TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 1, pp. 273-286, February 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

LINEAR 2-ARBORICITY OF THE COMPLETE GRAPH

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Abstract. A *linear k-forest* is a graph whose components are paths with lengths at most k. The minimum number of linear k-forests needed to decompose a graph G is the *linear k-arboricity* of G and denoted by $la_k(G)$. In this paper, we settle the cases left in determining the linear 2-arboricity of the complete graph K_n. Mainly, we prove that $la_2(K_{12t+10}) = la_2(K_{12t+11}) = 9t + 8$ for any $t \geq 0$.

1. INTRODUCTION

Throughout this paper, all graphs considered are finite, undirected, loopless, and without multiple edges.

A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition $G_1, \ldots, G_d$, then we say that G can be decomposed into $G_1, \ldots, G_d$ or $G_1, \ldots, G_d$ decompose G.

A *complete graph* is a graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n. A *linear k-forest* is a graph whose components are paths with lengths at most k. The *linear k-arboricity* of a graph G, denoted by $la_k(G)$, is the minimum number of linear k-forests needed to decompose G.

The notion of linear k-arboricity was defined by Habib and Peroche in [9]. It is a natural generalization of *edge coloring*. Clearly, a linear 1-forest is induced by a matching and $la_1(G) = \chi'(G)$ which is the *chromatic index* of a graph G. It is also a refinement of the concept of *linear arboricity*, introduced earlier by Harary in [11], in which the paths have no length constraints.

In 1982, Habib and Peroche [10] made the following conjecture:

Received February 8, 2006, accepted March 13, 2007.
Communicated by Xuding Zhu.
2000 *Mathematics Subject Classification*: 05C38, 05C70.
Key words and phrases: Linear k-forest, Linear k-arboricity, Complete graph.

Conjecture 1.1. If G is a graph with maximum degree $\Delta(G)$ and $k \geq 2$, then

$$la_k(G) \leq \begin{cases} \left\lceil \dfrac{\Delta(G) \cdot |V(G)|}{2\left\lfloor \frac{k \cdot |V(G)|}{k+1} \right\rfloor} \right\rceil & \text{if } \Delta(G) = |V(G)| - 1 \text{ and} \\ \left\lceil \dfrac{\Delta(G) \cdot |V(G)| + 1}{2\left\lfloor \frac{k \cdot |V(G)|}{k+1} \right\rfloor} \right\rceil & \text{if } \Delta(G) < |V(G)| - 1. \end{cases}$$

So far, quite a few results on the verification of this conjecture have been obtained in the literature, especially for some graphs with particular properties, see [1, 2, 3, 4, 5, 8, 12, 13]. Among them, Bermond et al. [1] determined the linear 2-arboricity of the complete graph K_n almost completely. They had the following result:

Theorem 1.2. *For* $n \not\equiv 10, 11 \pmod{12}$, $la_2(K_n) = \left\lceil \frac{n(n-1)}{2\lfloor \frac{2n}{3} \rfloor} \right\rceil$.

Later, Chen et al. [4] derived a similar result by using the ideas from *latin squares*. They claimed that the following theorem is proved.

Theorem 1.3. $la_2(K_{3u}) = \left\lceil \frac{3(3u-1)}{4} \right\rceil$, $la_2(K_{3u+1}) = \left\lceil \frac{3(3u+1)}{4} \right\rceil$, *and* $la_2(K_{3u+2}) = \left\lceil \frac{(3u+2)(3u+1)}{2(2u+1)} \right\rceil$ *except possibly if* $3u + 1 \in \{49, 52, 58\}$.

Unfortunately, their result mentioned in Corollary 4.7 of [4] that $la_2(K_{12t+11}) = 9t + 9$ is not coherent to the theorem they proved, the expected linear 2-arboricity of K_{12t+11} is $9t + 8$.

In this paper, we will prove that $la_2(K_{12t+10}) = la_2(K_{12t+11}) = 9t+8$ for any $t \geq 0$. Thus, the exact value of $la_2(K_n)$ is completely determined. Furthermore, the results obtained are coherent with the corresponding cases of Conjecture 1.1.

2. Preliminaries

First, we need some definitions. A graph G is *m-partite* if $V(G)$ can be partitioned into m independent sets called *partite sets* of G. When $m = 2$, we also say that G is bipartite. A *complete m-partite graph* is an m-partite graph G such that the edge $uv \in E(G)$ if and only if u and v are in different partite sets. When $m \geq 2$, we write $K_{n_1,n_2,\ldots,n_m}$ for the complete m-partite graph with partite sets of sizes $n_1, n_2, \ldots, n_m$.

Let $S = \{1, 2, \ldots, \nu\}$ be a set of ν elements. A *latin square of order ν* is a $\nu \times \nu$ array in which each cell contains a single element from S, such that each element occurs exactly once in each row and exactly once in each column. A latin square $L = [\ell_{ij}]$ is *idempotent* if $\ell_{ii} = i$ for all $1 \le i \le \nu$, and *commutative* if $\ell_{ij} = \ell_{ji}$ for all $1 \le i, j \le \nu$. In [6], the following result has been mentioned.

Theorem 2.1. *An idempotent commutative latin square of order ν exists if and only if ν is odd.*

An *incomplete latin square* of order ν, denoted by ILS$(\nu; b_1, b_2, \ldots, b_k)$, is a $\nu \times \nu$ array A with entries from a set B of size ν, where $B_i \subseteq B$ for $1 \le i \le k$ with $|B_i| = b_i$, and $B_i \cap B_j = \emptyset$ for $1 \le i \ne j \le k$. Moreover,

1. each cell of A is empty or contains an element of B;
2. the subarrays indexed by $B_i \times B_i$ are empty (and called *holes*); and
3. the elements in row or column b are exactly those of $B - B_i$ if $b \in B_i$, and of B otherwise.

A *partitioned incomplete latin square* PILS$(\nu; b_1, b_2, \ldots, b_k)$ is an incomplete latin square of order ν with $b_1 + b_2 + \cdots + b_k = \nu$. Figure 1 is an example of a commutative PILS$(8; 2, 2, 2, 2)$. It is worthy of noting that, Fu and Fu [7] proved that:

Theorem 2.2. *For any $k \ge 3$, a commutative partitioned incomplete latin square PILS$(2k; 2, 2, \ldots, 2)$ exists.*

Next, we state some properties of $la_k(G)$.

		8	6	7	3	4	5
		5	7	4	8	3	6
8	5			1	7	6	2
6	7			8	2	5	1
7	4	1	8			2	3
3	8	7	2			1	4
4	3	6	5	2	1		
5	6	2	1	3	4		

Fig. 1. A commutative PILS$(8; 2, 2, 2, 2)$.

Lemma 2.3. *If H is a subgraph of G, then $la_k(H) \le la_k(G)$.*

Lemma 2.4. *If a graph G is the edge-disjoint union of two subgraphs G_1 and G_2, then $la_k(G) \le la_k(G_1) + la_k(G_2)$.*

Lemma 2.5. $la_k(G) \geq \max\left\{\left\lceil \frac{\Delta(G)}{2} \right\rceil, \left\lceil \frac{|E(G)|}{\left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor} \right\rceil\right\}$.

Lemmas 2.3 and 2.4 are evident by the definition of linear k-arboricity. Since any vertex of a linear k-forest in a graph G has degree at most 2 and a linear k-forest in G has at most $\left\lfloor \frac{k|V(G)|}{k+1} \right\rfloor$ edges, we have Lemma 2.5.

3. Main Results

In what follows, for convenience, we use an $n \times n$ array to represent a linear k-forest decomposition of $K_fig.3n, n$ or K_n, which also shows an upper bound of $la_k(K_{n,n})$ or $la_k(K_n)$. Figure 2 is an example of $K_{12,12}$ with $la_2(K_{12,12}) \leq 9$. The entry w_{ij} in row i and column j means that the edge u_iv_j belongs to the linear 2-forest labelled by w_{ij}. In fact, $la_2(K_{12,12}) = 9$ since $la_2(K_{12,12}) \geq \left\lceil \frac{144}{\left\lfloor \frac{2 \cdot 24}{3} \right\rfloor} \right\rceil = 9$ by Lemma 2.5.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
u_1	1	8	4	4	3	3	8	6	9	2	5	7
u_2	1	4	6	6	2	2	7	7	9	3	5	8
u_3	2	4	7	9	1	1	5	9	5	3	6	8
u_4	2	5	7	1	8	4	4	3	3	8	6	9
u_5	3	5	8	1	4	6	6	2	2	7	7	9
u_6	3	6	8	2	4	7	9	1	1	5	9	5
u_7	8	6	9	2	5	7	1	8	4	4	3	3
u_8	7	7	9	3	5	8	1	4	6	6	2	2
u_9	5	9	5	3	6	8	2	4	7	9	1	1
u_{10}	4	3	3	8	6	9	2	5	7	1	8	4
u_{11}	6	2	2	7	7	9	3	5	8	1	4	6
u_{12}	9	1	1	5	9	5	3	6	8	2	4	7

Fig. 2. The array shows that $la_2(K_{12,12}) \leq 9$.

As we have seen in $W = [w_{ij}]$, a number occurs in each row and each column at most twice and furthermore if $w_{ij} = w_{i'j'}$ where $i \neq i'$ and $j \neq j'$, then $w_{ij'} \neq w_{ij}$ and $w_{i'j} \neq w_{ij}$. The condition on K_n is similar except the array $W = [w_{ij}]$ is symmetric, i.e., $w_{ij} = w_{ji}$ for all $i \neq j$, and w_{ii} is empty for each $i \in \{1, 2, \ldots, n\}$.

Now, we are ready to obtain the main results.

Proposition 3.1. $la_2(K_{11}) = 8$.

Proof. We construct the array in Figure 3 to show that $la_2(K_{11}) \leq 8$. On the other hand, by Lemma 2.5, $la_2(K_{11}) \geq \left\lceil \frac{55}{\lfloor \frac{2 \cdot 11}{3} \rfloor} \right\rceil = 8$. ■

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}
v_1		2	1	7	5	4	8	6	5	3	1
v_2	2		8	5	2	7	4	4	6	1	3
v_3	1	8		6	4	5	6	8	7	3	2
v_4	7	5	6		4	2	1	3	8	5	7
v_5	5	2	4	4		6	7	7	8	1	3
v_6	4	7	5	2	6		1	3	2	7	4
v_7	8	4	6	1	7	1		2	3	8	5
v_8	6	4	8	3	7	3	2		1	6	5
v_9	5	6	7	8	8	2	3	1		4	6
v_{10}	3	1	3	5	1	7	8	6	4		2
v_{11}	1	3	2	7	3	4	5	5	6	2	

Fig. 3. The array shows that $la_2(K_{11}) \leq 8$.

Proposition 3.2. $la_2(K_{23}) = 17$.

Proof. It is clear that K_{23} is an edge-disjoint union of $K_{12} \cup K_{11}$ and $K_{12,11}$. First, we decompose $(K_{12} \cup K_{11}) - M$ into 8 linear 2-forests where M is a matching of size 3 in K_{12}. Then, from the result $la_2(K_{12,12}) = 9$, we find a way to decompose $K_{12,11} \cup G[M]$ into 9 linear 2-forests where $G[M]$ is a subgraph of K_{23} induced by M.

Hence, we obtain the array in Figure 4 which shows that $la_2(K_{23}) \leq 8+9 = 17$ by Lemma 2.4. On the other hand, by Lemma 2.5, $la_2(K_{23}) \geq \left\lceil \frac{253}{\lfloor \frac{2 \cdot 23}{3} \rfloor} \right\rceil = 17$. ■

Proposition 3.3. $la_2(K_{n,n,n}) = \lceil \frac{3n}{2} \rceil$ *for any* $n \geq 0$.

Proof. Assume that the partite sets of $K_{n,n,n}$ are $V_1 = \{v_{1[1]}, v_{1[2]}, \ldots, v_{1[n]}\}$, $V_2 = \{v_{2[1]}, v_{2[2]}, \ldots, v_{2[n]}\}$, and $V_3 = \{v_{3[1]}, v_{3[2]}, \ldots, v_{3[n]}\}$. First, for all $1 \leq$

$\alpha \neq \beta \leq 3$, we use the notation $G(V_\alpha, V_\beta)$ to denote the subgraph of $K_{n,n,n}$ induced by V_α and V_β. Then $G(V_\alpha, V_\beta)$ is a complete bipartite graph $K_{n,n}$ and it is well-known that the edges of $K_{n,n}$ can be partitioned into n perfect matchings.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}	v_{20}	v_{21}	v_{22}	v_{23}
v_1		2	1	7	5	4	8	6	5	3	1	15	9	16	12	12	11	11	16	14	17	10	13
v_2	2		16	5	8	7	4	4	6	1	3	2	9	12	14	14	10	10	15	15	17	11	13
v_3	1	16		6	7	5	6	8	7	3	2	4	10	12	15	17	9	9	13	17	13	11	14
v_4	7	5	6		17	2	1	3	8	5	7	4	10	13	15	9	16	12	12	11	11	16	14
v_5	5	8	7	17		6	3	2	1	4	8	6	11	13	16	9	12	14	14	10	10	15	15
v_6	4	7	5	2	6		1	3	2	7	4	5	11	14	16	10	12	15	17	9	9	13	17
v_7	8	4	6	1	3	1		2	3	8	5	7	16	14	17	10	13	15	9	16	12	12	11
v_8	6	4	8	3	2	3	2		1	6	5	7	15	15	17	11	13	16	9	12	14	14	10
v_9	5	6	7	8	1	2	3	1		4	6	8	13	17	13	11	14	16	10	12	15	17	9
v_{10}	3	1	3	5	4	7	8	6	4		2	1	12	11	11	16	14	17	10	13	15	9	16
v_{11}	1	3	2	7	8	4	5	5	6	2		3	14	10	10	15	15	17	11	13	16	9	12
v_{12}	15	2	4	4	6	5	7	7	8	1	3		17	9	9	13	17	13	11	14	16	10	12
v_{13}	9	9	10	10	11	11	16	15	13	12	14	17		2	1	7	5	4	8	6	5	3	1
v_{14}	16	12	12	13	13	14	14	15	17	11	10	9	2		8	5	2	7	4	4	6	1	3
v_{15}	12	14	15	15	16	16	17	17	13	11	10	9	1	8		6	4	5	6	8	7	3	2
v_{16}	12	14	17	9	9	10	10	11	11	16	15	13	7	5	6		4	2	1	3	8	5	7
v_{17}	11	10	9	16	12	12	13	13	14	14	15	17	5	2	4	4		6	7	7	8	1	3
v_{18}	11	10	9	12	14	15	15	16	16	17	17	13	4	7	5	2	6		1	3	2	7	4
v_{19}	16	15	13	12	14	17	9	9	10	10	11	11	8	4	6	1	7	1		2	3	8	5
v_{20}	14	15	17	11	10	9	16	12	12	13	13	14	6	4	8	3	7	3	2		1	6	5
v_{21}	17	17	13	11	10	9	12	14	15	15	16	16	5	6	7	8	8	2	3	1		4	6
v_{22}	10	11	11	16	15	13	12	14	17	9	9	10	3	1	3	5	1	7	8	6	4		2
v_{23}	13	13	14	14	15	17	11	10	9	16	12	12	1	3	2	7	3	4	5	5	6	2	

Fig. 4. The array shows that $la_2(K_{23}) \leq 17$.

Next, we find that the edges of a union of any two perfect matchings in $G(V_1, V_2)$, $G(V_2, V_3)$, and $G(V_3, V_1)$ respectively can produce 3 linear 2-forests of $K_{n,n,n}$. Figure 5 shows an example of $K_{7,7,7}$. Hence, $la_2(K_{n,n,n}) \leq \lceil \frac{n}{2} \cdot 3 \rceil = \lceil \frac{3n}{2} \rceil$. On the other hand, by Lemma 2.5, $la_2(K_{n,n,n}) \geq \lceil \frac{3n}{2} \rceil$. ■

Proposition 3.4. $la_2(K_{35}) = 26$.

Proof. It is clear that K_{35} is an edge-disjoint union of $K_{12} \cup K_{12} \cup K_{11}$ and $K_{12,12,11}$. First, we decompose $(K_{12} \cup K_{12} \cup K_{11}) - (M_1 \cup M_2)$ into 8 linear 2-forests where M_1 and M_2 are matchings of size 3 in different K_{12}'s. Then, from the result $la_2(K_{n,n,n}) = \lceil \frac{3n}{2} \rceil$ in Proposition 3.3, we find a way to decompose

$K_{12,12,11} \cup (G[M_1] \cup G[M_2])$ into 18 linear 2-forests where $G[M_1]$ and $G[M_2]$ are subgraphs of K_{35} induced by M_1 and M_2. Hence, we obtain the array in Figure 6 which shows that $la_2(K_{35}) \leq 8 + 18 = 26$ by Lemma 2.4. On the other hand, by Lemma 2.5, $la_2(K_{35}) \geq \left\lceil \frac{595}{\lfloor \frac{2\cdot 35}{3} \rfloor} \right\rceil = 26$. ∎

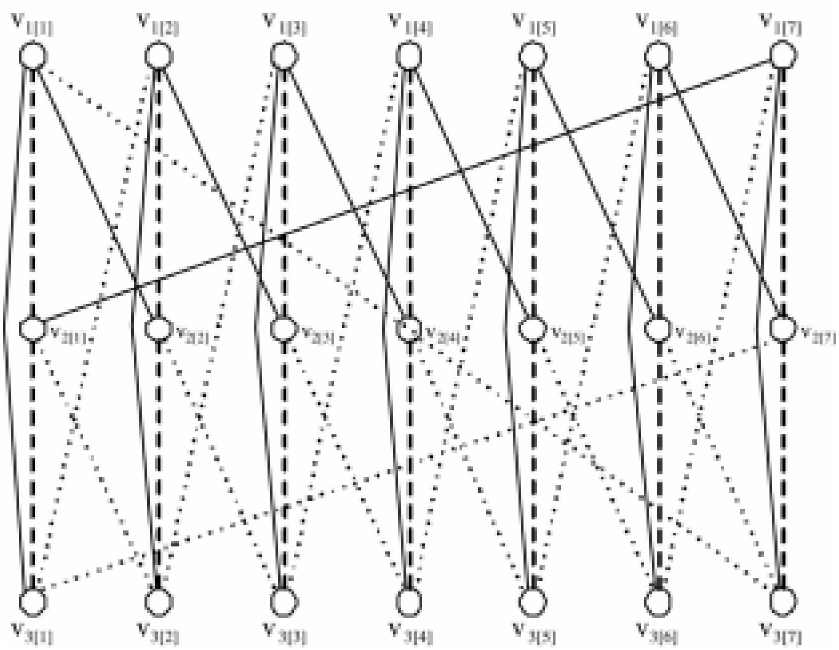

Fig. 5. Three linear 2-forests in $K_{7,7,7}$.

Proposition 3.5. $la_2(K_{59}) = 44$.

Proof. Since K_{59} is an edge-disjoint union of $K_{20} \cup K_{19} \cup K_{20}$ and $K_{20,19,20}$, we first decompose $(K_{20} \cup K_{19} \cup K_{20}) - (E_1 \cup E_2 \cup E_3)$ into 14 linear 2-forests where E_1, E_3 are edge subsets of size 8 in different K_{20}'s and E_2 is an edge subset of size 3 in K_{19}.

Then, from the result $la_2(K_{n,n,n}) = \lceil \frac{3n}{2} \rceil$ in Proposition 3.3, we find a way to decompose $K_{20,19,20} \cup (G[E_1] \cup G[E_2] \cup G[E_3])$ into 30 linear 2-forests where $G[E_1]$, $G[E_2]$, and $G[E_3]$ are subgraphs of K_{59} induced by E_1, E_2, and E_3 respectively.

Hence, we obtain the array in Figure 7 which shows that $la_2(K_{59}) \leq 14 + 30 = 44$ by Lemma 2.4, where B_1, B_2 are the arrays in Figure 8 and C, D_1, D_2, D_3 are the arrays in Figure 9. Moreover, the arrays ${D_1}^T$, ${D_2}^T$, and ${D_3}^T$ are the transposes of D_1, D_2, and D_3 respectively. On the other hand, by Lemma 2.5, $la_2(K_{59}) \geq \left\lceil \frac{1711}{\lfloor \frac{2\cdot 59}{3} \rfloor} \right\rceil = 44$. ∎

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}	v_{20}	v_{21}	v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	v_{28}	v_{29}	v_{30}	v_{31}	v_{32}	v_{33}	v_{34}	v_{35}
v_1		2	1	7	5	4	8	6	5	3	1	25	10	26	19	23	13	20	22	17	16	14	24	11	9	10	12	13	15	16	20	18	23	21	26
v_2	2		22	5	8	7	4	4	6	1	3	2	11	10	26	19	23	13	20	22	17	16	14	25	24	9	10	12	13	15	16	20	18	23	21
v_3	1	22		6	7	5	6	8	7	3	2	4	25	11	10	26	19	23	13	20	21	17	16	14	26	24	9	10	12	13	15	16	20	18	23
v_4	7	5	6		19	2	1	3	8	5	7	4	14	25	11	10	26	19	23	13	20	22	17	16	21	26	24	9	10	12	13	15	16	20	18
v_5	5	8	7	19		6	3	2	1	4	8	6	16	14	25	11	10	26	18	23	13	20	22	17	23	21	26	24	9	10	12	13	15	16	20
v_6	4	7	5	2	6		1	3	2	7	4	5	17	16	14	25	11	10	26	19	23	13	20	22	18	23	21	26	24	9	10	12	13	15	16
v_7	8	4	6	1	3	1		2	3	8	5	7	22	17	16	14	25	11	10	26	19	23	13	20	20	18	23	21	26	24	9	10	12	13	15
v_8	6	4	8	3	2	3	2		1	6	5	7	20	22	17	15	14	25	11	10	26	19	23	13	16	20	18	23	21	26	24	9	10	12	13
v_9	5	6	7	8	1	2	3	1		4	6	8	13	20	22	17	16	14	25	11	10	26	19	23	15	16	20	18	23	21	26	24	9	10	12
v_{10}	3	1	3	5	4	7	8	6	4		2	1	23	12	20	22	17	16	14	25	11	10	26	19	13	15	16	20	18	23	21	26	24	9	10
v_{11}	1	3	2	7	8	4	5	5	6	2		3	19	23	13	20	22	17	16	14	25	11	10	26	12	13	15	16	20	18	23	21	26	24	9
v_{12}	25	2	4	4	6	5	7	7	8	1	3		26	19	23	13	20	22	17	16	14	25	11	9	10	12	13	15	16	20	18	23	21	26	24
v_{13}	10	11	25	14	16	17	22	20	13	23	19	26		13	2	2	7	4	4	6	1	3	5	8	9	24	25	21	22	18	19	17	15	14	12
v_{14}	26	10	11	25	14	16	17	22	20	12	23	19	13		4	1	5	6	8	7	3	2	6	7	11	9	24	25	21	22	18	19	17	15	14
v_{15}	19	26	10	11	25	14	16	17	22	20	13	23	2	4		16	5	7	7	8	1	3	4	6	12	11	9	24	25	21	22	18	19	17	15
v_{16}	23	19	26	10	11	25	14	15	17	22	20	13	2	1	16		4	8	6	5	3	1	7	5	14	12	11	9	24	25	21	22	18	19	17
v_{17}	13	23	19	26	10	11	25	14	16	17	22	20	7	5	5	4		1	3	2	7	4	2	6	15	14	12	11	9	24	25	21	22	18	19
v_{18}	20	13	23	19	26	10	11	25	14	16	17	22	4	6	7	8	1		2	3	8	5	1	3	17	15	14	12	11	9	24	25	21	22	18
v_{19}	22	20	13	23	18	26	10	11	25	14	16	17	4	8	7	6	3	2		1	6	5	3	2	19	17	15	14	12	11	9	24	25	21	22
v_{20}	17	22	20	13	23	19	26	10	11	25	14	16	6	7	8	5	2	3	1		4	6	8	1	18	19	17	15	14	12	11	9	24	25	21
v_{21}	16	17	21	20	13	23	19	26	10	11	25	14	1	3	1	3	7	8	6	4		2	5	4	22	18	19	17	15	14	12	11	9	24	25
v_{22}	14	16	17	22	20	13	23	19	26	10	11	25	3	2	3	1	4	5	5	6	2		7	8	21	22	18	19	17	15	14	12	11	9	24
v_{23}	24	14	16	17	22	20	13	23	19	26	10	11	5	6	4	7	2	1	3	8	5	7		10	25	21	22	18	19	17	15	14	12	11	9
v_{24}	11	25	14	16	17	22	20	13	23	19	26	9	8	7	6	5	6	3	2	1	4	8	10		24	25	21	22	18	19	17	15	14	12	11
v_{25}	9	24	26	21	23	18	20	16	15	13	12	10	9	11	12	14	15	17	19	18	22	21	25	24		2	1	7	5	4	8	6	5	3	1
v_{26}	10	9	24	26	21	23	18	20	16	15	13	12	24	9	11	12	14	15	17	19	18	22	21	25	2		8	5	2	7	4	4	6	1	3
v_{27}	12	10	9	24	26	21	23	18	20	16	15	13	25	24	9	11	12	14	15	17	19	18	22	21	1	8		6	4	5	6	8	7	3	2
v_{28}	13	12	10	9	24	26	21	23	18	20	16	15	21	25	24	9	11	12	14	15	17	19	18	22	7	5	6		4	2	1	3	8	5	7
v_{29}	15	13	12	10	9	24	26	21	23	18	20	16	22	21	25	24	9	11	12	14	15	17	19	18	5	2	4	4		6	7	7	8	1	3
v_{30}	16	15	13	12	10	9	24	26	21	23	18	20	18	22	21	25	24	9	11	12	14	15	17	19	4	7	5	2	6		1	3	2	7	4
v_{31}	20	16	15	13	12	10	9	24	26	21	23	18	19	18	22	21	25	24	9	11	12	14	15	17	8	4	6	1	7	1		2	3	8	5
v_{32}	18	20	16	15	13	12	10	9	24	26	21	23	17	19	18	22	21	25	24	9	11	12	14	15	6	4	8	3	7	3	2		1	6	5
v_{33}	23	18	20	16	15	13	12	10	9	24	26	21	15	17	19	18	22	21	25	24	9	11	12	14	5	6	7	8	8	2	3	1		4	6
v_{34}	21	23	18	20	16	15	13	12	10	9	24	26	14	15	17	19	18	22	21	25	24	9	11	12	3	1	3	5	1	7	8	6	4		2
v_{35}	26	21	23	18	20	16	15	13	12	10	9	24	12	14	15	17	19	18	22	21	25	24	9	11	1	3	2	7	3	4	5	5	6	2	

Fig. 6. The array shows that $la_2(K_{35}) \leq 26$.

B_1	D_1	D_2
D_1^T	C	D_3
D_2^T	D_3^T	B_2

Fig. 7. A partition of a 59×59 array into nine subarrays.

B_1

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}	v_{20}
v_1		15	3	6	6	1	2	13	10	14	4	11	10	9	5	8	7	8	12	43
v_2	15		40	11	12	10	7	9	3	5	6	8	2	14	13	1	3	4	5	11
v_3	3	40		4	1	13	12	5	8	6	2	9	11	13	8	9	10	14	3	7
v_4	6	11	4		37	4	3	3	2	8	9	14	12	10	13	7	9	10	1	5
v_5	6	12	1	37		7	14	7	2	12	1	11	8	5	3	13	10	9	4	4
v_6	1	10	13	4	7		34	8	11	9	5	3	6	2	10	12	5	14	9	3
v_7	2	7	12	3	14	34		12	11	1	8	7	6	9	5	14	4	1	10	13
v_8	13	9	5	3	7	8	12		31	14	4	13	5	2	11	10	8	6	6	1
v_9	10	3	8	2	2	11	11	31		15	14	4	13	5	6	6	12	9	1	7
v_{10}	14	5	6	8	12	9	1	14	15		11	10	13	4	3	7	6	2	2	8
v_{11}	4	6	2	9	1	5	8	4	14	11		15	14	3	7	12	2	13	10	6
v_{12}	11	8	9	14	11	3	7	13	4	10	15		4	6	2	2	1	5	8	12
v_{13}	10	2	11	12	8	6	6	5	13	13	14	4		8	9	1	11	3	7	2
v_{14}	9	14	13	10	5	2	9	2	5	4	3	6	8		14	3	12	11	7	1
v_{15}	5	13	8	13	3	10	5	11	6	3	7	2	9	14		11	1	4	12	9
v_{16}	8	1	9	7	13	12	14	10	6	7	12	2	1	3	11		4	5	13	10
v_{17}	7	3	10	9	10	5	4	8	12	6	2	1	11	12	1	4		7	14	13
v_{18}	8	4	14	10	9	14	1	6	9	2	13	5	3	11	4	5	7		11	12
v_{19}	12	5	3	1	4	9	10	6	1	2	10	8	7	7	12	13	14	11		14
v_{20}	43	11	7	5	4	3	13	1	7	8	6	12	2	1	9	10	13	12	14	

B_2

	v_{40}	v_{41}	v_{42}	v_{43}	v_{44}	v_{45}	v_{46}	v_{47}	v_{48}	v_{49}	v_{50}	v_{51}	v_{52}	v_{53}	v_{54}	v_{55}	v_{56}	v_{57}	v_{58}	v_{59}
v_{40}		19	6	6	13	2	1	10	10	14	12	11	4	9	5	8	7	8	3	15
v_{41}	19		5	12	3	1	13	7	2	8	6	4	14	13	9	10	1	4	7	11
v_{42}	6	5		22	12	3	3	2	4	8	9	14	13	10	1	7	9	10	12	11
v_{43}	6	12	22		7	14	7	2	8	4	13	11	12	5	3	1	10	9	13	4
v_{44}	13	3	12	7		25	8	11	6	9	5	3	9	2	10	4	5	14	1	10
v_{45}	2	1	3	14	25		4	11	6	13	8	7	10	9	5	14	12	13	4	7
v_{46}	1	13	3	7	8	4		28	5	14	12	1	6	2	11	10	8	6	5	9
v_{47}	10	7	2	2	11	11	28		1	1	14	12	13	5	6	6	4	9	8	3
v_{48}	10	2	4	8	6	6	5	1		15	14	12	7	8	9	13	11	3	11	2
v_{49}	14	8	8	4	9	13	14	1	15		11	10	2	12	3	7	6	2	6	5
v_{50}	12	6	9	13	5	8	12	14	14	11		15	10	3	7	4	2	1	2	6
v_{51}	11	4	14	11	3	7	1	12	12	10	15		8	6	2	2	13	5	9	8
v_{52}	4	14	13	12	9	10	6	13	7	2	10	8		7	4	1	14	11	3	5
v_{53}	9	13	10	5	2	9	2	5	8	12	3	6	7		14	3	4	11	1	14
v_{54}	5	9	1	3	10	5	11	6	9	3	7	2	4	14		11	13	12	8	1
v_{55}	8	10	7	1	4	14	10	6	13	7	4	2	1	3	11		12	5	9	13
v_{56}	7	1	9	10	5	12	8	4	11	6	2	13	14	4	13	12		7	10	3
v_{57}	8	4	10	9	14	13	6	9	3	2	1	5	11	11	12	5	7		14	12
v_{58}	3	7	12	13	1	4	5	8	11	6	2	9	3	1	8	9	10	14		16
v_{59}	15	11	11	4	10	7	9	3	2	5	6	8	5	14	1	13	3	12	16	

Fig. 8. Two subarrays B_1 and B_2 of the array in Figure 7.

D_1	D_2
C	D_3

	v_{21}	v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	v_{28}	v_{29}	v_{30}	v_{31}	v_{32}	v_{33}	v_{34}	v_{35}	v_{36}	v_{37}	v_{38}	v_{39}
v_1	1	15	18	19	21	22	24	25	27	28	32	30	35	33	38	36	41	39	44
v_2	42	9	16	18	19	21	22	24	25	27	28	32	30	35	33	38	36	41	39
v_3	44	42	15	16	18	19	21	22	24	25	27	28	32	30	35	33	38	36	41
v_4	39	44	42	15	16	18	19	21	22	24	25	27	28	32	30	35	33	38	36
v_5	41	39	44	42	15	16	18	19	21	22	24	25	27	28	32	30	35	33	38
v_6	36	41	39	44	42	15	16	18	19	21	22	24	25	27	28	32	30	35	33
v_7	38	36	41	39	44	42	15	16	18	19	21	22	24	25	27	28	32	30	35
v_8	33	38	36	41	39	44	42	15	16	18	19	21	22	24	25	27	28	32	30
v_9	35	33	38	36	41	39	44	42	15	16	18	19	21	22	24	25	27	28	32
v_{10}	30	35	33	38	36	41	39	44	42	11	16	18	19	21	22	24	25	27	28
v_{11}	32	30	35	33	38	36	41	39	44	42	7	16	18	19	21	22	24	25	27
v_{12}	28	32	30	35	33	38	36	41	39	44	42	6	16	18	19	21	22	24	25
v_{13}	27	28	32	30	35	33	38	36	41	39	44	42	15	16	18	19	21	22	24
v_{14}	25	27	28	32	30	35	33	38	36	41	39	44	42	15	16	18	19	21	22
v_{15}	24	25	27	28	32	30	35	33	38	36	41	39	44	42	15	16	18	19	21
v_{16}	22	24	25	27	28	32	30	35	33	38	36	41	39	44	42	15	16	18	19
v_{17}	21	22	24	25	27	28	32	30	35	33	38	36	41	39	44	42	15	16	18
v_{18}	19	21	22	24	25	27	28	32	30	35	33	38	36	41	39	44	42	15	16
v_{19}	18	19	21	22	24	25	27	28	32	30	35	33	38	36	41	39	44	42	15
v_{20}	16	18	19	21	22	24	25	27	28	32	30	35	33	38	36	41	39	44	42
v_{21}		16	13	10	5	6	10	4	5	9	15	3	14	7	6	8	11	12	11
v_{22}	16		7	13	11	10	6	12	11	6	2	14	3	2	14	12	8	4	1
v_{23}	13	7		9	10	4	13	3	8	2	11	4	14	5	12	3	6	7	1
v_{24}	10	13	9		7	11	8	9	2	5	4	2	6	3	12	11	13	1	6
v_{25}	5	11	10	7		14	9	2	7	10	14	12	4	6	3	8	3	1	2
v_{26}	6	10	4	11	14		12	7	1	13	5	8	10	3	8	7	9	2	9
v_{27}	10	6	13	8	9	12		5	14	8	1	3	7	11	9	4	12	2	7
v_{28}	4	12	3	9	2	7	5		1	13	6	11	5	10	11	14	4	6	8
v_{29}	5	11	8	2	7	1	14	1		9	12	13	12	8	10	4	6	13	14
v_{30}	9	6	2	5	10	13	8	13	9		3	5	4	7	1	2	14	12	3
v_{31}	15	2	11	4	14	5	1	6	12	3		15	9	10	4	5	1	11	13
v_{32}	3	14	4	2	12	8	3	11	13	5	15		1	9	7	1	7	9	12
v_{33}	14	3	14	6	4	10	7	5	12	4	9	1		11	13	9	2	3	8
v_{34}	7	2	5	3	6	3	11	10	8	7	10	9	11		1	6	14	5	13
v_{35}	6	14	12	12	3	8	9	11	10	1	4	7	13	1		13	2	10	5
v_{36}	8	12	3	11	8	7	4	14	4	2	5	1	9	6	13		10	14	10
v_{37}	11	8	6	13	3	9	12	4	6	14	1	7	2	14	2	10		8	5
v_{38}	12	4	7	1	1	2	2	6	13	12	11	9	3	5	10	14	8		4
v_{39}	11	1	1	6	2	9	7	8	14	3	13	12	8	13	5	10	5	4	

	v_{40}	v_{41}	v_{42}	v_{43}	v_{44}	v_{45}	v_{46}	v_{47}	v_{48}	v_{49}	v_{50}	v_{51}	v_{52}	v_{53}	v_{54}	v_{55}	v_{56}	v_{57}	v_{58}	v_{59}
v_1	16	44	31	41	19	38	34	35	22	32	37	29	25	26	40	23	28	20	42	17
v_2	17	16	44	31	41	19	38	34	35	22	32	37	29	25	26	40	23	28	20	43
v_3	43	17	16	44	31	41	19	38	34	35	22	32	37	29	25	26	39	23	28	20
v_4	20	43	17	16	44	31	41	19	38	34	35	22	32	37	29	25	26	40	23	28
v_5	28	20	43	17	16	44	31	41	19	38	34	35	22	32	36	29	25	26	40	23
v_6	23	28	20	43	17	16	44	31	41	19	38	34	35	22	32	37	29	25	26	40
v_7	40	23	28	20	43	17	16	44	31	41	19	38	33	35	22	32	37	29	25	26
v_8	26	40	23	28	20	43	17	16	44	31	41	19	38	34	35	22	32	37	29	25
v_9	25	26	40	23	28	20	43	17	16	44	30	41	19	38	34	35	22	32	37	29
v_{10}	29	25	26	40	23	28	20	43	17	16	44	31	41	19	38	34	35	22	32	37
v_{11}	37	29	25	26	40	23	28	20	43	17	16	44	31	41	19	38	34	35	22	32
v_{12}	32	37	29	25	26	40	23	27	20	43	17	16	44	31	41	19	38	34	35	22
v_{13}	22	32	37	29	25	26	40	23	28	20	43	17	16	44	31	41	19	38	34	35
v_{14}	35	22	32	37	29	24	26	40	23	28	20	43	17	16	44	31	41	19	38	34
v_{15}	34	35	22	32	37	29	25	26	40	23	28	20	43	17	16	44	31	41	19	38
v_{16}	38	34	35	21	32	37	29	25	26	40	23	28	20	43	17	16	44	31	41	19
v_{17}	19	38	34	35	22	32	37	29	25	26	40	23	28	20	43	17	16	44	31	41
v_{18}	41	18	38	34	35	22	32	37	29	25	26	40	23	28	20	43	17	16	44	31
v_{19}	31	41	19	38	34	35	22	32	37	29	25	26	40	23	28	20	43	17	16	44
v_{20}	44	31	41	19	38	34	35	22	32	37	29	25	26	40	23	28	20	43	17	15
v_{21}	2	17	18	20	21	23	24	26	27	29	31	30	34	33	37	36	40	39	43	42
v_{22}	42	5	17	18	20	21	23	24	26	27	29	31	30	34	33	37	36	40	39	43
v_{23}	43	42	15	17	18	20	21	23	24	26	27	29	31	30	34	33	37	36	40	39
v_{24}	39	43	42	15	17	18	20	21	23	24	26	27	29	31	30	34	33	37	36	40
v_{25}	40	39	43	42	15	17	18	20	21	23	24	26	27	29	31	30	34	33	37	36
v_{26}	36	40	39	43	42	15	17	18	20	21	23	24	26	27	29	31	30	34	33	37
v_{27}	37	36	40	39	43	42	15	17	18	20	21	23	24	26	27	29	31	30	34	33
v_{28}	33	37	36	40	39	43	42	15	17	18	20	21	23	24	26	27	29	31	30	34
v_{29}	34	33	37	36	40	39	43	42	3	17	18	20	21	23	24	26	27	29	31	30
v_{30}	30	34	33	37	36	40	39	43	42	15	17	18	20	21	23	24	26	27	29	31
v_{31}	31	30	34	33	37	36	40	39	43	42	8	17	18	20	21	23	24	26	27	29
v_{32}	29	31	30	34	33	37	36	40	39	43	42	10	17	18	20	21	23	24	26	27
v_{33}	27	29	31	30	34	33	37	36	40	39	43	42	15	17	18	20	21	23	24	26
v_{34}	26	27	29	31	30	34	33	37	36	40	39	43	42	15	17	18	20	21	23	24
v_{35}	24	26	27	29	31	30	34	33	37	36	40	39	43	42	15	17	18	20	21	23
v_{36}	23	24	26	27	29	31	30	34	33	37	36	40	39	43	42	15	17	18	20	21
v_{37}	21	23	24	26	27	29	31	30	34	33	37	36	40	39	43	42	15	17	18	20
v_{38}	20	21	23	24	26	27	29	31	30	34	33	37	36	40	39	43	42	15	17	18
v_{39}	18	20	21	23	24	26	27	29	31	30	34	33	37	36	40	39	43	42	15	17

Fig. 9. Four subarrays C, D_1, D_2 and D_3 of the array in Figure 7.

Proposition 3.6. $la_2(K_{12t+11}) = 9t + 8$ *for any* $t \geq 3$ *and* $t \neq 4$.

Proof. We prove this proposition by using the techniques from latin squares proposed by Chen et al. [4]. First, assume that t is odd. Then let the 23×23 array in Figure 4 be partitioned into four subarrays P, Q, Q^T, R as shown in Figure 10, where P, Q, R are 12×12, 12×11, 11×11 arrays respectively, and Q^T is the transpose of Q. Moreover, let the 12×12 array in Figure 2 be denoted by W.

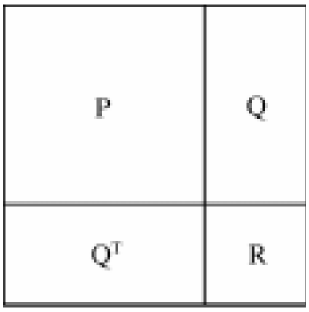

Fig. 10. Four subarrays of the array in Figure 4 or Figure 6.

From Theorem 2.1, we can find an idempotent commutative latin square of order t. By using $L = [\ell_{ij}]$ to denote this idempotent commutative latin square, we can construct a $(12t+11) \times (12t+11)$ symmetric array as shown in Figure 11 to show that $la_2(K_{12t+11}) \leq 9t + 8$, where, for $1 \leq x \leq t$,

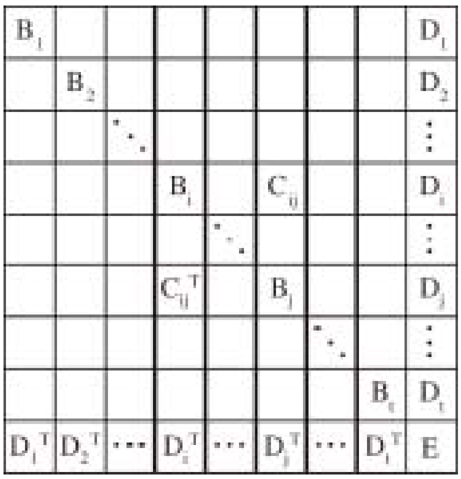

Fig. 11. A $(12t + 11) \times (12t + 11)$ symmetric array.

1. B_x is a 12×12 array;
2. the entry $B_x(r,s)$ in B_x equals $P(r,s)$ in P if $P(r,s) \in \{1,2,\ldots,8\}$;
3. $B_x(r,s) = P(r,s) + (x-1) \cdot 9$ if $P(r,s) \notin \{1,2,\ldots,8\}$;
4. the 12×12 array $C_{ij} = W + 8 + (\ell_{ij} - 1) \cdot 9$, for $1 \leq i,j \leq t$;
5. the 12×11 array $D_x = Q + (x-1) \cdot 9$;
6. the 11×11 array $E = R$; and
7. the arrays ${C_{ij}}^T$ and ${D_x}^T$ are the transposes of C_{ij} and D_x respectively.

Next, if t is even, then let the 35×35 array in Figure 6 be partitioned into four subarrays P, Q, Q^T, R as shown in Figure 10, where P, Q, R are 24×24, 24×11, 11×11 arrays respectively, and Q^T is the transpose of Q. From Theorem 2.2, then we can find a commutative PILS$(2k; 2, 2, \ldots, 2)$ such that $t = 2k$. By using $L = [\ell_{ij}]$ to denote this commutative PILS$(2k; 2, 2, \ldots, 2)$, we can construct a $(12t+11) \times (12t+11)$ symmetric array as shown in Figure 12 to show that $la_2(K_{12t+11}) \leq 9t + 8$, where, for $1 \leq x \leq k$,

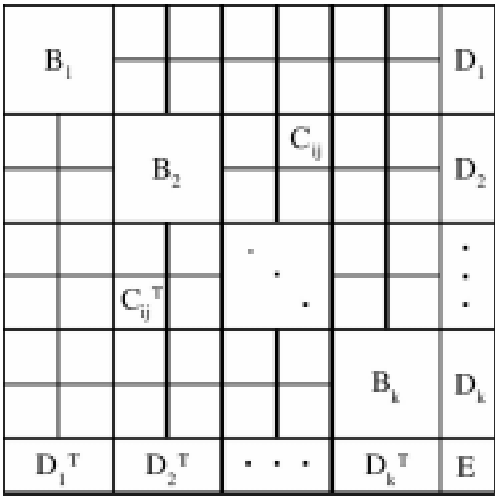

Fig. 12. A $(12t+11) \times (12t+11)$ symmetric array.

1. B_x is a 24×24 array;
2. the entry $B_x(r,s)$ in B_x equals $P(r,s)$ in P if $P(r,s) \in \{1,2,\ldots,8\}$;
3. $B_x(r,s) = P(r,s) + (x-1) \cdot 18$ if $P(r,s) \notin \{1,2,\ldots,8\}$;
4. the 12×12 array $C_{ij} = W + 8 + (\ell_{ij} - 1) \cdot 9$, for $1 \leq i,j \leq 2k$;
5. the 24×11 array $D_x = Q + (x-1) \cdot 18$;
6. the 11×11 array $E = R$; and

7. the arrays ${C_{ij}}^T$ and ${D_x}^T$ are the transposes of C_{ij} and D_x respectively.

On the other hand, by Lemma 2.5, $la_2(K_{12t+11}) \geq \left\lceil \frac{(12t+11)(12t+10)}{2\left\lfloor \frac{2(12t+11)}{3} \right\rfloor} \right\rceil = 9t+8$. This concludes the proof. ∎

Corollary 3.7. $la_2(K_{12t+10}) = la_2(K_{12t+11}) = 9t+8$ *for any* $t \geq 0$.

Proof. By Propositions 3.1 $\sim$ 3.2 and 3.4 $\sim$ 3.6, $la_2(K_{12t+11}) = 9t+8$ for any $t \geq 0$. Moreover, from Lemmas 2.3 and 2.5, $9t+8 = la_2(K_{12t+11}) \geq la_2(K_{12t+10}) \geq \left\lceil \frac{(12t+10)(12t+9)}{2\left\lfloor \frac{2(12t+10)}{3} \right\rfloor} \right\rceil = 9t+8$ for any $t \geq 0$. ∎

Finally, we conclude this paper by the following theorem, which provides the answers of the unsolved cases in Theorem 1.2. Furthermore, the results obtained on $la_2(K_n)$ are coherent with the corresponding cases of Conjecture 1.1.

Theorem 3.8. $la_2(K_n) = \left\lceil \frac{n(n-1)}{2\left\lfloor \frac{2n}{3} \right\rfloor} \right\rceil$ *for* $n \equiv 10, 11 \pmod{12}$.

Proof. We can assume that $n = 12t+10$ or $n = 12t+11$ for any $t \geq 0$. Since $\left\lceil \frac{n(n-1)}{2\left\lfloor \frac{2n}{3} \right\rfloor} \right\rceil = 9t+8$ when $n = 12t+10$ or $n = 12t+11$ for any $t \geq 0$, from Corollary 3.7, then the assertion holds. ∎

References

1. J. C. Bermond, J. L. Fouquet, M. Habib and B. Peroche, On linear k-arboricity, *Discrete Math.*, **52** (1984), 123-132.
2. G.-J. Chang, Algorithmic aspects of linear k-arboricity, *Taiwanese J. Math.*, **3** (1999), 73-81.
3. G.-J. Chang, B.-L. Chen, H.-L. Fu and K.-C. Huang, Linear k-arboricities on trees, *Discrete Applied Math.*, **103** (2000), 281-287.
4. B.-L. Chen, H.-L. Fu and K.-C. Huang, Decomposing graphs into forests of paths with size less than three, *Australas. J. Combin.*, **3** (1991), 55-73.
5. B.-L. Chen and K.-C. Huang, On the linear k-arboricity of K_n and $K_{n,n}$, *Discrete Math.*, **254** (2002), 51-61.
6. J. Denes and A. D. Keedwell, *Latin squares and their applications*, Academic Press Inc., New York.

7. C.-M. Fu and H.-L. Fu, On the intersections of latin squares with holes, *Utilitas Mathematica*, **35** (1989), 67-74.
8. H.-L. Fu and K.-C. Huang, The linear 2-arboricity of complete bipartite graphs, *Ars Combin.*, **38** (1994), 309-318.
9. M. Habib and B. Peroche, La k-arboricité linéaire des arbres, *Ann. Discrete Math.*, **17** (1983), 307-317.
10. M. Habib and B. Peroche, Some problems about linear aboricity, *Discrete Math.*, **41** (1982), 219-220.
11. F. Harary, Covering and packing in graphs I, *Ann. New York Acad. Sci.*, **175** (1970), 198-205.
12. C. Thomassen, Two-coloring the edges of a cubic graph such that each monochromatic component is a path of length at most 5, *J. Combin. Theorey, Ser B.*, **75** (1999), 100-109.
13. Chih-Hung Yen and Hung-Lin Fu, Linear 3-arboricity of the balanced complete multipartite graph, *JCMCC*, **60** (2007), 33-46.

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