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NOTE ON F-IMPLICIT GENERALIZED VECTOR VARIATIONAL INEQUALITIES

Yen-Cherng Lin and Mu-Ming Wong*

Abstract. In this paper, we deal with weak and strong solutions to F-implicit generalized vector variational inequalities and F-implicit generalized (weak) vector variational inequalities. Several results of the existence for the weak solutions and strong solutions to both problems are derived.

1. Introduction and Preliminaries

Let X be arbitrary real normed space with dual space $X^\star$ and $(\cdot,\cdot)$ be the dual pair of $X^\star$ and X. Let K be a nonempty closed convex set of X. The mappings $F : K \to \mathbb{R}$ and $g : K \to K$ and $T : K \to 2^{X^*}$ are given. In 2006, Zeng et. al.[3] introduced and discussed the F-Implicit generalized variational inequality problem: find $\bar{x} \in K$ such that

$$\sup_{s\in T(\bar{x})} (s, y - g(\bar{x})) + F(y) - F(g(\bar{x})) \geq 0$$

for all $y \in K$.

We will generalize some results of the F-Implicit generalized variational inequality problem for the vector case with moving cone in the Section 2.

Let X, Y be arbitrary real Housdorff topological vector spaces. Let K be a nonempty set of X, $C : K \to 2^Y$ a set-valued mapping such that for each $x \in K$, $C(x)$ is a proper closed convex and pointed cone with apex at the origin and $intC(x) \neq \emptyset$, that is, for each $x \in K$, $C(x)$ is proper closed with $intC(x) \neq \emptyset$ and satisfied (1) $\lambda C(x) \subseteq C(x)$, $\forall \lambda > 0$;(2) $C(x) + C(x) \subseteq C(x)$; and (3) $C(x) \cap (-C(x)) = \{0\}$.

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*Corresponding author.

For each $x \in K$, we can define relations "$\geq_{C(x)}$" and "$\ngeq_{C(x)}$" as follows: (1) $y \geq_{C(x)} z \Leftrightarrow y - z \in C(x)$; and (2) $y \ngeq_{C(x)} z \Leftrightarrow y - z \notin C(x)$. The notation "$z \leq_{C(x)} y$" is equivalent to "$y \geq_{C(x)} z$" and "$z \nleq_{C(x)} y$" is equivalent to "$y \ngeq_{C(x)} z$". Throughout this paper, for each $x \in K$, the relation "$y\rho_x z$" represents one and only one of the following relations: (1) $y \geq_{C(x)} z$, (2) $y \nleq_{\mathrm{int}C(x)} z$, for all $y, z \in Y$. The relation "$y\rho_x^c z$" represents one and only one of the corresponding relations: (1) $y \ngeq_{C(x)} z$, (2) $y \leq_{\mathrm{int}C(x)} z$, for all $y, z \in Y$.

Let $L(X, Y)$ be the space of all continuous linear mappings from X to Y. The mappings $F : K \to Y$, $g : K \to K$, $A : L(X, Y) \to L(X, Y)$ and $T : K \to 2^{L(X,Y)}$ are given. Now, we consider the F-implicit generalized vector variational inequality (FIGVVI) as follows: find an $\bar{x} \in K$ such that for each $y \in K$, there is $\bar{s} \in T(\bar{x})$ satisfying

$$\{(A\bar{s}, y - g(\bar{x})) + F(y) - F(g(\bar{x}))\}\rho_{\bar{x}}0 \tag{1}$$

and we call such a solution a weak solution to FIGVVI. For the case that $\bar{s}$ does not depend on y, that is, find an $\bar{x} \in K$ with some $\bar{s} \in T(\bar{x})$ such that

$$\{(A\bar{s}, y - g(\bar{x})) + F(y) - F(g(\bar{x}))\}\rho_{\bar{x}}0 \tag{2}$$

for all $y \in K$, we call such solution a strong solution to FIGVVI.

In this paper, we aim to derive some solvabilities for these two kinds of F-implicit generalized vector variational inequality problem. We note that existence and solution methods for similar problems in scalar and vector cases have been investigated extensively in for example [6-29]. Now let us list the basic assumptions for the mappings $F : K \to Y$, $g : K \to K$, $A : L(X, Y) \to L(X, Y)$, $T : K \to 2^{L(X,Y)}$, $C : K \to 2^Y$ and $\nu : K \times K \to Y$ as follows.

Hypothesis (H_0). *Let X, Y be real Hausdorff topological vector spaces, K a nonempty closed convex subset of X. The set-valued mapping $C : K \to 2^Y$ is such that for any fixed $x \in K$, $C(x)$ is a proper closed convex and pointed cone with apex at the origin and $intC(x) \neq \emptyset$.*

Hypothesis (H_a). *The mappings $A : L(X, Y) \to L(X, Y)$, $F : K \to Y$, $g : K \to K$, $T : K \to 2^{L(X,Y)}$ and $\nu : K \times K \to Y$. Suppose that*

(a_1) *for each $x \in K$, there is $s \in T(x)$ such that for all $y \in K$,*

$$F(y) - F(g(x)) - (\nu(x, y) - (As, y - g(x))) \in C(x);$$

(a_2) *there is a nonempty compact convex subset D of K, such that for every $x \in K \setminus D$, there is $y \in D$ such that for all $s \in T(x)$,*

$$(As, y - g(x))\rho_x^c(F(g(x)) - F(y)).$$

Hypothesis (H_b) *The vector-valued mapping* $\nu : K \times K \to Y$ *is such that*

(b_1) $\nu(x,x) \in C(x)$ *for all* $x \in K$*;*

(b_2) *for each* $x \in K$*, the set* $\{y \in K : \nu(x,y) \notin C(x)\}$ *is convex.*

We note that the hypothesis (a_1) is equivalent to the statement "for each $x \in K$, there is $s \in T(x)$ such that for all $y \in K$, $\nu(x,y) - (As, y - g(x)) \leq_{C(x)} F(y) - F(g(x))$" and is weaker than (1) and (2) because the value of $\nu(x,y)$ need not belong to $C(x)$. For more detail, we refer the reader to Example 2.1 below. Let us first recall the following results.

Fan's Lemma. ([1]). *Let* K *be a nonempty subset of Hausdorff topological vector space* X*. Let* $G : K \to 2^X$ *be a KKM mapping such that for any* $y \in K$*,* $G(y)$ *is closed and* $G(y^*)$ *is compact for some* $y^* \in K$*. Then there exists* $x^* \in K$ *such that* $x^* \in G(y)$ *for all* $y \in K$*.*

Definition 1.1. ([5]). Let Ω be a vector space, Σ a topological vector space, K a nonempty convex subset of Ω, $C : K \to 2^\Sigma$ a set-valued mapping such that for each $x \in K$, $C(x)$ is a proper closed convex and pointed cone with apex at the origin and $intC(x) \neq \emptyset$. For any fixed $x \in K$, $\varphi : K \to \Sigma$ is said to be

(1) *$C(x)$-convex* if $\varphi(tx_1 + (1-t)x_2) \leq_{C(x)} t\varphi(x_1) + (1-t)\varphi(x_2)$ for every $x_1,\ x_2 \in K$ and $t \in [0,1]$;

(2) *properly quasi $C(x)$-convex* if we have either

$$\varphi(tx_1 + (1-t)x_2) \leq_{C(x)} \varphi(x_1)$$

or

$$\varphi(tx_1 + (1-t)x_2) \leq_{C(x)} \varphi(x_2)$$

for every $x_1,\ x_2 \in K$ and $t \in [0,1]$.

The following definition can also be found in [5].

Definition 1.2. Let Ω be a vector space, Σ a topological vector space, K a nonempty convex subset of Ω, $C : K \to 2^\Sigma$ a set-valued mapping such that for any fixed $x \in K$, $C(x)$ is a proper closed convex and pointed cone with apex at the origin and $intC(x) \neq \emptyset$, A a nonempty subset of Σ, then for any fixed $x \in K$,

(1) a point $z \in A$ is called a *minimal point* of A with respect to the cone $C(x)$ if $A \cap (z - C(x)) = \{z\}$; $\mathrm{Min}^{C(x)} A$ is the set of all minimal points of A with respect to the cone $C(x)$;

(2) a point $z \in A$ is called a *maximal point* of A with respect to the cone $C(x)$ if $A \cap (z + C(x)) = \{z\}$; $\mathrm{Max}^{C(x)} A$ is the set of all maximal points of A with respect to the cone $C(x)$;

(3) a point $z \in A$ is called a *weakly minimal point* of A with respect to the cone $C(x)$ if $A \cap (z - intC(x)) = \emptyset$; $\mathrm{Min}_w^{C(x)} A$ is the set of all weakly minimal points of A with respect to the cone $C(x)$; and

(4) a point $z \in A$ is called a *weakly maximal point* of A with respect to the cone $C(x)$ if $A \cap (z + intC(x)) = \emptyset$; $\mathrm{Max}_w^{C(x)} A$ is the set of all weakly maximal points of A with respect to the cone $C(x)$.

Lemma 1.1. *Let Σ be an arbitrary real Housdorff topological vector space, K a nonempty set, $C : K \to 2^{\Sigma}$ a set-valued mapping such that for any fixed $x \in K$, $C(x)$ is a proper closed convex and pointed cone with apex at the origin. For any fixed $x \in K$, If $y \geq_{C(x)} 0$ and $y \leq_{C(x)} 0$, then $y = 0$.*

Lemma 1.2. ([4]). *Let X, Y and Z be real topological vector spaces, K and C be nonempty subsets of X and Y, respectively. Let $F : K \times C \to 2^Z$, $S : K \to 2^Y$ be multi-valued mappings. If both F and S are upper semicontinuous with compact values, then the multi-valued mapping $T : K \to 2^Z$ defined by $T(x) = \bigcup_{y \in S(x)} F(x, y) = F(x, S(x))$ is upper semicontinuous with nonempty compact values.*

2. F-Implicit Generalized Vector Variational Inequalities Problems

Now, we state and show our main results of solvabilities for F-implicit generalized vector variational inequalities problems with moving cone.

Theorem 2.1. *Let X, Y, K, C, A, F, g, T and ν satisfy hypothesis (H_0), (H_a) and (H_b). Suppose that for each $y \in K$, the set*

(3) $$\{x \in K : (As, y - g(x))\rho_x^c(F(g(x)) - F(y)) \text{ for all } s \in T(x)\}$$

is open in K. Then FIGVVI has a weak solution. That is, there is an $\bar{x} \in K$ such that for all $y \in K$, there is $\bar{s} \in T(\bar{x})$ satisfying

$$\{(A\bar{s}, y - g(\bar{x})) + F(y) - F(g(\bar{x}))\}\rho_{\bar{x}} 0.$$

Proof. Define $\Omega : K \to 2^D$ by

$$\Omega(y) = \{x \in D : (As, y - g(x))\rho_x(F(g(x)) - F(y)) \text{ for some } s \in T(x)\}$$

for all $y \in K$. From condition (3) we see that for each $y \in K$, the set $\Omega(y)$ is closed in K and hence it is compact in D because of the compactness of D.

Next, we claim that the family $\{\Omega(y) : y \in K\}$ has the finite intersection property, then the whole intersection $\cap_{y \in K} \Omega(y)$ is nonempty and any element in the intersection $\cap_{y \in K}\Omega(y)$ is a weak solution of FIGVVI. Indeed, for any given nonempty finite subset N of K. Let $D_N = co\{D \cup N\}$, the convex hull of $D \cup N$. Then D_N is a compact convex subset of K. Define the mappings $\mathfrak{S},\ \mathfrak{R} : D_N \to 2^{D_N}$, respectively, by

$$\mathfrak{S}(y) = \{x \in D_N : (As, y - g(x))\rho_x(F(g(x)) - F(y)) \text{ for some } s \in T(x)\},$$

and

$$\mathfrak{R}(y) = \{x \in D_N : \nu(x, y) \in C(x)\},$$

for each $y \in D_N$. From the conditions (b_1) of (H_b) and (a_1) of (H_a), we have

$$\nu(y, y) \in C(y) \text{ for all } y \in D_N, \tag{4}$$

and for each $y \in K$, there is an $s \in T(y)$ such that

$$F(y) - F(g(y)) + (As, y - g(y)) - \nu(y, y) \in C(y).$$

Hence $(As, y - g(y)) + F(y) - F(g(y)) \in C(y)$ and then $y \in \mathfrak{S}(y)$ for all $y \in D_N$. We can easily see that $\mathfrak{S}$ has closed values in D_N. Since, for each $y \in D_N$, $\Omega(y) = \mathfrak{S}(y) \cap D$, if we prove that the whole intersection of the family $\{\mathfrak{S}(y) : y \in D_N\}$ is nonempty, we can deduce that the family $\{\Omega(y) : y \in K\}$ has finite intersection property because $N \subset D_N$ and due to the condition (a_2) of (H_a). In order to deduce the conclusion of our theorem, we can apply the Fan's Lemma if we claim that $\mathfrak{S}$ is a KKM mapping. Indeed, if $\mathfrak{S}$ is not a KKM mapping, neither is $\mathfrak{R}$ since $\mathfrak{R}(y) \subset \mathfrak{S}(y)$ for each $y \in D_N$. Then there is a nonempty finite subset M of D_N such that

$$coM \not\subset \cup_{u \in M}\mathfrak{R}(u).$$

Thus there is an element $\bar{u} \in coM \subset D_N$ such that $\bar{u} \notin \mathfrak{R}(u)$ for all $u \in M$, that is, $0 \not\leq_{C(\bar{u})} \nu(\bar{u}, u)$ for all $u \in M$. By (b_2) of (H_b), we have

$$\bar{u} \in coM \subset \{y \in K : \nu(\bar{u}, y) \notin C(\bar{u})\}$$

and hence $\nu(\bar{u}, \bar{u}) \notin C(\bar{u})$ which contradicts (4). Hence $\mathfrak{R}$ is a KKM mapping, and so is $\mathfrak{S}$. This completes the proof. ∎

If condition (3) is replaced by stronger conditions as follows:

$$\begin{aligned} &F \textit{ is continuous on } K, \textit{ the mappings } A : L(X,Y) \to L(X,Y), \\ &\quad g : K \to K \textit{ are continuous and} \\ &T : K \to 2^{L(X,Y)} \textit{ is upper semicontinuous} \\ &\quad \textit{with nonempty compact values,} \end{aligned} \tag{5}$$

then from Lemma 1.2, we know that the condition (5) of Theorem 2.1 is always true. Hence we have the following corollary.

Corollary 2.1. *Under the framework of Theorem 2.1 except condition (3). Suppose that the mappings $A : L(X, Y) \to L(X, Y)$, $F : K \to Y$ and $g : K \to K$ are continuous, $T : K \to 2^{L(X,Y)}$ is upper semicontinuous with nonempty compact values. Then there exists an $\bar{x} \in K$ which is a weak solution to FIG VVI. That is, there is an $\bar{x} \in K$ such that for all $y \in K$, there is $\bar{s} \in T(\bar{x})$ satisfying*

$$\{(A\bar{s}, y - g(\bar{x})) + F(y) - F(g(\bar{x}))\}\rho_{\bar{x}}0.$$

Remark. If the condition (5) holds, then condition (3) of Theorem 2.1 can be omitted automatically. Furthermore, we note that if $F,\ A,\ g,\ T$ are single-valued continuous mappings, then the condition (3) of Theorem 2.1 is fulfilled. Hence, we have the following corollary in the Hausdorff topological vector spaces settings with moving cone.

Corollary 2.2. *Let $X,\ Y,\ K,\ C$ satisfy hypothesis (H_0). Let the mappings $F : K \to Y$ and $g : K \to K$ be continuous, $T : K \to Y$ be continuous and the mapping ν satisfy hypothesis (H_b). Suppose that*

(1) $F(y) - F(g(x)) - \nu(x, y) + (T(x), y - g(x)) \in C(x)\ \forall x, y \in K$;

(2) *there is a nonempty compact convex subset D of K, such that for every $x \in K \setminus D$, there is a $y \in D$ such that*

$$((T(x), y - g(x)))\rho_x^c(F(g(x)) - F(y)).$$

Then there is an $\bar{x} \in K$ such that

$$(T(\bar{x}), y - g(\bar{x})) + F(y) - F(g(\bar{x}))\rho_{\bar{x}}0 \tag{6}$$

for all $y \in K$. Furthermore, if the mapping C is closed; that is, C satisfies the condition

$$\text{net } \alpha_\tau \to \alpha_0,\ \text{net } \beta_\tau \to \beta_0,\ \beta_\tau \in C(\alpha_\tau)\ \Rightarrow\ \beta_0 \in C(\alpha_0), \tag{7}$$

then the set of all solutions of the inequality (6) is compact.

Remark. For the compactness of the solution set in Corollary 2.2, we can use the continuities of $T,\ F$ and g, the condition (7) and the argument of Corollary 2.3 below to prove it, so we omit the proof.

We note that if we take $C(x) = C$ for all $x \in K$, then Corollary 2.2 can be reduced to [6, Theorem 3.2]. In order to discuss the results of existence for the strong solution to FIGVVI, we introduce the geometric conditions (GC_K) for the mappings A, T, g and F with respect to the set K as follows.

Geometric Conditions (GC_K)

(G_1) $Max^{C(\bar{x})} \bigcup_{s\in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{x\in K}\{(As, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\} \subset Min_w^{C(\bar{x})} \bigcup_{x\in K}\{(As, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\} + C(\bar{x}), \forall s \in T(\bar{x})$;

(G_2) for any fixed $x \in K$, if $\delta \in Max^{C(\bar{x})} \bigcup_{s\in T(\bar{x})}\{(As, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\}$ and δ cannot be compared with $(A\bar{s}, x - g(\bar{x})) - F(g(\bar{x})) + F(x)$ which does not equal to δ, then $\delta\rho_{\bar{x}}0$; and

(G_3) if $Max^{C(\bar{x})} \bigcup_{s\in T(\bar{x})} Min_w^{C(\bar{x})} \bigcup_{x\in K}\{(As, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\} \subset B(\bar{x})$, there exists an $s \in T(\bar{x})$ such that $Min_w^{C(\bar{x})} \bigcup_{x\in K}\{(As, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\} \subset B(\bar{x})$, where $B(x)$ is either $C(x)$ or $Y \setminus (-intC(x))$ that corresponds to ρ_x.

The conditions (GC_K) are obviously fulfilled that if $Y = \mathbb{R}$.

Theorem 2.2. *Let X, Y, K, C, D, A, F, g and ν satisfy hypothesis (H_0), (H_a) and (H_b). If the condition (3) of Theorem 2.1 holds, then we have a weak solution $\bar{x}$ to FIGVVI. In addition, if K is compact, $T(\bar{x})$ is convex, F is $C(\bar{x})$-convex and continuous on K, the mapping $A : L(X, Y) \to L(X, Y)$ is continuous, $T : K \to 2^{L(X,Y)}$ is upper semicontinuous with nonempty compact values and the mapping $s \to -(As, x - g(\bar{x}))$ is properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $x \in K$. Assume that the conditions (GC_K) hold. Then $\bar{x}$ is a strong solution to FIGVVI, that is, there exists $\bar{s} \in T(\bar{x})$ such that*

$$\{(A\bar{s}, x - g(\bar{x})) + F(x) - F(g(\bar{x}))\}\rho_{\bar{x}}0$$

for all $x \in K$.

Proof. Since F is $C(\bar{x})$-convex on K, the mapping $x \to (As, x - g(\bar{x})) - F(g(\bar{x})) + F(x)$ is $C(\bar{x})$-convex on K. Since the mapping $s \to -(As, x - g(\bar{x}))$ is properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $x \in K$, so is the mapping $s \to -(As, x - g(\bar{x})) + F(g(\bar{x})) - F(x)$ for each $x \in K$.

From Theorem 2.1, we know that $\bar{x} \in K$ such that for all $x \in K$, there is $\bar{s} \in T(\bar{x})$ such that (1) holds. Then $\forall\gamma \in \mathrm{Min} \cup_{x\in K} \mathrm{Max} \cup_{s\in T(\bar{x})} \{(As, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\}$, by ($G_2$) we have $\gamma\rho_{\bar{x}}0$. From condition (G_1), the convexity of $T(\bar{x})$ and [2, Theorem 3.1], we have for every $\alpha \in \mathrm{Max} \cup_{s\in T(\bar{x})} \mathrm{Min}_w \cup_{x\in K} \{(As, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\}$, $\alpha\rho_{\bar{x}}0$. This implies that

$$\mathrm{Max} \cup_{s\in T(\bar{x})} \mathrm{Min}_w \cup_{x\in K} \{(As, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\} \subset B(\bar{x}).$$

From (G_3), there is an $\bar{s} \in T(\bar{x})$ such that $\mathrm{Min}_w \cup_{x\in K} \{(A\bar{s}, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\} \subset B(\bar{x})$. Hence, we know that $\forall \tau \in \bigcup_{x\in K}\{(A\bar{s}, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\}$, we have $\tau \rho_{\bar{x}} 0$. Hence there exists $\bar{s} \in T(\bar{x})$ such that

$$\{(A\bar{s}, x - g(\bar{x})) - F(g(\bar{x})) + F(x)\}\rho_{\bar{x}} 0$$

for all $x \in K$ and $\bar{x}$ is a strong solution to FIGVVI. ■

Corollary 2.3. *Under the framework of Theorem 2.2 except X, g and C. Let X be a real Banach space, $g : K \to K$ is continuous and $C : K \to 2^Y$ is a closed mapping. Then the set of all strong solutions to FIGVVI is compact.*

Proof. Let us claim that the solution set to FIGVVI is compact. It is sufficient to show that the solution set is closed due to the coercivity condition (4) of Theorem 2.1. To this end, let Γ denote the solution set of FIGVVI. Suppose that $\{x_n\} \subset \Gamma$ which converges to some p. Fix $y \in K$. For each $n \in \mathbb{N}$, there is $s_n \in T(x_n)$ such that

$$(As_n, y - g(x_n))\rho_{x_n}(F(g(x_n)) - F(y)). \tag{8}$$

Since T is upper semicontinuous with nonempty compact values and the set $\{x_n\} \cup \{p\}$ is compact, it follows that $T(\{x_n\} \cup \{p\})$ is compact. Therefore, without loss of generality, we may assume that the sequence $\{s_n\}$ converges to some s. Then $s \in T(p)$ and

$$(As_n, y - g(x_n)) - F(g(x_n)) + F(y) \in B(x_n).$$

We note that

$$\begin{aligned} &-F(g(x_n)) + F(y) + (As_n, y - g(x_n)) \\ &= -F(g(x_n)) + F(y) + (As_n - As, y - g(x_n)) + (As, y - g(x_n)) \\ &= -F(g(x_n)) + F(y) + (As_n - As, y - g(x_n)) \\ &\quad +(As, (y - g(x_n)) - (y - g(p))) + (As, y - g(p)). \end{aligned} \tag{9}$$

Since $\{x_n\}\cup\{p\}$ is compact and g is continuous, $g(\{x_n\}\cup\{p\})$ is also compact. Hence it is bounded. Thus $(As_n - As, y - g(x_n)) \to 0$ as $n \to \infty$ and

$$(As, (y - g(x_n)) - (y - g(p))) = (As, g(p) - g(x_n)) \to 0, \ n \to \infty$$

by the continuity of g. Since F is continuous and the condition (7), from (8) and (9) we have

$$\begin{aligned} &-F(g(p)) + F(y) + (As, y - g(p)) \\ &= \lim_{n\to\infty} -F(g(x_n)) + F(y) + (As_n, y - g(x_n)) \in B(p). \end{aligned}$$

We then obtain

$$\{(As, y - g(p)) + F(y) - F(g(p))\}\rho_p 0.$$

Hence $p \in \Gamma$ and Γ is closed. ■

We would like to point out that the conditions ((GC_K) and (7) of Theorem 2.2 are fulfilled if we take $Y = \mathbb{R}$, and $C(x) = [0, \infty)$ for each $x \in K$. The following is a concrete example for Theorem 2.1 and Theorem 2.2.

Example 2.1. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$, $K = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, $F \equiv const.$,

$$C(x) = \begin{cases} \{(y_1, y_2) \in Y : 0 \leq y_1 \leq y_2\}, & 0 \leq x_1 \neq x_2 \leq 1, \\ \{(y_1, y_2) \in Y : y_1 \geq 0, y_2 \geq 0\}, & 0 \leq x_1 = x_2 \leq 1, \end{cases}$$

for all $x = (x_1, x_2) \in K$ and $D = \{(y_1, y_2) \in K : y_1 + y_2 \leq 1\}$. Choose $A : L(X, Y) \to L(X, Y)$ and g to be identity mapping on K. Let $T : K \to 2^{L(X,Y)}$ be defined by

$$T(x) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & \displaystyle\int_0^{x_2+1} 2t dt \end{bmatrix} \right\} \subset L(X, Y)$$

for every $x \in K$. Choose $\nu : K \times K \to Y$ by

$$\nu(x, y) = \begin{cases} (0, 0) & y_2 \geq x_2, 0 \leq x_1 \leq x_2 \leq 1 \\ (0, (x_2 + 1)^2(y_2 - x_2)), & y_2 \geq x_2, 0 \leq x_2 < x_1 \leq 1, \\ (0, (x_2 + 1)^2(y_2 - x_2)), & y_2 \leq x_2. \end{cases}$$

Then all conditions of both Theorem 2.1 and Theorem 2.2 are satisfied. By Theorem 2.2, the FIGVVI has a strong solution. A simple geometric discussion tells us that $\bar{x} = (0, 0)$ is a strong solution to FIGVVI.

Next, we consider the result of existence theorem for the strong solutions to FIGVVI without compactness. The proof can be obtained by standard argument so it will be omitted.

Theorem 2.3. *Let X be a finite dimensional real Banach space, Y, K, C, D, A, F, g, T and ν as in Theorem 2.1. Under the assumptions of Theorem 2.1, we have a weak solution $\bar{x}$ to FIGVVI. In addition, if $T(\bar{x})$ is convex, F is $C(\bar{x})$-convex and continuous on K, $C : K \to 2^Y$ is a closed mapping, the mappings $A : L(X, Y) \to L(X, Y)$, $g : K \to K$ are continuous, $T : K \to 2^{L(X,Y)}$ is upper semicontinuous with nonempty compact values and the mapping $s \to -(As, x - g(\bar{x}))$ is properly quasi $C(\bar{x})$-convex on $T(\bar{x})$ for each $x \in K$. Assume that for some $r > \|g(\bar{x})\|$,* *the conditions(GC_{K_r}) with respect to the set $K_r \doteq \bar{B}(0,r) \cap K$ are satisfied. Then $\bar{x}$ is also a strong solution to FIGVVI, that is, there exists $\bar{s} \in T(\bar{x})$ such that*

$$\{(A\bar{s}, x - g(\bar{x})) + F(x) - F(g(\bar{x}))\}\rho_{\bar{x}}0$$

for all $x \in K$. Furthermore, the set of all strong solutions to FIGVVI is compact.

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Yen-Cherng Lin
Department of Occupational Safety and Health,
China Medical University,
Taichung 40421,
Taiwan

Mu-Ming Wong
Department of Applied Mathematics,
Chung Yuan Christian University,
Chung-Li 32023,
Taiwan
E-mail: mmwong@cycu.edu.tw