TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 2, pp. 719-732, April 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

MINIMAX THEOREMS FOR VECTOR-VALUED MAPPINGS IN ABSTRACT CONVEX SPACES

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Abstract. In this paper, we introduce the concepts of C-quasiconcave mappings and properly C-quasiconcave mappings in abstract convex spaces. By using the Fan-Browder type fixed point theorem and the maximal element theorem, we establish some minimax theorems for vector-valued mappings in abstract convex spaces. We also give some examples to illustrate our results.

1. INTRODUCTION

It is well known that Ky Fan minimax theorem (see [11]) and Ky Fan minimax inequality (see [12]) play very important roles in many fields, such as variational inequalities, game theory, mathematical economics, control theory, and fixed point theory. Ky Fan minimax theorems and Ky Fan minimax inequalities relative to real-valued functions and vector-valued mappings have been studied extensively by many authors (see, for example, [1, 2 5-10, 13-15, 18-20, 22, 28-33] and the references therein).

At the beginning, the investigation of minimax theorems for vector-valued mappings were mainly devoted to study in topological vector spaces under various hypotheses. With the development of KKM theory, there have appeared a number of spaces without any linear structures, which seem to lead, in a natural way, to the investigation of minimax theorems in these more general settings. Chen [4] proved a generalized Fan's section theorem and a generalized Browder's fixed point theorem for set-valued mappings in H-spaces. Based on these two results, he obtained

Received December 31, 2008, accepted June 15, 2009.
Communicated by J. C. Yao.
2000 *Mathematics Subject Classification*: 49J35; 47J20.
Key words and phrases: Abstract convex space, C-quasiconcave, Properly C-quasiconcave, Fixed point, Maximal element.
This work was supported by the National Natural Science Foundation of China (10671135, 70831005) and the Natural Science Foundation of Henan Education Department of China (2008B110012).
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a minimax inequality for vector-valued mappings in H-spaces. Chang et al. [3] presented a generalized section theorem and a generalized fixed point theorem in W-spaces, and applied these results to establish minimax inequalities for vector-valued mappings in W-spaces.

On the other hand, Park [23] introduced the new concept of abstract convex space which includes convex spaces, H-spaces, G-convex spaces, and W-spaces as special cases. With this new concept, he obtained some new coincidence theorems and fixed point theorems in abstract convex spaces. Very recently, Park [24, 25, 26, 27] further studied KKM theory with applications in abstract convex spaces. It is worth mentioning that some minimax theorems for real-valued functions have been proved in abstract convex spaces by Park [25]. However, to the best of our knowledge, there is no paper dealing with the minimax theorems for vector-valued mappings in abstract convex spaces.

Motivated and inspired by the works mentioned above, in this paper, we introduce the concepts of C-quasiconcave mappings and properly C-quasiconcave mappings in abstract convex spaces. By using the Fan-Browder type fixed point theorem and the maximal element theorem due to Park [25], we establish some new minimax theorems for vector-valued mappings in abstract convex spaces. We also give some examples to illustrate our results.

2. Preliminaries

Definition 2.1. ([21]). Let Z be a topological vector space, C a pointed closed convex cone with nonempty interior int C, B a nonempty subset of Z. A point $z \in B$ is said to be

(i) a minimal point of B if $B \cap (z - C) = \{z\}$;
(ii) a weakly minimal point of B if $B \cap (z - \text{int}\, C) = \emptyset$;
(iii) a maximal point of B if $B \cap (z + C) = \{z\}$;
(iv) a weakly maximal point of B if $B \cap (z + \text{int}\, C) = \emptyset$.

We denote, by $\text{Min}B$, $\text{Min}_w B$, $\text{Max}B$ and $\text{Max}_w B$, the set of all of the minimal points, the set of all of the weakly minimal points, the set of all of the maximal points, and the set of all of the weakly maximal points of B, respectively. It is easy to verify that

$$\text{Min}B \subset \text{Min}_w B \text{ and } \text{Max}B \subset \text{Max}_w B. \tag{2.1}$$

Obviously, when $Z = \mathbb{R}$ and $C = [0, +\infty)$, the (weakly) minimal/maximal point of B in Z reduces to the minimum/maximum of B in $\mathbb{R}$.

Lemma 2.1. ([21]). *Let B be a nonempty compact subset of a Hausdorff topological vector space Z with a pointed closed convex cone C. Then*

(i) $\mathrm{Min}B \neq \emptyset$,
(ii) $B \subset \mathrm{Min}B + C$,
(iii) $\mathrm{Max}B \neq \emptyset$,
(iv) $B \subset \mathrm{Max}B - C$.

Let G_1, G_2 be two Hausdorff topological spaces and let $T : G_1 \to 2^{G_2}$ be a set-valued mapping with nonempty values. T is said to be upper semicontinuous if, for each $x_0 \in G_1$ and for each open set V containing $T(x_0)$, there exists a neighborhood U of x_0 such that $T(U) \subset V$. Moreover, if T is compact-valued, then T is upper semicontinuous if and only if, for each net $\{x_i\} \subset G_1$ such that $x_i \to x_0 \in G_1$ and for each $y_i \in T(x_i)$, there exist $y_0 \in T(x_0)$ and a subnet $\{y_{i_j}\}$ of $\{y_i\}$ such that $y_{i_j} \to y_0$. For more details, we refer to [14, 17, 22].

Lemma 2.2. ([2]). *Let E and Z be two Hausdorff topological spaces, $\emptyset \neq X \subset E$, and $F : X \to 2^Z$ be a set-valued mapping. If X is compact and F is upper semicontinuous and compact-valued, then $F(X) = \bigcup_{x\in X} F(x)$ is compact.*

Lemma 2.3. *Let E be a Hausdorff topological space and X be a nonempty compact subset of E. Let Z be a Hausdorff topological vector space and C be a pointed closed convex cone in Z with int $C \neq \emptyset$. Let $f : X \times X \to Z$ be a continuous mapping. Then the set-valued mapping $F : X \to 2^Z$ defined by*

$$F(x) = \mathrm{Max}_w f(x, X),\ \forall x \in X$$

is upper semicontinuous and compact-valued.

Proof. For each $x \in X$, since $f(x, X)$ is nonempty compact, by Lemma 2.1 (iii), we know that $\mathrm{Max}f(x, X) \neq \emptyset$ and so $F(x) = \mathrm{Max}_w f(x, X) \neq \emptyset$. Moreover, $F(x)$ is compact since it is contained in the compact set $f(x, X)$ and is closed. In fact, let $\{z_i\}$ be a net in $F(x)$ such that $z_i \to z_0$. We have $z - z_i \notin \text{int } C$ for each $z \in f(x, X)$, and so $z - z_0 \notin \text{int } C$. Thus, $z_0 \in F(x)$. Therefore, F is compact-valued.

Let $x_0 \in X$ and $\{x_i\}$ be a net in X such that $x_i \to x_0$ and $z_i \in F(x_i)$. For suitable $y_i \in X$, we have $z_i = f(x_i, y_i) \in f(X, X)$. From compactness, there exist $y_0 \in X$ and $z_0 \in Z$ and $\{y_{i_j}\}$, $\{z_{i_j}\}$, subnets of $\{y_i\}$ and $\{z_i\}$, respectively, such that $y_{i_j} \to y_0$, $z_{i_j} \to z_0$. By contradiction, suppose that $z_0 \notin F(x_0)$. Hence, there exists $y' \in X$ and $c \in \text{int } C$ such that $z_0 = f(x_0, y') - c$. Note that $z_0 = \lim z_{i_j} = \lim f(x_{i_j}, y_{i_j}) = f(x_0, y_0)$. We have

$$f(x_{i_j}, y') - z_{i_j} = (z_0 - z_{i_j} + f(x_{i_j}, y') - f(x_0, y')) + c,$$

where the right-hand side lies in int C for j large enough. Consequently, $z_{i_j} \notin F(x_{i_j})$ for such values of j, which is a contradiction. Thus, F is upper semicontinuous. This completes the proof.

Let Z be a Hausdorff topological vector space and C be a pointed closed convex cone in Z with int $C \neq \emptyset$. For any fixed $a \in Z$ and $e \in \text{int}\, C$, we define a functional $h_{e,a} : Z \to \mathbb{R}$ by

$$h_{e,a}(z) = \min\{t \in \mathbb{R} : z \in a + te - C\}, \ \forall z \in Z.$$

By [16] and [21], we have the following results:

(i) $h_{e,a}$ is a continuous functional;

(ii) $h_{e,a}$ is monotonically increasing, i.e., $z_1 - z_2 \in C \Longrightarrow h_{e,a}(z_1) \geq h_{e,a}(z_2)$;

(iii) $h_{e,a}$ is strictly monotonically increasing, i.e., $z_1 - z_2 \in \text{int}\, C \Longrightarrow h_{e,a}(z_1) > h_{e,a}(z_2)$;

(iv) $h_{e,a}(z) < r \Longleftrightarrow z \in a + re - \text{int}\, C$;

(v) $h_{e,a}(z) \leq r \Longleftrightarrow z \in a + re - C$;

(vi) $h_{e,a}(z) \geq r \Longleftrightarrow z \notin a + re - \text{int}\, C$;

(vii) $h_{e,a}(z) > r \Longleftrightarrow z \notin a + re - C$.

Let $\langle D \rangle$ denote the set of all nonempty finite subset of a set D.

Definition 2.2. ([23]). An abstract convex space $(E, D; \Gamma)$ consists of nonempty sets E, D, and a set-valued mapping $\Gamma : \langle D \rangle \to 2^E$ with nonempty values. Denote $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$.

An abstract convex space $(E, D; \Gamma)$ with any topology on E is called an abstract convex topological space. Moreover, if E is a Hausdorff topological space, $(E, D; \Gamma)$ will be called an abstract convex Hausdorff topological space.

It is obvious that any topological vector space E is an abstract convex topological space with $\Gamma = \text{co}$, where co is the convex hull in vector spaces. In specially, $(\mathbb{R}; \text{co})$ is an abstract convex topological space. For more examples of abstract convex spaces, we refer to [23].

Let $(E, D; \Gamma)$ be an abstract convex space. For any $D' \subset D$, the $\Gamma-$convex hull of D' is denoted and defined by

$$\text{co}_\Gamma D' = \bigcup\{\Gamma_A | A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a $\Gamma-$convex subset of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$; that is, $\text{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D' \rangle})$ itself is an abstract convex space called a subspace of $(E, D; \Gamma)$. In such case, a subset X of E is said to be $\Gamma-$convex if $\text{co}_\Gamma(X \cap D) \subset X$; in other words, X is

$\Gamma-$convex relative to $D' = X \cap D$. In case $E = D$, let $(E;\Gamma) = (E, E;\Gamma)$. When $(E;\Gamma) = (\mathbb{R};\mathrm{co})$, the Γ-convex subset reduces to the ordinary convex subset.

Let $(E, D;\Gamma)$ be an abstract convex space and Z a set. For a set-valued mapping $F : E \to 2^Z$ with nonempty values, if a set-valued mapping $G : D \to 2^Z$ satisfies

$$F(\Gamma_A) \subset G(A) = \bigcup_{y\in A} G(y),\ \forall A \in \langle D\rangle,$$

then G is called a KKM map with respect to F. A KKM map $G : D \to 2^E$ is a KKM map with respect to the identity map 1_E. A set-valued mapping $F : E \to 2^Z$ is said to have the KKM property and called a $\mathfrak{R}$-map, if for any KKM map $G : D \to 2^Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{R}(E, Z) = \{F : E \to 2^Z | F \text{ is a } \mathfrak{R}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{RC}$-map is defined for closed-valued maps G, and a $\mathfrak{RO}$-map is defined for open-valued maps G. It is known that for a G-convex space $(E, D;\Gamma)$, we have the identity map $1_E \in \mathfrak{RC}(E, E) \cap \mathfrak{RO}(E, E)$. For more details, we refer to [23] and the references therein.

Now, we give the following Fan-Browder type fixed point theorem, which will be used in Section 3.

Lemma 2.4. *Let $(E;\Gamma)$ be an abstract convex topological space with $1_E \in \mathfrak{RC}(E, E)$ and $X \subset E$ be nonempty compact and Γ-convex. If the set-valued mapping $T : X \to 2^X$ satisfies the following conditions:*

(i) $\forall x \in X$, *$T(x)$ is nonempty and Γ-convex;*

(ii) $\forall y \in X$, *$T^{-1}(y)$ is open in X.*

Then T has a fixed point $\bar{x} \in X$; that is, $\bar{x} \in T(\bar{x})$.

Proof. Since $X \subset E$ is nonempty and Γ-convex, by Lemma 1 in [25], $(X;\Gamma|_{\langle X\rangle})$ is also an abstract convex topological space. In fact, $(X;\Gamma|_{\langle X\rangle})$ is a subspace. Since $1_E \in \mathfrak{RC}(E, E)$, by Lemma 2 in [25], $1_X \in \mathfrak{RC}(X, X)$. Note that X is compact. By Theorem 11 in [25], T has a fixed point $\bar{x} \in X$; that is, $\bar{x} \in T(\bar{x})$. This completes the proof.

By using the above Fan-Browder type fixed point theorem, we can get the following maximal element theorem, which will be used in Section 3.

Lemma 2.5. *Let $(E;\Gamma)$ be an abstract convex topological space with $1_E \in \mathfrak{RC}(E, E)$ and $X \subset E$ be nonempty compact and Γ-convex. If the set-valued mapping $T : X \to 2^X$ satisfies the following conditions:*

(i) $\forall x \in X$, *$T(x)$ is Γ-convex;*

(ii) $\forall y \in X$, *$T^{-1}(y)$ is open in X;*

(iii) $\forall x \in X$, $x \notin T(x)$.

Then T has a maximal element $\bar{x} \in X$; that is, $T(\bar{x}) = \emptyset$.

Proof. Suppose that $T(x) \neq \emptyset$ for each $x \in X$. Then all the conditions of Lemma 2.4 are satisfied. By Lemma 2.4, there exists $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$, which violates the condition (iii) of Lemma 2.5. Therefore, T has a maximal element $\bar{x} \in X$. This completes the proof.

Definition 2.3. Let X be a nonempty Γ-convex subset of an abstract convex space $(E;\Gamma)$. Let Z be a topological vector space and C be a pointed closed convex cone of Z with int $C \neq \emptyset$. Then $f : X \to Z$ is said to be

(i) C-quasiconcave if, for any $z \in Z$, the set $\{x \in X : f(x) \in z + \text{int } C\}$ is Γ-convex;

(ii) properly C-quasiconcave if, for any $A \in \langle X \rangle$ and any $x \in \Gamma_A$, there exists $x' \in A$ such that $f(x) \in f(x') + C$;

(iii) properly C-quasiconvex if $-f$ is properly C-quasiconcave.

Remark 2.1. (a) If $Z = \overline{\mathbb{R}}$ and $C = [0, +\infty]$, then Definition 2.3 (i) reduces to the definition of quasiconcave functions introduced by Park [25]; (b) If E is a topological vector space, $\Gamma = \text{co}$ and $|A| = 2$ (where $|A|$ denotes the cardinality of A), then Definition 2.3 (ii) reduces to the definition of properly quasi-C-concave mappings introduced by Li and Wang [18]; (c) If E is a topological vector space, $\Gamma = \text{co}$ and $|A| = 2$, then Definition 2.3 (iii) reduces to the definition of properly quasi-C-convex mappings introduced by Ferro [14].

Example 2.1 Let $(E;\Gamma) = (\mathbb{R};\text{co})$, $X = [0,1]$, $Z = \mathbb{R}$, and $C = [0, +\infty)$. For any nonempty convex subset B of X with $B \neq X$, define a function $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \in B; \\ 0, & x \notin B. \end{cases}$$

Then f is C-quasiconcave. In fact, for any $r \in \mathbb{R}$,

$$\{x \in X : f(x) > r\} = \begin{cases} \emptyset, & r \geq 1, \\ B, & 0 \leq r < 1, \\ X, & r < 0 \end{cases}$$

is convex and so it is Γ-convex. It follows that f is C-quasiconcave. Furthermore, we show that f is properly C-quasiconcave. In fact, for any $A \in \langle X \rangle$ and $x \in \mathrm{co}A$, if $A \subset B$, then $\mathrm{co}A \subset B$ and so $f(x) = f(x') = 1$ for all $x' \in A$; if $A \not\subset B$, then there exists $x' \in A$ such that $x' \notin B$ and so $f(x') = 0$. Thus, for any $A \in \langle X \rangle$ and $x \in \mathrm{co}A$, we have $f(x) \in f(x') + C$ and this implies that f is properly C-quasiconcave.

Lemma 2.6. *Let X be a nonempty Γ-convex subset of an abstract convex space $(E;\Gamma)$. Let Z be a topological vector space and C be a pointed closed convex cone of Z with int $C \neq \emptyset$. If $f : X \to Z$ is properly C-quasiconcave and $g : Z \to \mathbb{R}$ is monotonically increasing, then $gf : X \to \mathbb{R}$ is C-quasiconcave.*

Proof. For any $r \in \mathbb{R}$, we show that $\{x \in X : gf(x) > r\}$ is $\Gamma-$convex. In fact, for any $A \in \langle\{x \in X : gf(x) > r\}\rangle$ and $\bar{x} \in \Gamma_A$, since $f : X \to Z$ is properly C-quasiconcave, there exists $x' \in A$ such that $f(\bar{x}) \in f(x') + C$. It follows that $f(\bar{x}) - f(x') \in C$ and so $gf(\bar{x}) \geq gf(x')$. Since $x' \in A$, we have $gf(x') > r$. It follows that $gf(\bar{x}) > r$ and so $\bar{x} \in \{x \in X : gf(x) > r\}$. By the arbitrariness of $\bar{x}$, we know that $\Gamma_A \subset \{x \in X : gf(x) > r\}$. Thus, $\{x \in X : gf(x) > r\}$ is Γ-convex and so $gf : X \to \mathbb{R}$ is C-quasiconcave. This completes the proof.

3. Minimax Theorems

Theorem 3.1. *Let $(E;\Gamma)$ be an abstract convex Hausdorff topological space with $1_E \in \mathfrak{RC}(E,E)$ and $X \subset E$ be nonempty compact and Γ-convex. Let Z be a Hausdorff topological vector space and C be a pointed closed convex cone in Z with int $C \neq \emptyset$. Let $f : X \times X \to Z$ be a mapping satisfying the following:*

(i) *f is continuous;*

(ii) *$f(x,\cdot)$ is properly C-quasiconcave for each $x \in X$.*

Then

$$\emptyset \neq \mathrm{Min}_w \bigcup_{x\in X} \mathrm{Max}_w f(x,X) \subset \mathrm{Max} \bigcup_{x\in X} f(x,x) + Z \setminus intC. \tag{3.1}$$

Proof. Since f is continuous and X is compact, by Lemmas 2.1-2.3, we know that

$$\mathrm{Min} \bigcup_{x\in X} \mathrm{Max}_w f(x,X) \neq \emptyset.$$

It follows from (2.1) that

$$\mathrm{Min_w} \bigcup_{\mathrm{x}\in \mathrm{X}} \mathrm{Max_w f(x,X)} \neq \emptyset.$$

Since $\text{Max}_w f(x,X) \neq \emptyset$ for all $x \in X$, there exists $y_x \in X$ such that $f(x, y_x) \in \text{Max}_w f(x,X)$. For any $\bar{z} \in \text{Min}_w \bigcup_{x\in X} \text{Max}_w f(x,X)$, by the definition of weakly minimal points, we have $f(x, y_x) \notin \bar{z} - \text{int}C$. Define a set-valued mapping $T : X \to 2^X$ by

$$T(x) = \{y \in X : f(x,y) \notin \bar{z} - \text{int}C\}, \ \forall x \in X.$$

Obviously, $T(x) \neq \emptyset$ for each $x \in X$. Now we show that T has a fixed point. In fact, for any $e \in \text{int}C$, put $a = \bar{z} - e$, and define a functional $h_{e,a} : Z \to \mathbb{R}$ by

$$h_{e,a}(z) = \text{Min}\{t \in \mathbb{R} : z \in a + te - C\}, \ \forall z \in Z.$$

Let $g(x, y) = h_{e,a} f(x, y)$. Then g is continuous. By the property of $h_{e,a}$, we know that, for each fixed $x \in X$,

$$y \in T(x) \Longleftrightarrow g(x,y) \geq 1.$$

Define a set-valued mapping $T_n : X \to 2^X$ by

$$T_n(x) = \{y \in X : g(x,y) > 1 - \frac{1}{n}\}, \quad \forall x \in X, \ n = 1, 2, \cdots.$$

Since $T(x) \subset T_n(x)$, we have $T_n(x) \neq \emptyset$ for each $x \in X$ and $n = 1, 2, \cdots$. The continuity of g implies that

$$T_n^{-1}(y) = \{x \in X : g(x,y) > 1 - \frac{1}{n}\}$$

is open in X for each $y \in X$ and $n = 1, 2, \cdots$. By Lemma 2.6, we have $g(x, \cdot)$ is C-quasiconcave for each $x \in X$ and so $T_n(x)$ is Γ-convex for each $x \in X$ and $n = 1, 2, \cdots$. From Lemma 2.4, there exists $x_n \in X$ such that $x_n \in T_n(x_n)$, i.e., $g(x_n, x_n) > 1 - \frac{1}{n}$ for $n = 1, 2, \cdots$. Since X is compact, without loss of generality, we assume that $\{x_n\}$ converges to $x_0 \in X$. Hence, $g(x_0, x_0) \geq 1$ and so $f(x_0, x_0) \notin \bar{z} - \text{int}C$. It follows from Lemma 2.1 that

$$\begin{aligned} \bar{z} &\in f(x_0, x_0) + Z \setminus \text{int}C \\ &\subset \bigcup_{x\in X} f(x,x) + Z \setminus \text{int}C \\ &\subset \text{Max} \bigcup_{x\in X} f(x,x) - C + Z \setminus \text{int}C \\ &= \text{Max} \bigcup_{x\in X} f(x,x) + Z \setminus \text{int}C. \end{aligned}$$

This completes the proof.

Remark 3.2. (a) If E is a Hausdorff topological vector space and $\Gamma = \text{co}$, then Theorem 3.1 reduces to Theorem 1 of Li and Wang [18]; (b) We would like to point out that the proof of Theorem 3.1 is quite different from the proof of Theorem 1 in [18].

Example 3.1 Let $E = Z = \mathbb{R}$, $\Gamma = \text{co}$, $X = [0, 1]$, and $C = [0, +\infty)$. Define $f : X \times X \to Z$ by

$$f(x, y) = x + y, \quad \forall (x, y) \in X \times X.$$

Then all the conditions of Theorem 3.1 are satisfied. In fact, for any $x \in X$, $A \in \langle X \rangle$ and $y \in \text{co}A$, there exists $y' \in A$ such that $y \geq y'$. Thus, $x + y \geq x + y'$. It follows that $f(x, y) \in f(x, y') + C$ and so $f(x, \cdot)$ is properly C-quasiconcave for each $x \in X$.

Relaxing the concavity condition in Theorem 3.1, we have the following minimax theorem for vector-valued mappings.

Theorem 3.2. *Let $(E; \Gamma)$ be an abstract convex Hausdorff topological space with $1_E \in \mathfrak{RC}(E, E)$ and $X \subset E$ be nonempty compact and Γ-convex. Let Z be a Hausdorff topological vector space and C be a pointed closed convex cone in Z with int $C \neq \emptyset$. Let $f : X \times X \to Z$ be a mapping satisfying the following:*

(i) *f is continuous;*

(ii) *$f(x, \cdot)$ is C-quasiconcave for each $x \in X$.*

Then

$$\emptyset \neq \text{Max}_w \bigcup_{x \in X} f(x, x) \subset \text{Min} \bigcup_{x \in X} \text{Max}_w f(x, X) + Z \setminus (-\mathit{int}C). \quad (3.2)$$

Proof. Since f is continuous and X is compact, by Lemma 2.1 (iii), we know that

$$\text{Max} \bigcup_{x \in X} f(x, x) \neq \emptyset.$$

By (2.1),

$$\text{Max}_w \bigcup_{x \in X} f(x, x) \neq \emptyset.$$

Let $\bar{z} \in \text{Max}_w \bigcup_{x \in X} f(x, x)$. We prove that there exists $x_0 \in X$ such that $\bar{z} \in f(x_0, y) + Z \setminus (-\text{int}C)$ for each $y \in X$. Since $\bar{z} \in \text{Max}_w \bigcup_{x \in X} f(x, x)$, we have $f(x, x) \notin \bar{z} + \text{int } C$ for each $x \in X$. Define a set-valued mapping $T : X \to 2^X$ by

$$T(x) = \{y \in X : f(x, y) \in \bar{z} + \text{int } C\}, \ \forall x \in X.$$

We have

(i) for each $x \in X$, $x \notin T(x)$;

(ii) for each $x \in X$, $T(x) = \{y \in X : f(x,y) \in \bar{z} + \text{int}C\}$ is Γ-convex by condition (ii);

(iii) for each $y \in X$, $T^{-1}(y) = \{x \in X : f(x,y) \in \bar{z} + \text{int}C\}$ is open in X by condition (i).

By Lemma 2.5, T has a maximal element; that is, there exists $x_0 \in X$ such that $T(x_0) = \emptyset$. Hence, $f(x_0, y) \notin \bar{z} + \text{int}C$ for each $y \in X$. It follows that

$$\bar{z} \in f(x_0, y) + Z \setminus (-\text{int}C), \ \forall y \in X. \tag{3.3}$$

Since $f(x_0, X)$ is compact, by Lemma 2.1 and (2.1), we can find $y_0 \in X$ such that

$$f(x_0, y_0) \in \text{Max}_w f(x_0, X) \subset \bigcup_{x\in X} \text{Max}_w f(x, X) \subset \text{Min} \bigcup_{x\in X} \text{Max}_w f(x, X) + C. \tag{3.4}$$

Hence, by (3.3) and (3.4), we have

$$\begin{aligned} \bar{z} &\in f(x_0, y_0) + Z \setminus (-\text{int}C) \\ &\subset \text{Min} \bigcup_{x\in X} \text{Max}_w f(x, X) + C + Z \setminus (-\text{int}C) \\ &= \text{Min} \bigcup_{x\in X} \text{Max}_w f(x, X) + Z \setminus (-\text{int}C). \end{aligned}$$

This completes the proof.

Remark 3.3. (a) If E is a Hausdorff topological vector space and $\Gamma = \text{co}$, then Theorem 3.2 reduces to Theorem 2 of Li and Wang [18]; (b) We would like to point that the proof of Theorem 3.2 is quite different from the proof of Theorem 2 in [18].

Example 3.2. Let $E = \mathbb{R}$, $\Gamma = \text{co}$, $X = [0, 1]$, $Z = \mathbb{R}^2$, and $C = \{(x, y) : x \geq 0, y \geq 0\}$. Define $f : X \times X \to Z$ by

$$f(x, y) = (x, y), \ \forall (x, y) \in X \times X.$$

Then all the conditions of Theorem 3.2 are satisfied. In fact, it is easy to see that f is continuous. We now show that $f(x, \cdot)$ is properly C-quasiconcave for each $x \in X$. For any $x \in X$, $A \in \langle X \rangle$ and $y \in \text{co}A$, there exists $y' \in A$ such that $y \geq y'$ and so $(x, y) \geq (x, y')$. It follows that $f(x, y) \in f(x, y') + C$. That is, $f(x, \cdot)$ is properly C-quasiconcave for each $x \in X$. Therefore, all the conditions of Theorem 3.2 are satisfied.

Theorem 3.3. *Let $(E;\Gamma)$ be an abstract convex Hausdorff topological space with $1_E \in \mathfrak{RC}(E,E)$ and $X \subset E$ be nonempty compact and Γ-convex. Let Z be a Hausdorff topological vector space and C be a pointed closed convex cone in Z with int $C \neq \emptyset$. Let $f : X \times X \to Z$ be a mapping satisfying the following:*

(i) *f is continuous;*

(ii) *$f(x,\cdot)$ is C-quasiconcave for each $x \in X$;*

(iii) *$Max_w \bigcup_{x\in X} f(x,x) \subset f(x,x) + C$ for each $x \in X$.*

Then

$$\text{Max}_w \bigcup_{x\in X} f(x,x) \subset \text{Min} \bigcup_{x\in X} \text{Max}_w f(x,X) + C. \tag{3.5}$$

Proof. We first show that there exists $x_0 \in X$ such that

$$f(x_0,x_0) \in \text{Max}_w f(x_0,X). \tag{3.6}$$

Define a set-valued mapping $T : X \to 2^X$ by

$$T(x) = \{y \in X : f(x,y) \in f(x,x) + \text{int}C\},\ \forall x \in X.$$

It is obvious that, for any $x \in X$, $x \notin T(x)$. By condition (ii), we know that $T(x)$ is Γ-convex for each $x \in X$. Moreover, since f is continuous,

$$T^{-1}(y) = \{x \in X : f(x,y) \in f(x,x) + \text{int}C\}$$

is open for each $y \in X$. By Lemma 2.5, T has a maximal element $x_0 \in X$, i.e., $T(x_0) = \emptyset$. Hence,

$$f(x_0,y) \notin f(x_0,x_0) + \text{int}\, C,\ \forall y \in X.$$

By the definition of weakly maximal points, we know that (3.6) holds. Now we prove that (3.5) is true. In fact, for each $\bar{z} \in \text{Max}_w \bigcup_{x\in X} f(x,x)$, it follows from condition (iii) and (3.6), we have

$$\begin{aligned} \bar{z} &\in f(x_0,x_0) + C \\ &\subset \text{Max}_w f(x_0,X) + C \\ &\subset \bigcup_{x\in X} \text{Max}_w f(x,X) + C \\ &\subset \text{Min} \bigcup_{x\in X} \text{Max}_w f(x,X) + C + C \\ &= \text{Min} \bigcup_{x\in X} \text{Max}_w f(x,X) + C, \end{aligned}$$

which implies that (3.5) is true. This completes the proof.

Remark 3.4. (a) If E is a Hausdorff topological vector space and $\Gamma = \text{co}$, then Theorem 3.3 reduces to Theorem 4 in Li and Wang [18]; (b) The proof of Theorem 3.3 is quite different from that of Theorem 4 in [18]; (c) Condition (iii) of Theorem 3.3 is similar to the assumption in Ferro [14]; (d) Condition (iii) is always true if f is real-valued (see [18]).

Remark 3.5. By (2.1), it is easy to see that (3.1), (3.2) and (3.5) can be rewritten, respectively, as follows:

$$\text{Min}_w \bigcup_{x\in X} \text{Max}_w f(x, X) \subset \text{Max}_w \bigcup_{x\in X} f(x, x) + Z \setminus \text{int}C,$$

$$\text{Max}_w \bigcup_{x\in X} f(x, x) \subset \text{Min}_w \bigcup_{x\in X} \text{Max}_w f(x, X) + Z \setminus (-\text{int}C),$$

$$\text{Max}_w \bigcup_{x\in X} f(x, x) \subset \text{Min}_w \bigcup_{x\in X} \text{Max}_w f(x, X) + C.$$

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