann, On inner products in linear metric spaces, *Ann. of Math.*, **36** (1935), 719-723.

13. K. Jun and Y. Lee, On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality, *Math. Inequal. Appl.*, **4** (2001), 93-118.

14. Z. Kaiser, On stability of the Cauchy equation in normed spaces over fields with valuation, *Publ. Math. Debrecen*, **64(1-2)** (2004), 189-200.

15. Pl. Kannappan, Quadratic functional equation and inner product spaces, *Results Math.*, **27** (1995), 368-372.

16. M. Mirzavaziri and M. S. Moslehian, A fixed point approach to stability of a quadratic equation, *Bull. Braz. Math. Soc.*, **37(3)** (2006), 361-376.

17. M. S. Moslehian and Th. M. Rassias, Stability of functional equation in non-Archimedean spaces, *Applicable Analysis and Discrete Mathematics*, **1** (2007), 325-334.

18. M. S. Moslehian, On the orthogonal stability of the Pexiderized quadratic equation, *J. Differ. Equations. Appl.*, **11(11)** (2005), 999-1004.

19. M. S. Moslehian and Gh. Sadegi, Stability of two types of cubic functional equations in non-Archimedean spaces, *Real Anal. Exchange*, **32(2)** (2008), 375-383.

20. F. Moradlou, H. Vaezi and C. Park, *Fixed Points and Stability of an Additive Functional Equation of n-Apollonius Type in C*-Algebras*, Abstract and Applied Analysis, 2008, doi:10.1155/2008/672618.

21. A. Najati and G. Z. Eskandani, Stability of a mixed additive and cubic functional equation in quasi-Banach spaces, *J. Math. Anal. Appl.*, **342** (2008), 1318-1331.

22. C. Park and F. Moradlou, Stability of Homomorphisms and Derivations in C*-ternary Rings, *Taiwanese J. Math.*, to appear.

23. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

24. F. Skof, Local properties and approximations of operators, *Rend. Sem. Mat. Fis. Milano*, **53** (1983), 113-129.

25. S. M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York, 1960.

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