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PARALLEL SURFACES IN THREE-DIMENSIONAL LORENTZIAN LIE GROUPS

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Abstract. A three-dimensional homogeneous Lorentzian manifold is either symmetric or locally isometric to a Lie group equipped with a left-invariant Lorentzian metric [4]. We completely classify surfaces with parallel second fundamental form in all non-symmetric homogeneous Lorentzian three-manifolds. Interesting differences arise with respect to the Riemannian case studied in [11, 12].

1. INTRODUCTION

Let (N, g) be a pseudo-Riemannian manifold. A submanifold M of (N, g) is said to be *parallel* if its second fundamental form is covariantly constant and so, the extrinsic invariants of M do not vary with the point. The study of parallel submanifolds of a given pseudo-Riemannian manifold (N, g), is an interesting problem which enriches our knowledge and understanding of its geometry. Note that parallel submanifolds are a natural extension of totally geodesic submanifolds, for which the second fundamental form vanishes identically.

In the Riemannian framework, several authors studied parallel and semi-parallel submanifolds, see for example [9, 1, 2]. A good survey can be found in [13]. In this context, a special case arises naturally, namely, parallel surfaces of three-dimensional homogeneous Riemannian manifolds. These spaces represent a natural generalization of three-dimensional real space forms and were completely classified

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in [15]. Curves and surfaces of a three-dimensional homogeneous Riemannian manifold of non-constant sectional curvature were investigated first in [10] and a complete classification of parallel surfaces was obtained in the works [11, 12] by the second author and J. Inoguchi.

In general, it is worthwhile to investigate whether and to what extent, results valid in Riemannian geometry can be extended to the pseudo-Riemannian case and in particular, to Lorentzian geometry. The study of parallel submanifolds in Lorentzian settings is rather recent and mainly limited to the case of an ambient space of constant sectional curvature (see for example [3, 14]). A brief description of the state of the art is given in [13]. Moreover, the second author and B.-Y. Chen classified parallel surfaces in three- and four-dimensional Lorentzian space forms in [8].

Recently, the first author studied homogeneous Lorentzian three-manifolds, proving that such a space is either symmetric or isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric in [4], and providing a full classification. The curvature of homogeneous Lorentzian three-manifolds was then completely described in [5].

In this paper, we shall give the complete classification of parallel surfaces in non-symmetric homogeneous Lorentzian three-manifolds. The case of parallel surfaces in symmetric Lorentzian three-spaces will be treated in a forthcoming paper [6]. The paper is organized in the following way. In Section 2 we collect some basic facts concerning parallel surfaces of a three-dimensional Lorentzian manifold, the classification of homogeneous Lorentzian three-spaces and algebraic restrictions to the existence of a parallel surface. In Section 3 we provide the classification of parallel surfaces of three-dimensional Lorentzian Lie groups and in Section 4 we formulate some remarks and our conclusions.

2. Preliminaries

2.1. On parallel surfaces

Let (N, g) be a three-dimensional homogeneous Lorentzian manifold and M a surface in N. Throughout the paper, we assume that M is *non-degenerate*, that is, the induced metric on M is non-degenerate. We will denote by ξ a fixed normal vector field on the surface, with $\langle \xi, \xi \rangle = \varepsilon$. Here, either $\varepsilon = -1$ or $\varepsilon = 1$, according to the surface being either Riemannian or Lorentzian, respectively. We shall call ξ a ε-unit normal (vector field).

Denote by ∇^M and ∇ the Levi Civita connections of M and N, respectively. Let X and Y denote vector fields tangent to M. The formula of Gauss gives a decomposition of the vector field $\nabla_X Y$ into a tangent and a normal component ([7, 17]):

$$\nabla_X Y = \nabla^M_X Y + h(X, Y)\xi. \tag{2.1}$$

This formula defines h, which is called the *second fundamental form*. If we define the shape operator S associated to ξ by

$$SX = -\nabla_X \xi, \tag{2.2}$$

then at every point $p \in M$, S is a symmetric endomorphism of the tangent plane T_pM and

$$\langle SX, Y\rangle = \varepsilon h(X, Y). \tag{2.3}$$

The well-known equations of Gauss and Codazzi are respectively given by

$$\langle R(X,Y)Z,W\rangle = \langle R^M(X,Y)Z,W\rangle - \varepsilon(h(X,Z)h(Y,W) - h(Y,Z)h(X,W)), \tag{2.4}$$

$$\langle R(X,Y)Z,\xi\rangle = \varepsilon((\nabla^M h)(Y,X,Z) - (\nabla^M h)(X,Y,Z)), \tag{2.5}$$

for X, Y, Z, W tangent to M, were R is the curvature tensor of the ambient space N, taken with the sign convention

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y], \tag{2.6}$$

R^M is the curvature tensor of the surface M and $\nabla^M h$ is defined by

$$(\nabla^M h)(X, Y, Z) = X(h(Y, Z)) - h(\nabla^M_X Y, Z) - h(Y, \nabla^M_X Z). \tag{2.7}$$

The surface M is said to be *totally geodesic* in N if $h = 0$ holds identically, *parallel* if $\nabla^M h = 0$ and *semi-parallel* if $R^M \cdot h = 0$, where

$$(R^M \cdot h)(X, Y, Z, W) = -h(R^M(X, Y)Z, W) - h(Z, R^M(X, Y)W). \tag{2.8}$$

Finally, we say that M is *totally umbilical* in N if S is a scalar multiple of the identity at every point. The following results can be easily obtained.

Lemma 1. *Any parallel surface in a Lorentzian manifold is semi-parallel. A surface in a three-dimensional Lorentzian manifold is semi-parallel if and only if it is either flat or totally umbilical.*

Proof. The fact that parallelism implies semi-parallelism can be proven as in the Riemannian case. In fact, the condition of semi-parallelism is an integrability condition for the condition of parallelism.

Now let M be a surface in a three-dimensional Lorentzian manifold N. Let ξ be an ε-unit normal to M with associated shape operator S. Let $\{E_1, E_2\}$ be a pseudo-orthonormal basis for the tangent distribution to M, with $\langle E_1, E_1\rangle = 1$,

$\langle E_1, E_2\rangle = 0$ and $\langle E_2, E_2\rangle = -\varepsilon$. Then, with respect to this basis, S takes the form

$$S = \begin{pmatrix} S_{11} & S_{12} \\ -\varepsilon S_{12} & S_{22} \end{pmatrix}.$$

A straightforward computation shows that $R^M \cdot h = 0$ if and only if $S_{12}\left\langle R^M(E_1, E_2)E_1, E_2\right\rangle = (S_{11} - S_{22})\left\langle R^M(E_1, E_2)E_1, E_2\right\rangle = 0$. Hence we obtain that either $R^M = 0$ or $S_{12} = S_{11} - S_{22} = 0$. In the first case M is a flat surface, in the second case the immersion is totally umbilical. ■

Lemma 2. *Let M be a surface in a three-dimensional Lorentzian manifold, ξ an ε-unit normal to M and $\{E_1, E_2\}$ a pseudo-orthonormal frame field with $\langle E_1, E_1\rangle = 1$, $\langle E_1, E_2\rangle = 0$ and $\langle E_2, E_2\rangle = -\varepsilon$, such that the shape operator associated to ξ takes the form*

$$S = \begin{pmatrix} S_{11} & S_{12} \\ -\varepsilon S_{12} & S_{22} \end{pmatrix},$$

with respect to $\{E_1, E_2\}$. Then M is parallel if and only if

$$\begin{cases} X(S_{11}) = -2\varepsilon S_{12}\left\langle \nabla^M_X E_1, E_2\right\rangle, \\ X(S_{12}) = (S_{22} - S_{11})\left\langle \nabla^M_X E_1, E_2\right\rangle, \\ X(S_{22}) = 2\varepsilon S_{12}\left\langle \nabla^M_X E_1, E_2\right\rangle, \end{cases} \tag{2.9}$$

for every tangent vector X.

As concerns the Gaussian curvature of a surface in a Lorentzian three-manifold, using (2.3) and (2.4) we easily obtain the following.

Lemma 3. *Let M be a surface in a Lorentzian manifold, ξ a ε-unit normal to M with shape operator S and $\{E_1, E_2\}$ a pseudo-orthonormal tangent frame to M. Then, the Gaussian curvature of M is given by*

$$K = \varepsilon(\det S - \langle R(E_1, E_2)E_1, E_2\rangle). \tag{2.10}$$

2.2. On Lorentzian homogeneous three-manifolds

Homogeneous Lorentzian three-spaces (N, g) where classified by the first author in [4]. Unless they are symmetric, they are Lie groups equipped with left-invariant Lorentzian metrics and are classified in the following theorem.

Theorem 1. [4] *Let (N, g) be a three-dimensional connected, simply connected, complete homogeneous Lorentzian manifold. If (N, g) is not symmetric, then $N = G$ is a three-dimensional Lie group and g is left-invariant. Moreover,* *there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$, with e_3 timelike, such that the Lie algebra of G is one of the following.*

- *Type $\mathfrak{g}_1$:*

$$(2.11)\qquad \begin{aligned} [e_1, e_2] &= \alpha e_1 - \beta e_3, \\ [e_1, e_3] &= -\alpha e_1 - \beta e_2, \\ [e_2, e_3] &= \beta e_1 + \alpha e_2 + \alpha e_3, \qquad \alpha \neq 0. \end{aligned}$$

In this case, $G = O(1,2)$ or $G = SL(2,\mathbb{R})$ if $\beta \neq 0$, while $G = E(1,1)$ if $\beta = 0$.

- *Type $\mathfrak{g}_2$:*

$$(2.12)\qquad \begin{aligned} [e_1, e_2] &= \gamma e_2 - \beta e_3, \\ [e_1, e_3] &= -\beta e_2 + \gamma e_3, \\ [e_2, e_3] &= \alpha e_1, \qquad \gamma \neq 0. \end{aligned}$$

In this case, $G = O(1,2)$ or $G = SL(2,\mathbb{R})$ if $\alpha \neq 0$, while $G = E(1,1)$ if $\alpha = 0$.

- *Type $\mathfrak{g}_3$:*

$$(2.13)\qquad \begin{aligned} [e_1, e_2] &= -\gamma e_3, \\ [e_1, e_3] &= -\beta e_2, \\ [e_2, e_3] &= \alpha e_1. \end{aligned}$$

The following Table 1 lists all the Lie groups G which admit a Lie algebra $\mathfrak{g}_3$, taking into account the different possibilities for α, β and γ:

Table 1.

G	α	β	γ
$O(1,2)$ *or* $SL(2,\mathbb{R})$	$+$	$+$	$+$
$O(1,2)$ *or* $SL(2,\mathbb{R})$	$+$	$-$	$-$
$SO(3)$ *or* $SU(2)$	$+$	$+$	$-$
$E(2)$	$+$	$+$	0
$E(2)$	$+$	0	$-$
$E(1,1)$	$+$	$-$	0
$E(1,1)$	$+$	0	$+$
H_3	$+$	0	0
H_3	0	0	$-$
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	0	0	0

- *Type* $\mathfrak{g}_4$*:*

$$(2.14)\qquad \begin{aligned} [e_1, e_2] &= -e_2 + (2\eta - \beta)e_3, \\ [e_1, e_3] &= -\beta e_2 + e_3, \\ [e_2, e_3] &= \alpha e_1, \qquad \eta = \pm 1. \end{aligned}$$

The following Table 2 describes all Lie groups G *admitting a Lie algebra* $\mathfrak{g}_4$*:*

Table 2.

G	α	β
$O(1,2)$ *or* $SL(2,\mathbb{R})$	$\neq 0$	$\neq \eta$
$E(1,1)$	0	$\neq \eta$
$E(1,1)$	< 0	η
$E(2)$	> 0	η
H_3	0	η

- *Type* $\mathfrak{g}_5$*:*

$$(2.15)\qquad \begin{aligned} [e_1, e_2] &= 0, \\ [e_1, e_3] &= \alpha e_1 + \beta e_2, \\ [e_2, e_3] &= \gamma e_1 + \delta e_2, \qquad \alpha + \delta \neq 0,\ \alpha\gamma + \beta\delta = 0. \end{aligned}$$

- *Type* $\mathfrak{g}_6$*:*

$$(2.16)\qquad \begin{aligned} [e_1, e_2] &= \alpha e_2 + \beta e_3, \\ [e_1, e_3] &= \gamma e_2 + \delta e_3, \\ [e_2, e_3] &= 0, \qquad \alpha + \delta \neq 0,\ \alpha\gamma - \beta\delta = 0. \end{aligned}$$

- *Type* $\mathfrak{g}_7$*:*

$$(2.17)\qquad \begin{aligned} [e_1, e_2] &= -\alpha e_1 - \beta e_2 - \beta e_3, \\ [e_1, e_3] &= \alpha e_1 + \beta e_2 + \beta e_3, \\ [e_2, e_3] &= \gamma e_1 + \delta e_2 + \delta e_3, \qquad \alpha + \delta \neq 0,\ \alpha\gamma = 0. \end{aligned}$$

Lie algebras of types $\mathfrak{g}_1$*,* $\mathfrak{g}_2$*,* $\mathfrak{g}_3$ *and* $\mathfrak{g}_4$ *correspond to unimodular groups, whereas Lie algebras of types* $\mathfrak{g}_5$*,* $\mathfrak{g}_6$ *and* $\mathfrak{g}_7$ *correspond to non-unimodular groups.*

Remark 1. In the list given in Theorem 1, we did not include the case when there exists a linear mapping l from $\mathfrak{g}$ to $\mathbb{R}$, such that

$$[x, y] = l(x)y - l(y)x, \tag{2.18}$$

for all $x, y \in \mathfrak{g}$. This case was already investigated by Nomizu [16], who proved that any Lorentzian metric on a Lie group G, whose Lie algebra satisfies (2.18), has constant sectional curvature, and this constant can be any real number (see Theorem 1 of [16]). In particular, G is symmetric. So, this possibility is included in the symmetric case.

We shall focus here our attention on proper homogeneous Lorentzian three-manifolds. So, we shall drop the cases corresponding to Lorentzian space forms and, more in general, to symmetric Lorentzian three-spaces. In [5], where the curvature of homogeneous Lorentzian three-manifolds has been completely described, the following classification results were proved.

Theorem 2. [5] *Let (G, g) be a connected, simply connected three-dimensional Lorentzian Lie group and $\mathfrak{g}$ its Lie algebra. (G, g) has constant sectional curvature if and only if one of the following cases occurs:*

- $\mathfrak{g}$ *is described by* (2.18),
- $\mathfrak{g}$ *is one of the following unimodular Lie algebras:*
  - *(a)* $\mathfrak{g} = \mathfrak{g}_3$ *and either* $G = E(1, 1)$ *with* $\alpha - \gamma = \beta = 0$, *or* $G = E(2)$ *with* $\alpha - \beta = \gamma = 0$;
  - *(b)* $\mathfrak{g} = \mathfrak{g}_3$ *and* $G = O(1, 2)$ *or* $SL(2, \mathbb{R})$ *with* $\alpha = \beta = \gamma \neq 0$;
  - *(c)* $\mathfrak{g} = \mathfrak{g}_4$ *and* $G = H_3$ *with* $\alpha = \beta - \eta = 0$;
- $\mathfrak{g}$ *is one of the following non-unimodular Lie algebras:*
  - *(d)* $\mathfrak{g} = \mathfrak{g}_5$ *with* $\beta + \gamma = 0 \neq \alpha = \delta$;
  - *(e)* $\mathfrak{g} = \mathfrak{g}_6$ *with either* $\beta - \gamma = 0 \neq \alpha = \delta$, *or* $\beta - \varepsilon\alpha = 0 = \gamma - \varepsilon\delta$ *(where* $\varepsilon = \pm 1$*)*;
  - *(f)* $\mathfrak{g} = \mathfrak{g}_7$ *with either* $\alpha = \gamma = 0 \neq \delta$, *or* $\gamma = 0 \neq \alpha = \delta$.

In particular, manifolds in the first case can have any constant sectional curvature, we have flat spaces in cases (a), (c) *and* (f), *positive sectional curvature in case* (d) *and negative sectional curvature in cases* (b) *and* (e).

Theorem 3. [5] *Let (G, g) be a three-dimensional connected, simply connected Lorentzian Lie group and $\mathfrak{g}$ its Lie algebra. (G, g) is symmetric if and only if one of the following occurs.*

- (G, g) *is a Lorentzian Lie group of constant sectional curvature (see Theorem* 2*).*
- $\mathfrak{g} = \mathfrak{g}_5$ *and either* $\alpha = \beta = \gamma = 0 \neq \delta$*, or* $\beta = \gamma = \delta = 0 \neq \alpha$*. In both cases,* G *is isometric to* $\mathbb{R} \times S^2_1$*.*
- $\mathfrak{g} = \mathfrak{g}_6$ *and either* $\alpha = \beta = \gamma = 0 \neq \delta$*, or* $\beta = \gamma = \delta = 0 \neq \alpha$*. In these cases,* G *is isometric to* $\mathbb{R} \times \mathbb{H}^2_1$ *and* $\mathbb{H}^2 \times \mathbb{R}$*, respectively.*
- $\mathfrak{g} = \mathfrak{g}_7$ *and* $\gamma = \delta = 0 \neq \alpha$*. In this case,* (M, g) *has a parallel null vector field.*

From now on, *we always assume that* G *is not a symmetric space (in particular, not a space form)* and so, G is not one of Lie groups listed in either Theorem 2 or 3.

2.3. Algebraic conditions

Using the results of [5], it can be easily seen that, with respect to pseudo-orthonormal frame fields $\{e_1, e_2, e_3\}$ for which (2.11)-(2.17) hold, the curvature components always satisfy $R_{1323} = -R_{1223}$. Here, R_{ABCD} is defined as $\langle R(e_A, e_B) e_C, e_D\rangle$. Thus, with respect to these frame fields, the curvature tensor of the corresponding Lorentzian Lie group is completely determined by

$$\begin{aligned} &D = -R_{1213}, \quad E = R_{1323} = -R_{1223}, \quad I = K_{12} - K_{13}, \\ &J = K_{12} - K_{23}, \quad K = K_{13} - K_{23}, \end{aligned} \tag{2.19}$$

where K_{ij} denotes the sectional curvature of the plane spanned by e_i and e_j. We then obtain the following algebraic restrictions for the existence of a parallel surface.

Lemma 4. *Let* (G, g) *be a connected, simply connected three-dimensional Lorentzian Lie group and* $\{e_1, e_2, e_3\}$ *the pseudo-orthonormal frame field used in Theorem 1. If* M *is a parallel surface in* G *with* ε*-unit normal* $\xi = ae_1 + be_2 + ce_3$*, then the following equations hold on* M*:*

$$\begin{cases} ac(b^2 I + a^2 J) - ab(a^2 + b^2 + c^2)D + a^2(a^2 + (b-c)^2)E = 0, \\ ab(c^2 I + a^2 K) + ac(a^2 - b^2 - c^2)D + a^2(a^2 - (b-c)^2)E = 0, \\ bc(b^2 I + a^2 J) - b^2(a^2 + b^2 + c^2)D + ab(a^2 + (b-c)^2)E = 0, \\ ab(c^2 J - b^2 K) - 2ab^2 cD - b(b-c)(a^2 - b^2 + c^2)E = 0, \\ bc(c^2 I + a^2 K) + c^2(a^2 - b^2 - c^2)D + ac(a^2 - (b-c)^2)E = 0, \\ ac(c^2 J - b^2 K) - 2abc^2 D - c(b-c)(a^2 - b^2 + c^2)E = 0. \end{cases} \tag{2.20}$$

Proof. We only include the proof of the first equation of system (2.20), because the others can be deduced analogously. Since $\xi = ae_1 + be_2 + ce_3$ is normal to M, the vector fields $X = ae_2 - be_1$ and $Y = ae_3 + ce_1$ are tangent to M. By the definitions of D and E, we then obtain

$$\begin{aligned} R(X,Y)X &= R(ae_2 - be_1, ae_3 + ce_1)(ae_2 - be_1) \\ &= (a^3E + a^2cK_{12} - a^2bD)e_1 + (a^2bE + abcK_{12} - ab^2D)e_2 \\ &\quad +(a^3K_{23} + a^2bE - a^2cE + abcD + a^2bE + ab^2K_{13})e_3. \end{aligned}$$

Since M is parallel, the equation of Codazzi gives $\langle R(X,Y)X, \xi\rangle = 0$ and this proves the result. ∎

The following result can now be verified by a straightforward calculation.

Lemma 5. *The solutions of* (2.20) *are the following:*

1. *if* $D = E = 0$*:*
   (a) $a = b = 0$,
   (b) $b = c = 0$,
   (c) $a = c = 0$,
   (d) $a = I = 0$,
   (e) $b = J = 0$,
   (f) $c = K = 0$,
   (g) $I = J = K = 0$;
2. *if* $D \neq 0$ *and* $E = 0$*:*
   (a) $a = 0$, $bc \neq 0$, $bcI - (b^2 + c^2)D = 0$,
   (b) $b = c = 0$,
   (c) $abc \neq 0$, $c^2J - b^2K - 2bcD = 0$,
   $2b^2c^2I + c^2(a^2 - b^2 - c^2)J + b^2(a^2 + b^2 + c^2)K = 0$;
3. *if* $D \neq 0$ *and* $E \neq 0$*:*
   (a) $b = 0$, $ac \neq 0$, $cD + aE = 0$, $acJ + (a^2 + c^2)E = 0$,
   (b) $c = 0$, $ab \neq 0$, $bD - aE = 0$, $abK + (a^2 - b^2)E = 0$,
   (c) $abc \neq 0$,
   $b^2cI + a^2cJ - b(a^2 + b^2 + c^2)D + a(a^2 + (b - c)^2)E = 0$,
   $bc^2I + a^2bK + c(a^2 + b^2 - c^2)D + a(a^2 - (b - c)^2)E$.

Hence, in order to classify parallel surfaces of a homogeneous Lorentzian three-manifold (N, g), one has to check different possibilities listed in Lemma 5, restricting to the ones compatible with the form of the curvature tensor of the ambient space (N, g).

3. Classification of Parallel Surfaces

In the sequel, by "a three-dimensional Lorentzian Lie group G_i" we shall mean a connected, simply connected three-dimensional Lie group G, equipped with a left-invariant Lorentzian metric g and having Lie algebra $\mathfrak{g}_i$.

3.1. Parallel surfaces of G_1

We start by recalling the following result on the Levi Civita connection and the curvature of a three-dimensional Lorentzian Lie group G_1.

Lemma 6. [5] *Let $\{e_1, e_2, e_3\}$ be the pseudo-orthonormal basis used in (2.11). Then*

$$\begin{array}{lll} \nabla_{e_1}e_1 = -\alpha e_2 - \alpha e_3, & \nabla_{e_2}e_1 = \frac{\beta}{2}e_3, & \nabla_{e_3}e_1 = \frac{\beta}{2}e_2, \\ \nabla_{e_1}e_2 = \alpha e_1 - \frac{\beta}{2}e_3, & \nabla_{e_2}e_2 = \alpha e_3, & \nabla_{e_3}e_2 = -\frac{\beta}{2}e_1 - \alpha e_3, \\ \nabla_{e_1}e_3 = -\alpha e_1 - \frac{\beta}{2}e_2, & \nabla_{e_2}e_3 = \frac{\beta}{2}e_1 + \alpha e_2, & \nabla_{e_3}e_3 = -\alpha e_2, \end{array} \tag{3.1}$$

and

$$\begin{array}{lll} R_{1212} = -2\alpha^2 - \frac{\beta^2}{4}, & R_{1313} = \frac{\beta^2}{4} - 2\alpha^2, & R_{2323} = \frac{\beta^2}{4}, \\ R_{1213} = 2\alpha^2, & R_{1223} = -\alpha\beta, & R_{1323} = \alpha\beta. \end{array} \tag{3.2}$$

By Lemmas 5 and 6, we now obtain the following.

Lemma 7. *Let M be a parallel surface of a three-dimensional Lorentzian Lie group G_1. Then, the structure constant β satisfies $\beta = 0$ and the ε-unit normal of M takes the form $\xi = e_1 + be_2 + be_3$.*

Proof. Using (3.2) and the notations introduced in the previous section, we have for G_1 that $I = -4\alpha^2$, $J = -2\alpha^2$, $K = 2\alpha^2$, $D = -2\alpha^2$ and $E = \alpha\beta$, with $\alpha \neq 0$.

We first assume that $E \neq 0$ and so, $\alpha\beta \neq 0$. Cases (3a) and (3b) of Lemma 5 imply that $\alpha = \beta = 0$, which gives a contradiction. In case (3c), the system becomes

$$\begin{cases} (-4b^2c - 2a^2c + 2b(a^2 + b^2 + c^2))\alpha + a(a^2 + (b-c)^2)\beta = 0, \\ (-4bc^2 - 2a^2b - 2c(a^2 - b^2 - c^2))\alpha + a(a^2 - (b-c)^2)\beta = 0. \end{cases}$$

Since $\alpha\beta \neq 0$, the determinant of this system of linear equations in α and β must vanish. A straightforward computation yields that this determinant equals $-2\varepsilon a(b-c)^3$, from which it follows $b = c$. Substituting this in the system we have $a^3\beta = 0$ and hence $\beta = 0$, which is again a contradiction with our assumption.

We now assume that $E = 0$. Since $\alpha \neq 0$, this implies that $\beta = 0$. Case (2a) of Lemma 5 cannot occur since it would imply that $a = 0$ and $b = c$, which would make ξ a null vector field. Cases (2b) and (2c) both reduce to $b = c$. Now $\langle \xi, \xi \rangle = \varepsilon$ yields $\varepsilon = 1$ and $a = \pm 1$. By changing the orientation if necessary, we may assume that $a = 1$. ∎

Theorems 2 and 3 imply that G_1 is never symmetric. We now prove the following.

Theorem 4. *Let M be a parallel surface in a three-dimensional Lorentzian Lie group G_1. Then $\beta = 0$, $\xi = e_1 + be_2 + be_3$ and the vector fields*

$$E_1 = (be_1 - e_2)/\sqrt{1+b^2}, \qquad E_2 = (be_1 + b^2 e_2 + (1+b^2)e_3)/\sqrt{1+b^2}$$

form a pseudo-orthonormal basis for the tangent plane at every point. Moreover, the function b satisfies

$$E_1(b) = E_2(b)$$

and

$$\begin{aligned} & E_1\left(\frac{E_1(b)}{\sqrt{1+b^2}} - \frac{2b}{1+b^2}\alpha\right) \\ & +2\left(\frac{E_1(b)}{\sqrt{1+b^2}} - \frac{2b}{1+b^2}\alpha\right) \\ & \left(\frac{b}{\sqrt{1+b^2}}E_1(b) - \frac{\alpha}{\sqrt{1+b^2}}\right) = 0. \end{aligned} \tag{3.3}$$

The surface is flat and parallel. Moreover, it is totally geodesic in the special case that

$$E_1(b) = E_2(b) = \frac{2b}{\sqrt{1+b^2}}\alpha.$$

Proof. Let M be a parallel surface with ε-unit normal ξ in G_1. According to Lemma 7, the unit normal takes the form $\xi = e_1 + be_2 + be_3$ and $\beta = 0$. Then the vector fields $E_1 = (be_1 - e_2)/\sqrt{1+b^2}$ and $E_2 = (be_1 + b^2 e_2 + (1+b^2)e_3)/\sqrt{1+b^2}$ form a pseudo-orthonormal basis for the tangent plane to M at every point. The integrability condition for the distribution spanned by E_1 and E_2 is

$$E_1(b) = E_2(b). \tag{3.4}$$

Remark that in this case,

$$[E_1, E_2] = \left(\frac{b}{1+b^2}E_1(b) - \frac{\alpha}{\sqrt{1+b^2}}\right)(E_2 - E_1). \tag{3.5}$$

From $SX = -\nabla_X \xi$, we obtain that the shape operator S is given by

$$S = \left(\frac{E_1(b)}{\sqrt{1+b^2}} - \frac{2b}{1+b^2}\alpha\right) \cdot \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \tag{3.6}$$

with respect to the basis $\{E_1, E_2\}$. If M is parallel, then it is either flat or totally umbilical, due to Lemma 1.

Case (i). *M is totally umbilical.* It follow from (3.6) that this only occurs if M is totally geodesic. This gives the special case mentioned in the theorem.

Case (ii). *M is flat.* From (3.6) and Lemma 3, we obtain that M is always flat. Lemma 2 yields that M is parallel if and only if

$$\begin{aligned} &E_i\left(\frac{E_i(b)}{\sqrt{1+b^2}} - \frac{2b}{1+b^2}\alpha\right) \\ &+2\left(\frac{E_i(b)}{\sqrt{1+b^2}} - \frac{2b}{1+b^2}\alpha\right) \\ &\left(\frac{b}{\sqrt{1+b^2}}E_i(b) - \frac{\alpha}{\sqrt{1+b^2}}\right) = 0 \end{aligned} \tag{3.7}$$

for $i = 1, 2$. By using (3.4) and (3.5), we see that it is sufficient to require that b satisfies (3.3). ∎

Remark 2. The second order equation (3.3) stated in Theorem 4 admits solutions, due to the existence and uniqueness theorem for ordinary differential equations.

The proof of Theorem 4 also implies at once the following.

Remark 3. Consider a Lie group G_1 with structure constant $\beta = 0$. Integral surfaces of a distribution spanned by E_1 and E_2 as given in Theorem 4, where b only satisfies $E_1(b) = E_2(b)$, are always flat and hence semi-parallel, but not necessarily parallel.

3.2. Parallel surfaces of G_2

We start with the following.

Lemma 8. [5]. *Let $\{e_1, e_2, e_3\}$ be the pseudo-orthonormal basis used in (2.12). Then*

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0, && \nabla_{e_2} e_1 = \gamma e_2 + \tfrac{\alpha}{2} e_3, && \nabla_{e_3} e_1 = \tfrac{\alpha}{2} e_2 - \gamma e_3,\\
&\nabla_{e_1} e_2 = (\tfrac{\alpha}{2} - \beta) e_3, && \nabla_{e_2} e_2 = -\gamma e_1, && \nabla_{e_3} e_2 = -\tfrac{\alpha}{2} e_1,\\
&\nabla_{e_1} e_3 = (\tfrac{\alpha}{2} - \beta) e_2, && \nabla_{e_2} e_3 = \tfrac{\alpha}{2} e_1, && \nabla_{e_3} e_3 = -\gamma e_1,
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&R_{1212} = -\tfrac{\alpha^2}{4} - \gamma^2, && R_{1313} = \tfrac{\alpha^2}{4} + \gamma^2, && R_{2323} = -\tfrac{3\alpha^2}{4} + \alpha\beta - \gamma^2,\\
&R_{1213} = \gamma(\alpha - 2\beta), && R_{1223} = 0, && R_{1323} = 0.
\end{aligned} \tag{3.9}$$

By using Lemmas 5 and 8 and proceeding as in the proof of Lemma 7, standard calculations give the following.

Lemma 9. *Let M be a parallel surface of a three-dimensional Lorentzian Lie group G_2. Then, there are two possibilities for the structure constants α, β and γ and the ε-unit normal $\xi = ae_1 + be_2 + ce_3$:*

(i) $b = c = 0$,

(ii) $a = 0$ *and* $\alpha = 2\beta$.

Theorems 2 and 3 yield that a Lorentzian Lie group G_2 is never symmetric. We are now ready to prove the following.

Theorem 5. *Let M be a parallel surface in a three-dimensional Lorentzian Lie group G_2. Then, one of the following statements holds.*

(a) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if $\alpha = 0$ and M is parallel, flat and minimal, but not totally geodesic.*

(b) *M is an integral surface of the distribution spanned by $\{e_1, ce_2 + be_3\}$, where b and c are real constants satisfying*

$$b^2 - c^2 = \varepsilon = \pm 1, \qquad bc = -\frac{\varepsilon\beta}{2\gamma}.$$

This case only occurs if $\alpha = 2\beta$ and M is totally geodesic.

Proof. Let M be a parallel surface with ε-unit normal $\xi = ae_1 + be_2 + ce_3$ in G_2. According to Lemma 9, there are two cases to consider.

Case (i). In this case, after a change of orientation if necessary, we have $\xi = e_1$. Hence, the vector fields e_2 and e_3 span the tangent plane at every point of the surface. By (3.8), this distribution is only integrable if $\alpha = 0$. The shape operator is then given by

$$S = \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix}$$

with respect to the pseudo-orthonormal basis $\{e_2, e_3\}$. A straightforward computation yields that $\nabla_{e_i} e_j$ is parallel to ξ for $i, j \in \{2, 3\}$. Together with the fact that the entries of S are constant, this implies that the surface is parallel. Since $\gamma \neq 0$, the surface is minimal but not totally geodesic, and has Gaussian curvature $K = \det S - R_{2323} = 0$, that is, M is flat.

Case (ii). We have $\xi = be_2 + ce_3$ and the vector fields $E_1 = e_1$ and $E_2 = ce_2 + be_3$ form a (pseudo-)orthonormal basis for the tangent plane at every point. The integrability condition for this distribution is

$$bE_1(c) - cE_1(b) = 2bc\gamma + \varepsilon\beta \tag{3.10}$$

and the shape operator with respect to the basis $\{E_1, E_2\}$ is given by

$$S = \begin{pmatrix} 0 & 2bc\gamma + \varepsilon\dfrac{\alpha}{2} \\ -2\varepsilon bc\gamma - \dfrac{\alpha}{2} & -\varepsilon(bE_2(c) - cE_2(b)) \end{pmatrix}. \tag{3.11}$$

We know from Lemma 1 that M must be either flat or totally umbilical.

Case (ii.1). *M is totally umbilical.* From (3.11), we see that this only occurs if M is totally geodesic. Now (3.10), (3.11) and the equality $b^2 - c^2 = \varepsilon$ yield that b and c are constants satisfying $bc = -(\varepsilon\alpha)/(4\gamma)$. This gives case (b) of the theorem.

Case (ii.2). *M is flat.* This is equivalent to requiring that $\langle R(E_1, E_2)E_1, E_2\rangle - \det S = 0$, which gives

$$4\gamma(bc)^2 + 2\varepsilon\alpha(bc) - \gamma = 0, \tag{3.12}$$

By (3.12) it follows that bc is constant, which, together with $b^2 - c^2 = \varepsilon$, implies that both b and c are constant. But then, (3.10) gives $bc = -(\varepsilon\beta)/(2\gamma)$, which contradicts (3.12). ∎

3.3. Parallel surfaces of G_3

The description of the Levi Civita connection and the curvature of a three-dimensional Lorentzian Lie group G_3 is resumed in the following

Lemma 10. [5] *Let $\{e_1, e_2, e_3\}$ be the pseudo-orthonormal basis used in (2.13). Then*

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, && \nabla_{e_2} e_1 = a_2 e_3, && \nabla_{e_3} e_1 = a_3 e_2, \\ &\nabla_{e_1} e_2 = a_1 e_3, && \nabla_{e_2} e_2 = 0, && \nabla_{e_3} e_2 = -a_3 e_1, \\ &\nabla_{e_1} e_3 = a_1 e_2, && \nabla_{e_2} e_3 = a_2 e_1, && \nabla_{e_3} e_3 = 0, \end{aligned} \tag{3.13}$$

where we put

$$a_1 = \frac{1}{2}(\alpha - \beta - \gamma), \quad a_2 = \frac{1}{2}(\alpha - \beta + \gamma), \quad a_3 = \frac{1}{2}(\alpha + \beta - \gamma), \tag{3.14}$$

and

$$\begin{aligned} &R_{1212} = -(a_1 a_2 + \gamma a_3), && R_{1313} = a_1 a_3 + \beta a_2, && R_{2323} = -(a_2 a_3 + \alpha a_3), \\ &R_{1213} = 0, && R_{1223} = 0, && R_{1323} = 0. \end{aligned} \tag{3.15}$$

By Lemmas 5 and 10 we easily get the following.

Lemma 11. *Let M be a parallel surface in a non-symmetric Lorentzian Lie group G_3. Then, the possibilities for the structure constants α, β and γ and the ε-unit normal $\xi = ae_1 + be_2 + ce_3$ are the following:*

(*i*) $a = b = 0$,
(*ii*) $b = c = 0$,
(*iii*) $a = c = 0$,
(*iv*) $a = 0$ *and* $(\beta - \gamma)(-\alpha + \beta + \gamma) = 0$,
(*v*) $b = 0$ *and* $(\alpha - \gamma)(\alpha - \beta + \gamma) = 0$,
(*vi*) $c = 0$ *and* $(\alpha - \beta)(\alpha + \beta - \gamma) = 0$.

We now prove the following classification result.

Theorem 6. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_3. Then, one of the following statements holds.*

(a) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if $\gamma = 0$ and M is flat and minimal, but not totally geodesic.*

(b) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if $\alpha = 0$ and M is flat and minimal, but not totally geodesic.*

(c) *M is an integral surface of the distribution spanned by $\{e_1, e_3\}$. This case only occurs if $\beta = 0$ and M is flat and minimal, but not totally geodesic.*

(d) *M is an integral surface of the distribution spanned by $\{E_1 = e_1, E_2 = ce_2 + be_3\}$, where b and c are functions on M satisfying $b^2 - c^2 = \varepsilon$ and*

$$E_1(b) = \beta c, \quad E_1(c) = \beta b, \quad E_2(b) = k_1 \varepsilon c, \quad E_2(c) = k_1 \varepsilon b,$$

for some real constant k_1. This case only occurs if $\beta = \gamma$ and M is flat.

(e) *M is an integral surface of the distribution spanned by $\{ce_2 + be_3, e_1\}$. Here, b and c are real constants satisfying*

$$b^2 = \frac{\gamma\varepsilon}{\gamma - \beta}, \quad c^2 = \frac{\beta\varepsilon}{\gamma - \beta}.$$

This case only occurs if $\alpha = \beta + \gamma$ and $\beta \neq \gamma$ and M is totally geodesic.

(f) *M is an integral surface of the distribution spanned by $\{E_1 = ce_1 + ae_3, E_2 = e_2\}$, where a and c are functions on the surface satisfying $a^2 - c^2 = \varepsilon$ and*

$$E_1(a) = k_2 \varepsilon c, \quad E_1(c) = k_2 \varepsilon a, \quad E_2(a) = -\alpha c, \quad E_2(c) = -\alpha a,$$

for some real constant k_2. This case only occurs if $\alpha = \gamma$ and M is flat.

(g) *M is an integral surface of the distribution spanned by $\{ce_1 + ae_3, e_2\}$. Here, a and c are real constants satisfying*

$$a^2 = -\frac{\gamma\varepsilon}{\alpha - \gamma}, \quad c^2 = -\frac{\alpha\varepsilon}{\alpha - \gamma}.$$

This case only occurs if $\beta = \alpha + \gamma$ and $\alpha \neq \gamma$ and M is totally geodesic.

(h) *M is an integral surface of the distribution spanned by $\{E_1 = be_1 - ae_2, E_2 = e_3\}$, where a and b are functions satisfying $a^2 + b^2 = 1$ and*

$$E_1(a) = \frac{k_3 b}{b^2 - a^2}, \quad E_1(b) = -\frac{k_3 a}{b^2 - a^2},$$
$$E_2(a) = \frac{b\alpha}{b^2 - a^2}, \quad E_2(b) = -\frac{a\alpha}{b^2 - a^2},$$

for some real constant k_3. This case only occurs if $\alpha = \beta$ and M is flat.

(i) *M is an integral surface of the distribution spanned by $\{be_1 - ae_2, e_3\}$, where a and b are constants satisfying*

$$a^2 = -\frac{\beta}{\alpha - \beta}, \quad b^2 = \frac{\alpha}{\alpha - \beta}.$$

This case only occurs if $\gamma = \alpha + \beta$ and $\alpha \neq \beta$ and M is totally geodesic.

Proof. According to Lemma 11, there are six different cases to consider.

Cases (i), (ii), (iii). Proceeding as in the first case in the proof of Theorem 5, we obtain cases (a), (b), (c), respectively.

Cases (iv). In this case $\xi = be_2 + ce_3$, with $b^2 - c^2 = \varepsilon$ and hence the vector fields $E_1 = e_1$ and $E_2 = ce_2 + be_3$ form a (pseudo-)orthonormal tangent frame field. Using (3.13), we find that the integrability condition for the distribution spanned by $\{E_1, E_2\}$ is given by

$$bE_1(c) - cE_1(b) = -\varepsilon a_1 - c^2 a_2 + b^2 a_3 \tag{3.16}$$

and the shape operator S takes the form

$$S = \begin{pmatrix} 0 & \varepsilon\alpha/2 \\ -\alpha/2 & -\varepsilon(bE_2(c) - cE_2(b)) \end{pmatrix} \tag{3.17}$$

with respect to the basis $\{E_1, E_2\}$. It follows from Lemma 11 that there are two cases to consider, namely $\beta = \gamma$ and $\alpha = \beta + \gamma$.

Case (iv.1): $\beta = \gamma$. In this case, (3.16) and (3.17) reduce to

$$S = \begin{pmatrix} 0 & \varepsilon\alpha/2 \\ -\alpha/2 & -\varepsilon(bE_2(c) - cE_2(b)) \end{pmatrix}, \quad bE_1(c) - cE_1(b) = \varepsilon\beta. \tag{3.18}$$

Since $b^2 - c^2 = \varepsilon$ we then obtain $E_1(b) = \beta c$ and $E_1(c) = \beta b$. It follows from Lemma 3 that M is flat. A straightforward computation shows that $\nabla_{E_i} E_j$ is parallel to ξ for $i, j \in \{1, 2\}$. Hence, (3.18) and Lemma 2 imply that M is parallel if and only if $bE_2(c) - cE_2(b) = k_1$ is constant. Combining this with $b^2 - c^2 = \varepsilon$ gives $E_2(b) = k_1\varepsilon c$ and $E_2(c) = k_1\varepsilon b$. This gives case (d).

Cases (iv.2). $\alpha = \beta + \gamma$. Now (3.16),(3.17) respectively reduce to

$$S = \begin{pmatrix} 0 & b^2\beta - c^2\gamma \\ -\varepsilon(b^2\beta - c^2\gamma) & -\varepsilon(bE_2(c) - cE_2(b)) \end{pmatrix}, \; bE_1(c) - cE_1(b) = b^2\beta - c^2\gamma. \tag{3.19}$$

Since M is parallel in G, M must be totally umbilical or flat. M is totally umbilical if and only if $b^2\beta - c^2\gamma$ and $bE_2(c) - cE_2(b) = 0$, which gives case (e).

By Lemma 3, M is flat if and only if $(b^2\beta - c^2\gamma)^2 = \beta\gamma$. This implies that b and c are constant and hence $S = 0$. Hence, this reduces to the totally umbilical case treated above.

Cases (v), (vi). These cases can be treated in a similar way as case (iv). ∎

The following remark is a direct consequence of the proof of Theorem 6

Remark 4. Consider a Lorentzian Lie group G_3 with structure constants $\beta = \gamma$. If we omit the equations for $E_2(b)$ and $E_2(c)$ in case (d) of Theorem 6, we have more examples of flat, and hence semi-parallel, surfaces. Similarly, case (f), respectively (h), allows us to construct examples of semi-parallel, non-parallel surfaces in G_3 with $\alpha = \gamma$, respectively $\alpha = \beta$.

3.4. Parallel surfaces of G_4

The Levi Civita connection and curvature tensor of a three-dimensional Lorentzian Lie group G_4 are described in the following

Lemma 12. [5] *Let* $\{e_1, e_2, e_3\}$ *be the pseudo-orthonormal basis used in* (2.14). *Then*

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, && \nabla_{e_2} e_1 = e_2 + b_2 e_3, && \nabla_{e_3} e_1 = b_3 e_2 - e_3, \\ &\nabla_{e_1} e_2 = b_1 e_3, && \nabla_{e_2} e_2 = -e_1, && \nabla_{e_3} e_2 = -b_3 e_1, \\ &\nabla_{e_1} e_3 = b_1 e_2, && \nabla_{e_2} e_3 = b_2 e_1, && \nabla_{e_3} e_3 = -e_1, \end{aligned} \tag{3.20}$$

where we put

$$b_1 = \frac{\alpha}{2} + \eta - \beta, \quad b_2 = \frac{\alpha}{2} - \eta, \quad b_3 = \frac{\alpha}{2} + \eta \tag{3.21}$$

and

$$\begin{aligned} &R_{1212} = (2\eta - \beta) b_3 - b_1 b_2 - 1, && R_{1313} = b_1 b_3 + \beta b_2 + 1, && R_{2323} = -(b_2 b_3 + \alpha b_1 + 1), \\ &R_{1213} = 2\eta - \beta + b_1 + b_2, && R_{1223} = 0, && R_{1323} = 0. \end{aligned} \tag{3.22}$$

As a consequence of Lemmas 5 and 12, we get the following algebraic restrictions for a parallel surface of G_4.

Lemma 13. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_4. Then there are the following possibilities for the structure constants α, β and η and the unit normal $\xi = ae_1 + be_2 + ce_3$:*

(*i*) $b = c = 0$,

(*ii*) $a = 0$ *and* $\alpha - 2\beta + 2\eta = 0$,

(*iii*) $a = 1$, $c = -\eta b \neq 0$ *and* $\alpha = 0$, $\beta \neq \eta$,

(*iv*) $a = 1$, $c = -\eta b \neq 0$ *and* $\alpha \neq 0$, $\alpha - \beta + \eta = 0$.

We now prove the following.

Theorem 7. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_4. Then one of the following statements holds.*

(a) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if $\alpha = 0$. M is parallel, flat and minimal, but not totally geodesic.*

(b) *M is an integral surface of the distribution spanned by $\{e_1, ce_2+be_3\}$, where b and c are constants satisfying $b^2-c^2=\varepsilon$ and $\beta b^2+2bc+(\beta-2\eta)c^2=0$. M is totally geodesic and has constant Gaussian curvature $K=-\varepsilon(\beta-\eta)$.*

Proof. We treat separately the four cases listed in Lemma 13.

Case (i). In this case, $\xi = e_1$ and the tangent plane to M is at every point spanned by e_2 and e_3. The distribution spanned by $\{e_2, e_3\}$ is only integrable if $\alpha = 0$. The shape operator S of the surface is given by

$$S = \begin{pmatrix} -1 & -\eta \\ \eta & 1 \end{pmatrix}$$

with respect to $\{e_2, e_3\}$. It is easy to verify that M is parallel, flat and minimal, but not totally geodesic. This gives case (a).

Case (ii). In this case, $\xi = be_2 + ce_3$, with $b^2 - c^2 = \varepsilon$. Hence, the tangent plane to M is at every point spanned by $E_1 = e_1$ and $E_2 = ce_2 + be_3$. Remark that $\langle E_1, E_1\rangle = 1$, $\langle E_1, E_2\rangle = 0$ and $\langle E_2, E_2\rangle = -\varepsilon$. The integrability condition for the distribution spanned by $\{E_1, E_2\}$ is

$$bE_1(c) - cE_1(b) = b^2\beta + 2bc + c^2(\beta - 2\eta) \tag{3.23}$$

and the shape operator with respect to $\{E_1, E_2\}$ is

$$S = \begin{pmatrix} 0 & b^2\beta + 2bc + c^2(\beta - 2\eta) \\ -\varepsilon(b^2\beta + 2bc + c^2(\beta - 2\eta)) & -\varepsilon(bE_2(c) - cE_2(b)) \end{pmatrix}. \tag{3.24}$$

Since M is either totally umbilical or flat, we distinguish two cases.

Case (ii.1). *M is totally umbilical.* It follows from (3.24) that this occurs only if M is totally geodesic. Together with (3.23), this implies that b and c are constants satisfying $b^2\beta + 2bc + c^2(\beta - 2\eta) = 0$ and $b^2 - c^2 = \varepsilon$. This gives case (b).

Case (ii.2). *M is flat.* Using Lemma 3 and (3.24), the condition of flatness can be expressed as a polynomial equation in b and c which, together with $b^2 - c^2 = \varepsilon$, implies that b and c are constant. Then it follows from (3.23) and (3.24) that M is totally geodesic. This case was already treated above.

Case (iii). In this case $\xi = e_1 + be_2 - \eta be_3$. Hence, the tangent plane to M is spanned by the pseudo-orthonormal vector fields $E_1 = (be_1 - e_2)/\sqrt{1+b^2}$ and $E_2 = (\eta be_1 + \eta b^2 e_2 + (1+b^2)e_3)/\sqrt{1+b^2}$. The integrability condition for the distribution spanned by E_1 and E_2 is

$$(1+2b^2)E_1(b) + \eta E_2(b) = \frac{2b^2(1-b^2)(1-\beta\eta)}{\sqrt{1+b^2}} \tag{3.25}$$

and the shape operator with respect to $\{E_1, E_2\}$ is given by

$$S = \begin{pmatrix} -\frac{1+b^2-\beta\eta b^2}{1+b^2} + \frac{E_1(b)}{\sqrt{1+b^2}} & \eta(1+2b^2)\left(\frac{1+b^2-\beta\eta b^2}{1+b^2} - \frac{E_1(b)}{\sqrt{1+b^2}}\right) \\ -\eta(1+2b^2)\left(\frac{1+b^2-\beta\eta b^2}{1+b^2} - \frac{E_1(b)}{\sqrt{1+b^2}}\right) & -(1+2b^2)\left(\frac{1+b^2+\beta\eta b^2}{1+b^2} + \eta\frac{E_2(b)}{\sqrt{1+b^2}}\right) \end{pmatrix}. \tag{3.26}$$

Case (iii.1). *M is totally umbilical.* From $S_{11} = S_{22}$ and $S_{12} = 0$, it follows that

$$E_1(b) = \frac{1+b^2-\beta\eta b^2}{\sqrt{1+b^2}}, \quad E_2(b) = -\eta\frac{1+b^2+\beta\eta b^2}{\sqrt{1+b^2}}. \tag{3.27}$$

Substituting this in (3.25) gives a non-trivial polynomial equation for b, so that b is constant. But this gives a contradiction with (3.27).

Case (iii.2). *M is flat.* From Lemma 3 and the expression for S above, we see that M is flat if and only if

$$\begin{aligned} &\left(\frac{1+b^2-\beta\eta b^2}{1+b^2} - \frac{E_1(b)}{\sqrt{1+b^2}}\right)\left(2 + \eta\frac{E_2(b)}{\sqrt{1+b^2}} - \frac{E_1(b)}{\sqrt{1+b^2}}\right) \\ &= \frac{4b^2(1+b^2)(1-\beta)(1+\eta)}{1+2b^2}. \end{aligned} \tag{3.28}$$

Moreover, since we assume that M is parallel, its mean curvature must be constant, i.e.,

$$-\frac{1+b^2-\beta\eta b^2}{1+b^2} + \frac{E_1(b)}{\sqrt{1+b^2}} - (1+2b^2)\left(\frac{1+b^2+\beta\eta b^2}{1+b^2} + \eta\frac{E_2(b)}{\sqrt{1+b^2}}\right) = C \tag{3.29}$$

for some real constant C. Taking into account (3.25) and (3.29), we can write $E_1(b)/\sqrt{1+b^2}$ and $E_2(b)/\sqrt{1+b^2}$ as rational functions of b. If we substitute these in (3.28), we obtain that b satisfies a non-trivial polynomial equation and hence is constant. From (3.25) we obtain $2b^2(1-b^2)(1-\beta\eta)$. Since we are in case (iii) of Lemma 13, the only possibility is that $b^2 = 1$. It follows now from (3.26) that the entries of S are constant. From Lemma 2 and the non-existence of totally umbilical surfaces in this case (cfr. case (iii.1)), we obtain that $\nabla^M_X E_1 = 0$ for all tangent vectors X. By putting X equal to E_1 and E_2 respectively and straightforward computations, we obtain contradictory equations.

Case (iv). This case can be treated by similar methods as those used in case (iii). ∎

3.5. Parallel surfaces of G_5

We start with the following.

Lemma 14. *Let* $\{e_1, e_2, e_3\}$ *be the pseudo-orthonormal basis used in* (2.15). *Then*

$$\begin{aligned}
&\nabla_{e_1} e_1 = \alpha e_3, && \nabla_{e_2} e_1 = \tfrac{\beta+\gamma}{2} e_3, && \nabla_{e_3} e_1 = -\tfrac{\beta-\gamma}{2} e_2,\\
&\nabla_{e_1} e_2 = \tfrac{\beta+\gamma}{2} e_3, && \nabla_{e_2} e_2 = \delta e_3, && \nabla_{e_3} e_2 = \tfrac{\beta-\gamma}{2} e_1,\\
&\nabla_{e_1} e_3 = \alpha e_1 + \tfrac{\beta+\gamma}{2} e_2, && \nabla_{e_2} e_3 = \tfrac{\beta+\gamma}{2} e_1 + \delta e_2, && \nabla_{e_3} e_3 = 0,
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
&R_{1212} = \alpha\delta - \frac{(\beta+\gamma)^2}{4}, && R_{1313} = -\alpha^2 - \frac{\beta(\beta+\gamma)}{2} - \frac{\beta^2-\gamma^2}{4}, && R_{2323} = -\delta^2 - \frac{\gamma(\beta+\gamma)}{2} + \frac{\beta^2-\gamma^2}{4},\\
&R_{1213} = 0, && R_{1223} = 0, && R_{1323} = 0.
\end{aligned} \tag{3.31}$$

Lemma 5 now implies the following.

Lemma 15. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_5. Then, the structure constants α, β, γ and δ and the unit normal $\xi = ae_1 + be_2 + ce_3$ satisfy one of the following conditions:*

(*i*) $a = b = 0$,

(*ii*) $b = c = 0$,

(*iii*) $a = c = 0$,

(*iv*) $a = 0$ *and* $\alpha = \beta = 0$,

(*v*) $b = 0$ *and* $\gamma = \delta = 0$.

We now prove the following classification result.

Theorem 8. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_5. Then M is one of the surfaces listed below.*

(a) *M is an integral surface of the distribution spanned by e_1 and e_2. M is flat but not totally geodesic.*

(b) *M is an integral surface of the distribution spanned by e_2 and e_3. This case only occurs if either $\beta = \gamma = 0$ or $\gamma = \delta = 0$. In the first case, M is totally geodesic and has constant Gaussian curvature $K = -\delta^2 \leq 0$. In the second case, M is flat and minimal, but not necessarily totally geodesic.*

(c) *M is an integral surface of the distribution spanned by e_1 and e_3. This case only occurs if either $\alpha = \beta = 0$ or $\beta = \gamma = 0$. In the first case, M is flat and minimal, but not necessarily totally geodesic. In the second case, M is totally geodesic and has constant Gaussian curvature $K = \alpha^2 \geq 0$.*

(d) *M is an integral surface of the distribution spanned by $\{E_1 = e_1, E_2 = ce_2 + be_3\}$, where b and c are functions satisfying $b^2 - c^2 = \varepsilon$ and*

$$E_1(b) = E_1(c) = 0, \quad E_2(b) = c(k_1 - c\delta), \quad E_2(c) = b(k_1 - c\delta),$$

for some real constant k_1. This case only occurs if $\alpha = \beta = 0$ and M is flat.

(e) *M is an integral surface of the distribution spanned by $\{E_1 = ce_1 + ae_3, E_2 = e_2\}$, where a and c are functions satisfying $a^2 - c^2 = \varepsilon$ and*

$$E_1(a) = -\varepsilon c(a^2 c\alpha - k_2), \quad E_1(c) = -\varepsilon a(a^2 c\alpha - k_2), \quad E_2(a) = E_2(c) = 0,$$

for some real constant k_2. This case only occurs if $\gamma = \delta = 0$ and M is flat.

Proof. According to Lemma 15, there are five case to consider.

Cases (i), (ii), (iii). These cases can be treated as the corresponding cases in previous theorems and they yield cases (a), (b), (c), respectively.

Case (iv). In this case, the unit normal on M is given by $\xi = be_2 + ce_3$, with $b^2 - c^2 = \varepsilon$. Hence, the tangent plane to M is at every point spanned by $E_1 = e_1$ and $E_2 = ce_2 + be_3$. The integrability condition for the distribution spanned by $\{E_1, E_2\}$ is $bE_1(c) - cE_1(b) = 0$, which, together with $b^2 - c^2 = \varepsilon$, gives

$$E_1(b) = E_1(c) = 0. \tag{3.32}$$

The shape operator with respect to $\{E_1, E_2\}$ is given by

$$S = \begin{pmatrix} 0 & \varepsilon\gamma/2 \\ -\gamma/2 & -\varepsilon(bE_2(c) - cE_2(b) + \varepsilon\delta c) \end{pmatrix}. \tag{3.33}$$

Since M is parallel, it must be either totally umbilical of flat.

Case (iv.1). *M is totally umbilical.* By (3.33) it follows that M is totally geodesic. But then $\gamma = 0$ and so, the ambient space G_5 is symmetric (Theorem 3).

Case (iv.2). *M is flat.* From (3.33) and Lemma 3, M is automatically flat. A straightforward computation shows that $\nabla_{E_i} E_j$ is parallel to ξ for $i, j \in \{1, 2\}$. Hence, M is parallel if and only if the entries of S are constant. This is equivalent to requiring $\varepsilon(bE_2(c) - cE_2(b) + c\varepsilon\delta) = k_1$, where k_1 is a real constant. Together with $b^2 - c^2 = \varepsilon$, we obtain $E_2(b) = c(k_1 - c\delta)$ and $E_2(c) = b(k_1 - c\delta)$, which gives case (d).

Case (v). This case can be treated in a similar way as case (iv). ∎

Remark 5. Consider G_5 with structure constants $\alpha = \beta = 0$. If we omit the equations for $E_2(b)$ and $E_2(c)$ in case (d) of Theorem 8, we obtain more examples of flat, and hence semi-parallel, surfaces. Similarly, from case (e) we can construct examples of semi-parallel, non-parallel surfaces in G_5 with $\gamma = \delta = 0$.

3.6. Parallel surfaces of G_6

The case of a three-dimensional Lie group G_6 is rather similar to the one of G_5. For this reason, we shall omit proofs in this subsection.

Lemma 16. [5] *Let $\{e_1, e_2, e_3\}$ be the pseudo-orthonormal basis used in* (2.16)*. Then*

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_2} e_1 = -\alpha e_2 - \frac{\beta-\gamma}{2} e_3, & \nabla_{e_3} e_1 = \frac{\beta-\gamma}{2} e_2 - \delta e_3, \\ \nabla_{e_1} e_2 = \frac{\beta+\gamma}{2} e_3, & \nabla_{e_2} e_2 = \alpha e_1, & \nabla_{e_3} e_2 = -\frac{\beta-\gamma}{2} e_1, \\ \nabla_{e_1} e_3 = \frac{\beta+\gamma}{2} e_2, & \nabla_{e_2} e_3 = -\frac{\beta-\gamma}{2} e_1, & \nabla_{e_3} e_3 = -\delta e_1, \end{array} \tag{3.34}$$

and

$$\begin{array}{lll} R_{1212} = -\alpha^2 + \frac{\beta^2-\gamma^2}{4} + \frac{\beta(\beta-\gamma)}{2}, & R_{1313} = \delta^2 + \frac{\beta^2-\gamma^2}{4} + \frac{\gamma(\beta-\gamma)}{2}, & R_{2323} = \alpha\delta + \frac{(\beta-\gamma)^2}{4}, \\ R_{1213} = 0, & R_{1223} = 0, & R_{1323} = 0. \end{array} \tag{3.35}$$

Algebraic restrictions to the existence of a parallel surface are then given by the following.

Lemma 17. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_6. Then there are the following possibilities for the structure constants α, β, γ and δ and the unit normal $\xi = ae_1 + be_2 + ce_3$:*

(i) $a = b = 0$,
(ii) $b = c = 0$,
(iii) $a = c = 0$,
(iv) $b = 0$ *and* $\alpha = \beta = 0$,
(v) $c = 0$ *and* $\gamma = \delta = 0$.

We then obtain the following classification result.

Theorem 9. *Let M be a parallel surface in a three-dimensional Lorentzian Lie group G_6. Then, one of the following statements holds.*

(a) *M is an integral surface of the distribution spanned by e_1 and e_2. This case only occurs if either $\alpha = \beta = 0$ or $\beta = \gamma = 0$. In the first case, M is parallel, flat and minimal, but not necessarily totally geodesic. In the second case, M is totally geodesic.*

(b) *M is an integral surface of the distribution spanned by e_2 and e_3. M is parallel and flat, but not necessarily totally geodesic.*

(c) *M is an integral surface of the distribution spanned by e_1 and e_3. This case only occurs if either $\beta = \gamma = 0$ or $\gamma = \delta = 0$. In the first case, M is totally geodesic. In the second case, M is parallel, flat and minimal, but not necessarily totally geodesic.*

(d) *M is an integral surface of the distribution spanned by $\{E_1 = ce_1 + ae_3, E_2 = e_2\}$, where a and c are functions satisfying $a^2 - c^2 = \varepsilon$ and*

$$E_1(a) = c(k_1 - \delta a), 1 \quad E_1(c) = a(k_1 - \delta a), \quad E_2(a) = E_2(c) = 0,$$

for some real constant k_1. This case only occurs if $\alpha = \beta = 0$ and M is parallel and flat.

(e) *M is an integral surface of the distribution spanned by $\{E_1 = be_1 - ae_2, E_2 = e_3\}$, where a and b are functions satisfying $a^2 + b^2 = 1$ and*

$$E_1(a) = b(k_2 + \alpha a), \quad E_1(b) = -a(k_2 + \alpha b), \quad E_2(a) = E_2(c) = 0,$$

for some real constant k_2. This case only occurs if $\gamma = \delta = 0$ and M is parallel and flat.

Remark 6. Consider a Lorentzian Lie group G_6 with $\alpha = \beta = 0$. Omitting the equations for $E_1(a)$ and $E_1(c)$ in case (d) of Theorem 9, we obtain more examples of flat, and hence semi-parallel surfaces. Similarly, case (e) allows us to construct examples of semi-parallel, non-parallel surfaces in G_6 with $\gamma = \delta = 0$.

3.7. Parallel surfaces of G_7

We first recall the following.

Lemma 18. [5]. *Let* $\{e_1, e_2, e_3\}$ *be the pseudo-orthonormal basis used in* (2.17). *Then*

$$\begin{aligned}
&\nabla_{e_1}e_1=\alpha e_2+\alpha e_3, && \nabla_{e_2}e_1=\beta e_2+\left(\beta+\tfrac{\gamma}{2}\right)e_3, && \nabla_{e_3}e_1=-\left(\beta-\tfrac{\gamma}{2}\right)e_2-\beta e_3,\\
&\nabla_{e_1}e_2=-\alpha e_1+\tfrac{\gamma}{2}e_3, && \nabla_{e_2}e_2=-\beta e_1+\delta e_3, && \nabla_{e_3}e_2=\left(\beta-\tfrac{\gamma}{2}\right)e_1-\delta e_3,\\
&\nabla_{e_1}e_3=\alpha e_1+\tfrac{\gamma}{2}e_2, && \nabla_{e_2}e_3=\left(\beta+\tfrac{\gamma}{2}\right)e_1+\delta e_2, && \nabla_{e_3}e_3=-\beta e_1-\delta e_2,
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&R_{1212}=\alpha\delta-\alpha^2-\beta\gamma-\tfrac{\gamma^2}{4}, && R_{1313}=\alpha\delta-\alpha^2-\beta\gamma+\tfrac{\gamma^2}{4}, && R_{2323}=-\tfrac{3}{4}\gamma^2,\\
&R_{1213}=\alpha^2-\alpha\delta+\beta\gamma, && R_{1223}=0, && R_{1323}=0.
\end{aligned} \tag{3.37}$$

By Lemmas 5 and 18, we easily obtain the following

Lemma 19. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_7. Then, the possibilities for the structure constants α, β,γ and δ and the unit normal $\xi = ae_1 + be_2 + ce_3$ are the following:*

(i) $b = c = 0$,

(ii) $a = 0$ *and* $\alpha = \beta = 0$,

(iii) $a = 1$, $b = c \neq 0$ *and* $\gamma = 0$, $\alpha(\alpha - \delta) \neq 0$.

We now prove the following.

Theorem 10. *Let M be a parallel surface in a non-symmetric three-dimensional Lorentzian Lie group G_7. Then, M is one of surfaces listed below.*

(a) *M is an integral surface of the distribution spanned by $\{e_2, e_3\}$. This case only occurs if either $\beta = \gamma = 0$ or $\gamma = \delta = 0$. In the first case, M is totally geodesic. In the second case, M is parallel and flat, but not necessarily totally geodesic.*

(b) *M is an integral surface of the distribution spanned by $\{E_1 = e_1, E_2 = ce_2 + be_3\}$, where b and c are functions satisfying $b^2 - c^2 = \varepsilon$ and*

$$E_1(b) = E_1(c) = 0, \quad E_2(b) = c((b-c)\delta - k_1), \quad E_2(c) = b((b-c)\delta - k_1)$$

for some real constant k_1. This case only occurs if $\alpha = \beta = 0$. M is flat, but not necessarily totally geodesic.

(c) *M is an integral surface of the distribution spanned by* $\{E_1 = (be_1 - e_2)/\sqrt{1+b^2}$ *and* $E_2 = (be_1 + b^2e_2 + (1+b^2)e_3)/\sqrt{1+b^2}\}$*, where b is a function satisfying* $E_1(b) = E_2(b)$ *and*

$$\begin{aligned} & E_1\left(\frac{E_1(b)}{\sqrt{1+b^2}} + \frac{b}{1+b^2}(\alpha-\delta)\right) \\ & +2\left(\frac{E_1(b)}{\sqrt{1+b^2}} + \frac{b}{1+b^2}(\alpha-\delta)\right) \\ & \left(\frac{b}{\sqrt{1+b^2}}E_1(b) - \frac{\delta}{\sqrt{1+b^2}}\right) = 0. \end{aligned}$$

The surface is flat and parallel. Moreover, it is totally geodesic in the special case that

$$E_1(b) = E_2(b) = \frac{b}{\sqrt{1+b^2}}(\delta-\alpha).$$

Proof. According to Lemma 19, there are three different cases to consider.

Cases (i), (ii). This cases can be treated as the corresponding ones in previous theorems.

Case (iii). In this case, the unit normal on M is given by $\xi = e_1 + be_2 + be_3$. Hence, the tangent plane at every point is spanned by the vector fields $E_1 = (be_1 - e_2)/\sqrt{1+b^2}$ and $E_2 = (be_1 + b^2e_2 + (1+b^2)e_3)/\sqrt{1+b^2}$. Remark that $\langle E_1, E_1\rangle = 1$, $\langle E_1, E_2\rangle = 0$ and $\langle E_2, E_2\rangle = -1$. The integrability condition for the distribution spanned by $\{E_1, E_2\}$ is

$$E_1(b) = E_2(b) \tag{3.38}$$

and the shape operator with respect to the basis $\{E_1, E_2\}$ is

$$S = \left(\frac{E_1(b)}{\sqrt{1+b^2}} + \frac{b}{1+b^2}(\alpha-\delta)\right)\cdot\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}. \tag{3.39}$$

Since M is parallel in G_7, M must be either totally umbilical or flat.

Case (iii.1). If M is totally umbilical, it follows from (3.39) that M is totally geodesic. This gives the special case mentioned in case (c).

Case (iii.2). This case can be treated in a similar way as in the proof of Theorem 4. ■

Remark 7. Consider a Lorentzian Lie group G_7 with structure constants $\alpha = \beta = 0$. If we omit the equations for $E_2(b)$ and $E_2(c)$ in case (b) of Theorem 10, we obtain more flat, and hence semi-parallel but not necessarily parallel, surfaces.

4. Remarks and Conclusions

It is worthwhile to compare the classification results given in this paper with the corresponding ones in the Riemannian framework, obtained by the second author and J. Inoguchi in [11, 12]. It turns out that, also from the point of view of the existence of parallel surfaces, homogeneous Lorentzian three-manifolds offer many more possibilities than their Riemannian analogues.

There is an alternative, more analytic way to represent several of the parallel surfaces of homogeneous Lorentzian three-spaces we found in theorems above. As an example, we illustrate here case (d) of Theorem 9. Assume that $\varepsilon = 1$ (when $\varepsilon = -1$, one can proceed in a similar way). Then, there exists a function φ such that $a = \cosh\varphi$ and $c = \sinh\varphi$. One can check that $[E_1, E_2] = 0$. Hence, it is possible to introduce pseudo-Euclidean coordinates (u, v) on M, such that $E_1 = \partial_u$ and $E_2 = \partial_v$. The equations stated in Theorem 9 are then equivalent to

$$\partial_u\varphi = k_1 - \delta\cosh\varphi, \quad \partial_v\varphi = 0.$$

One can solve the equations above by direct integration and state that M is given by an isometric immersion $f : U \subseteq \mathbb{E}^2_1 \to G : (u, v) \mapsto f(u, v)$, such that $f_*\partial_u = \sinh\varphi(u)e_1 + \cosh\varphi(u)e_3$ and $f_*\partial_v = e_2$.

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