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EXISTENCE THEOREM ON VARIATIONAL INEQUALITY PROBLEM WITH LOCAL INTERSECTION PROPERTY

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Abstract. Existence theorem for a variational inequality problem with local intersection property has been obtained in topological space by relaxing the property of open inverse values from the result of Vetrivel and Nanda [7].

1. Introduction

Interesting and valuable results as application of fixed point theorem are studied extensively in the field of variational inequality.

In this direction, an existence theorem for a variational inequality problem was discussed by Gwinner [2], which is, an infinite dimensional version of Walras excess demand theorem (see also Zeidler [9]), as follows:

Theorem 1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty compact convex subsets of Hausdorff locally convex topological vector spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Let $f : \mathcal{A}\times\mathcal{B} \to \mathbb{R}$ be continuous. Let $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a multifunction. Suppose that*

- (i) *for each $y \in \mathcal{B}, \{x \in \mathcal{A} : f(x, y) < t\}$ is convex for all $t \in \mathbb{R}$,*
- (ii) *$\mathcal{T}$ is an upper semicontinuous multifunction with nonempty compact convex values. Then there exists $x_0 \in \mathcal{A}$ and $y_0 \in \mathcal{T}(x_0)$ such that $f(x_0, y_0) \leq f(x, y_0)$ for all $x \in \mathcal{A}$.*

Later, in 2000, Vetrivel and Nanda [7] proved the same result for multifunction with open inverse values in the setting of same space in the line of Trafdar and Yuan [6]. To prove the result, they used results due to Lassonde [4] and Horwarth [3].

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Recently, Ding [1] proved a result in which he used local intersection property in place of property of open inverse values.

Inspired from the results of Ding [1], Vetrivel and Nanda [7], and others, an existence theorem for a variational inequality without open inverse values in topological space and the result of Lassonde [4], for Kakutani factorizable multifunction has been established. The main tool which here used to prove the result are due to Horvath [3] and Shioji [5].

2. Preliminaries

In the material to be presented here, the following definitions have been used:

Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty sets. The collection of all non-empty subsets of $\mathcal{X}$ is denoted by $2^{\mathcal{X}}$.

A multifunction or set-valued function from $\mathcal{X}$ to $\mathcal{Y}$ is defined to be a function that assigns to each elements of $\mathcal{X}$ a non-empty subset of $\mathcal{Y}$.

If $\mathcal{T}$ is a multifunction from $\mathcal{X}$ to $\mathcal{Y}$, then it is designated as $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}}$, and for every $x \in \mathcal{X}$, $\mathcal{T}x$ is called a value of $\mathcal{T}$.

For $\mathcal{A} \subseteq \mathcal{X}$, the image of $\mathcal{A}$ under $\mathcal{T}$, denoted by $\mathcal{T}(\mathcal{A})$, is defined as

$$\mathcal{T}(\mathcal{A}) = \bigcup_{x \in \mathcal{A}} \mathcal{T}x$$

For $\mathcal{B} \subseteq \mathcal{Y}$, the preimage or inverse image of $\mathcal{B}$ under $\mathcal{T}$, denoted by $\mathcal{T}^{-1}(\mathcal{B})$, is defined as

$$\mathcal{T}^{-1} = \{x \in \mathcal{X} : \mathcal{T}x \cap \mathcal{B} \neq \emptyset\}$$

If $y \in \mathcal{Y}$, then $\mathcal{T}^{-1}(y)$ is called a inverse value of $\mathcal{T}$. If it is open, then it called open inverse value.

A multivalued function $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}}$ is upper semicontinuous (usc)(lower semicontinuous(lsc)) if $\mathcal{T}^{-1}(\mathcal{B}) = \{x \in \mathcal{X} : \mathcal{T}x \cap \mathcal{B} \neq \emptyset\}$ is closed(open) for each closed (open) subset $\mathcal{B}$ of $\mathcal{Y}$. If $\mathcal{T}$ is both usc and lsc, then it is continuous .

A multifunction $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}}$ is said to be a compact multifunction, if $\mathcal{T}(\mathcal{X})$ is contained in a compact subset of $\mathcal{Y}$.

It is known that if $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}}$ is an upper semicontinuous multifunction with compact values, then $\mathcal{T}(\mathcal{K})$ is compact in $\mathcal{Y}$ whenever $\mathcal{K}$ is compact subset of $\mathcal{X}$.

Let Δ_n be the standard $n-$dimensional simplex with vertices $e_0, e_1, e_2, ...e_n$. If $\mathcal{J}_n = \{0, 1, 2, ..., n\}$. We denote by $\Delta_{\mathcal{J}} = Co\{e_j : j \in \mathcal{J}\}\ for\ any\ non-empty\ subset\ \mathcal{J}\ of\ \mathcal{J}_n.$

A topological space $\mathcal{X}$ is said to be contractible, if the identity mapping $\mathcal{I}_{\mathcal{X}}$ of $\mathcal{X}$ is homotopic to a constant function. A topological space is said to be an acyclic space if all of its reduced *C*ech homology groups over the rationals vanish. In particular, any contractible space is acyclic, and hence, any convex or star-shaped set in a topological vector space is acyclic. For a topological space $\mathcal{X}$, we shall denote by *ka*$(\mathcal{X})$, the family of all compact acyclic subsets of X.

Following results due to Horvath [3] and Shioji [5, Lemma 1] are needed in the sequel:

Theorem 2.1. [3]. *Let $\mathcal{X}$ be a topological space. For any nonempty subset $\mathcal{J}$ of $\{0,1,...,n\}$, let $\Gamma_{\mathcal{J}}$ be a nonempty contractible subset of X. If $\emptyset \neq \mathcal{J} \subset \mathcal{J}' \subset \{0,1,..,n\}$ implies $\Gamma_{\mathcal{J}} \subset \Gamma_{\mathcal{J}'}$, then there exists a single valued continuous function $f : \Delta_n \to \mathcal{X}$ such that $g[\Delta_{\mathcal{J}}] \subseteq \Gamma_{\mathcal{J}}$ for all nonempty subset $\mathcal{J}$ of $\{0,1,..,n\}$.*

Theorem 2.2. [5]. *Let Δ_n be an $n-$dimensional simplex with the Euclidean topology and $\mathcal{X}$ a compact topological space. Let $\phi : \mathcal{X} \to \Delta_n$ be a single-valued continuous mapping and $\mathcal{T} : \Delta_n \to ka(\mathcal{X})$ be a upper semicontinuous set-valued mapping. Then there exists a point $x_0 \in \Delta_n$ such that $x_0 \in \phi(\mathcal{T}(x_0))$.*

Besides Theorem 2.1 and Theorem 2.2, the following local intersection property Theorem 2.2 due to Ding [1, Lemma 1] will also be used. Before starting it, the following notations have been recalled [1].

Let $\mathcal{X}$ and $\mathcal{Y}$ be two topological spaces and $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}} \cup \{\emptyset\}$ a set-valued mapping. $\mathcal{T}$ is said to have local intersection property, if for each $x \in \mathcal{X}$ with $\mathcal{T}(x) \neq \emptyset$, there exists an open neighborhood $\mathcal{N}(x)$ of x such that $\bigcap_{z\in\mathcal{N}(x)} \mathcal{T}(z) \neq \emptyset$. It is not hard to see that each map with open inverse property has the local intersection property but the example given in [8, p. 63], shows that the converse is not true.

Theorem 2.3. [1]. *Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces and $\mathcal{T} : \mathcal{X} \to 2^{\mathcal{Y}}$ a set-valued mapping. Then the following conditions are equivalent:*

(*i*) *$\mathcal{T}$ has the local intersection property,*

(*ii*) *for each $y \in \mathcal{Y}$, $\mathcal{T}^{-1}(y)$ contain a open set $\mathcal{O}_y \subset \mathcal{X}$ (which may be empty) such that $\mathcal{X} = \bigcup_{y\in\mathcal{Y}} \mathcal{O}_y$,*

3. Main Result

Theorem 3.1. *Let $\mathcal{A}$ as in Theorem 1.1 and $\mathcal{B}$ be an arbitrary subset of topological spaces $\mathcal{Y}$. Let $f : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ be continuous. Let $\mathcal{T} : \mathcal{A} \to \mathcal{B}$ be a multifunction. Suppose that*

(*i*) *for each $y \in \mathcal{B}, \{x \in \mathcal{A} : f(x,y) < t\}$ is convex for all $t \in \mathbb{R}$;*

(*ii*) *$\mathcal{T}$ has local intersection property;*

(iii) *for every open set* $\mathcal{U} \subset \mathcal{A}$*, the set* $\cap\{\mathcal{T}u : u \in \mathcal{U}\}$ *is empty or contractible;*

(iv) $\mathcal{T}(\mathcal{A})$ *is compact and contractible.*

Then there exist $x_0 \in \mathcal{A}$ *and* $y_0 \in \mathcal{T}(x_0)$ *such that* $f(x_0, y_0) \leq f(x, y_0)$ *for all* $x \in \mathcal{A}$*.*

Proof. By (ii) and Theorem 2.3, for each $y \in \mathcal{T}(\mathcal{A})$, there exists an open set $O_y \in \mathcal{T}(\mathcal{A})$ (which may be empty) such that $\mathcal{O}_\dagger \in \mathcal{T}^{-\infty}(\dagger)$ and $\mathcal{A} = \bigcup_{\dagger\in\mathcal{T}(\mathcal{A})} \mathcal{O}_\dagger = \bigcup_{\dagger\in\mathcal{T}(\mathcal{A})} \mathcal{T}^{-\infty}(\dagger)$. Since $\mathcal{A}$ is compact, these exists a finite set $\{y_0, y_1, y_2, ..., y_n\} \subset \mathcal{T}(\mathcal{A})$ such that $\mathcal{A} = \bigcup_{\rangle=\prime}^{\backslash} \mathcal{O}_{\dagger_\rangle}$. Now, for each nonempty subset $\mathcal{J}$ of $\mathcal{N} = \{\prime, \infty, \in, ...\backslash\}$, define

$$\Gamma_{\mathcal{J}} = \begin{cases} \cap\{\mathcal{T}(x) : x \in \bigcap_{j\in\mathcal{J}} \mathcal{O}_{y_j}\}, & \text{if} \quad \bigcap_{j\in\mathcal{J}} \mathcal{O}_{y_j} \neq \emptyset, \\ \mathcal{T}(\mathcal{A}) & \text{, otherwise} \end{cases}$$

Evidently, if $x \in \bigcap_{j\in\mathcal{J}} \mathcal{O}_{y_j} \subset \bigcap_{j\in\mathcal{J}} \mathcal{T}^{-1}(y_j)$, then $\{y_j : j \in \mathcal{J}\} \subset \mathcal{T}(x)$. By (iii), each $\Gamma_{\mathcal{J}}$ is nonempty contractible and it is clear that $\Gamma_{\mathcal{J}} \subseteq \Gamma_{\mathcal{J}'}$, whenever $\emptyset \neq \mathcal{J} \subset \mathcal{J}' \subset \mathcal{N}$.

By Theorem 2.1, there exists a single valued continuous function $f : \Delta_n \to \mathcal{T}(\mathcal{A})$ such that $f[\Delta_{\mathcal{J}}] \subseteq \Gamma_{\mathcal{J}}$, for all $\phi \neq \mathcal{J} \subset \mathcal{N}$.

Let $\{\phi_0, \phi_1, ..., \phi_n\}$ be a continuous partition of unity subordinated to the open covering $\{\mathcal{O}_{y_i}\}_{i\in\mathcal{N}}$ i.e., for each $i \in \mathcal{N}$, $\phi_i : \mathcal{A} \to [0, 1]$ is continuous; $\{x \in \mathcal{A} : \phi_i(x) \neq 0\} \subset \mathcal{O}_{y_i} \subset \mathcal{T}^{-1}(y_i)$ such that $\sum_{i=0}^n \phi_i(x) = 1$ for all $x \in \mathcal{A}$.

Define $\phi : \mathcal{A} \to \Delta_n$ by

$$\phi(x) = (\phi_0(x), \phi_1(x), \phi_2(x), ..., \phi_n(x)) \quad for \;\; all \;\; x \in \mathcal{A}.$$

Then, ϕ is continuous. Then, $\phi(x) \subset \Delta_{\mathcal{J}(x)}$ for all $x \in \mathcal{A}$, where $\mathcal{J}(x) : \{j \in \mathcal{N} : \phi_j(x) \neq 0\}$. Therefore, we have

$$f(\phi(x)) \in f(\Delta_{\mathcal{J}(x)}) \subseteq \Gamma_{\mathcal{J}(x)} \subseteq \mathcal{T}(x), \quad for \;\; all \;\; x \in \mathcal{A}. \tag{3.1}$$

Consider $\mathcal{G} : \mathcal{T}(\mathcal{A}) \to \mathcal{A}$ defined by $\mathcal{G}(y) = \{z \in \mathcal{A} : f(z, y) \leq f(w, y)$ for all $w \in \mathcal{A}\}$. For each $y \in \mathcal{T}(\mathcal{A})$, $\mathcal{G}(y)$is nonempty since f assumes its minimum on the compact set $\mathcal{A}$. Also, it is closed and hence compact. Further, $\mathcal{G}(y)$ is convex. Indeed, let z_1 and $z_2 \in \mathcal{A}$ be such that $f(z_i, y) \leq f(w, y)$ for all $w \in \mathcal{A}$ and $i = 1, 2$. Since any convex or star-shaped set in a topological vector space is acyclic. So, $\mathcal{G}(y)$ is acyclic. By the assumption on f, $f(\lambda z_1 + (1-\lambda)z_2, y) \leq f(w, y)$ for all $w \in \mathcal{A}$. Since f is continuous, the graph of $\mathcal{G}$, $Gr(\mathcal{G}) = \{(y, z) : y \in \mathcal{T}(\mathcal{A}), z \in \mathcal{G}(y)\}$ is a closed subset of the compact set $\mathcal{T}(\mathcal{A}) \times \mathcal{A}$. Then it follows that $\mathcal{G}$ is upper semicontinuous.

Thus, by the above discussion $\mathcal{G}$ is upper semicontinuous with nonempty compact acyclic values and $f : \Delta_n \to \mathcal{T}(\mathcal{A})$ is continuous, it follows that the composition mapping $\mathcal{G} \circ f : \Delta \to \mathcal{A}$ is also upper semicontinuous with nonempty compact acyclic values. Since $\phi : \mathcal{A} \to \Delta_n$ is continuous and hence, Theorem 2.2 guarantees the existence of a point $x_0 \in \Delta_n$ such that $x_0 \in \phi(\mathcal{G} \circ f(x_0))$. Let $y_0 \in f(x_0)$, then we have

$$y_0 = f(x_0) \in f(\phi(\mathcal{A} \circ f(x_0))) = f(\phi(\mathcal{G}(y_0))),$$

so that there exists $x_0 \in \mathcal{G}(y_0)$ such that $y_0 = f(\phi(x_0)) \subset \mathcal{T}(x_0)$. This completes the proof.

■

Next, recall the following remark given by Ding [1]:

Remark 3.2. [1]. If $\mathcal{F}^{-1}(y)$ is open in $\mathcal{A}$ for each $x \in \mathcal{A}$ with $\mathcal{F}(x) \neq \emptyset$, we take $y \in \mathcal{F}(x)$ and let $\mathcal{N}(x) = \mathcal{F}^{-1}(y)$. Then $\mathcal{N}(x)$ is a open neighbourhood of x and $y \in \bigcap_{z \in \mathcal{N}(x)} \mathcal{F}(z)$. Hence, $\mathcal{F}$ has the local intersection property.

With the Remark 3.2 and the fact that any nonempty convex or star-shaped subset of a topological space is contractible [1], Theorem 3.1, in turn, generalizes the result of Vetrivel and Nanda [7].

References

1. X. P. Ding, A coincidence theorem involving contractible spaces, *Appl. Math. Lett.*, **10(3)** (1997), 53-56.
2. J. Gwinner, On fixed points and variational inequalities - a circular tour, *Nonlinear Anal. Theory Methods Appl.*, **5(5)** (1981), 565-583.
3. C. D. Horvath, Contractibility and generalized convexity, *J. Math. Anal. Appl.*, **156(2)** (1991), 341-357.
4. M. Lassonde, Fixed points for kakutani factorizable multifunctions, *J. Math. Anal. Appl.*, **152(1)** (1990), 46-60.
5. N. Shioji, A further generalization of Knaster-Kuratoski-Masurkiewicz theorem, *Proc. Amer. Math. Soc.*, **111** (1991), 187-195.
6. E. Tarafdar and X. Z. Yuan, A remark on coincidence theorems, *Proc. Amer. Math. Soc.*, **122(3)** (1994), 875-859.
7. V. Vetrivel and S. Nanda, A remark on gwinner's existence theorem on variational inequality problem, *Internat. J. Math. Math. Sci.*, **24(8)** (2000), 573-575.
8. X. Wu and S. Shen, A further generelization of Yannelis-Prabhakar's continuous selection theorem and its applications, *J. Math. Anal. Appl.*, **197** (1996), 61-74.

9. E. Zeidler, *Nonlinear Functional Analysis and its Applications. III, Variational methods and optimization*, Translated from the German by Leo F. Boron., Springer-Verlag, New York, Berlin, 1985.

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