

## ON NEAR RELATIVE PRIME NUMBER IN A SEQUENCE OF POSITIVE INTEGERS

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**Abstract.** Let  $A$  be a sequence of positive integers. An element  $\alpha$  of  $A$  is called an  $s$ -near relative prime number ( $s$ -near relprime in short) if  $\alpha$  is coprime to any distinct element in  $A$  except exactly  $s$  elements of  $A$ . In this paper, we study the existence of an arithmetic sequence with no 1-near relprimes.

### 1. INTRODUCTION AND PRELIMINARIES

The study of primes plays the most important role in Number Theory. It is well-known that the number of primes is infinite and also for each positive integer  $n$  there are  $n$  consecutive integers which are not primes. Therefore, we may have a very long sequence of integers which contains no primes. But, it is quite possible to find an integer  $a$  in a sequence of integers  $A$  such that for each  $b \in A$  and  $b \neq a$ , the greatest common divisor of  $a$  and  $b$  is 1. For convenience, in what follows, we say  $a$  and  $b$  are coprime and  $a$  is a relprime of  $A$ . A very fundamental argument can show that for any sequence of at most 16 consecutive positive integers contains at least one relprime. But, for 17 and larger, the existence of relprimes is in doubt. First, in [4], Pillai proved that for  $l = 17, 18, \dots, 430$ , there exists a sequence of  $l$  consecutive integers which has no relprimes. Then, in [2], Evans generalized the result of Pillai and showed that for  $l \geq 17$ , there exists a sequence of  $l$  consecutive integers which has no relprimes. Quite recently, the work was extended to study the existence of relprime in a sequence  $A = \langle a, a + k, a + 2k, \dots, a + (n - 1)k \rangle$  for  $k \geq 2$ . Ohtomo and Tamari were able to prove the following theorem.

**Theorem 1.1.** ([3]). *For every positive  $k$ , there is a positive integer  $l_0(k)$  such that for all integer  $n \geq l_0(k)$ , there exists a sequence  $A = \langle a, a + k, a + 2k, \dots, a + (n - 1)k \rangle$  which has no relprimes.*

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By observation, it is not difficult to see that in a sequence of positive integers  $A$ , even we can not find a relprime, we are able to find a number in  $A$  which is very close to being a relprime. For example, in  $A = \langle 2184, 2185, \dots, 2200 \rangle$ , a relprime does not exist. But, if we are looking for an integer in  $A$  which is coprime to almost any other number except one, then 2189 and 2197 are the integers, we call them 1-near relprimes. See Figure 1. (It is easily seen that the prime factors 2, 3, 5, 7, 11 and 13 occur at least twice.) The definition of an  $s$ -near relprime can be defined accordingly.

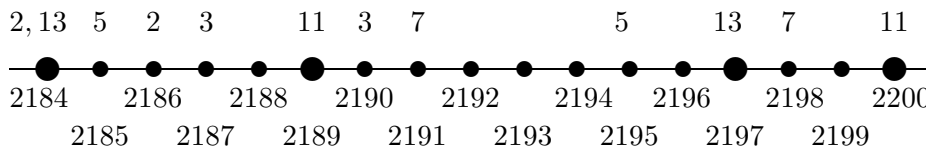


Figure 1: a sequence with two 1-near relprimes

**The Chinese Remainder Theorem.** Suppose  $n_1, n_2, \dots, n_r$  are coprime positive integers. Let  $a_1, a_2, \dots, a_r$  be any integers. Then there is a number  $x$  whose remainder on division by  $n_i$  is  $a_i$ , for each  $i$ . That is, the system of linear congruences  $x \equiv a_i \pmod{n_i}$  has a solution.

Therefore, in order to find a sequence which contains no  $s$ -near relprimes, it suffices to arrange the prime factors less than the length of the sequence such that each of them occurs at least  $s + 2$  times. Note that such sequence can be obtained by using the Chinese Remainder Theorem. For example, in Figure 2, we have  $a_1 \equiv 0 \pmod{2}$ ,  $a_1 \equiv 2 \pmod{3}$ ,  $a_1 \equiv 0 \pmod{5}$ ,  $a_1 \equiv 5 \pmod{7}$ ,  $a_1 \equiv 0 \pmod{11}$ , and  $a_1 \equiv 10 \pmod{13}$  after arranging the prime factors in  $\{2, 3, 5, 7, 11, 13\}$ . In fact, there are infinite many solutions for  $a_1$ , say 27830, 57860, ...

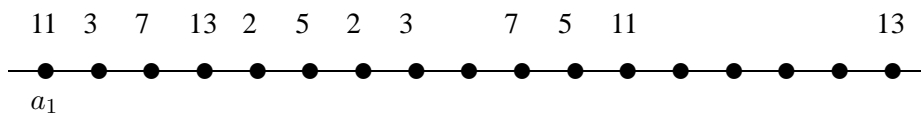


Figure 2: a sequence with no relprimes which is obtained by using the first 6 prime factors

Indeed, comparing to find a sequence with no relprimes, finding a sequence of consecutive integers which contains no  $s$ -near relprime is getting harder. By using of a computer program, we can construct a sequence of 41 consecutive integers starting with 249, 606, 071, 931 such that no 1-near relprimes exist. In other words, every integer contains at least 2 non-coprimes. As to 2-near relprime case, the sequence of consecutive integers without 2-near relprimes that we are able to find is

a sequence of 115 integers with leading number

18, 513, 236, 242, 025, 789, 239, 118, 205, 871, 632, 181, 843, 428.

To extent the study of Theorem 1.1, we can also investigate the existence of a sequence  $\langle a, a+k, a+2k, \dots, a+(n-1)k \rangle$  such that no 1-near relprimes exist where  $k > 1$ . To see the difficulty of this study, we consider  $A$  a set of consecutive odd integers. Then, by using a computer program, we can find a sequence of 229 odd integers starting from 17867559870915810441505496179807231100678713 which contains no relprimes. Compare to finding consecutive integers, this is a sequence of huge numbers and we are not able to find a sequence of smaller integers satisfying this condition at this moment.

Note that if  $(a, k) \neq 1$  holds, then clearly  $A$  has no  $s$ -near relprimes. So, we only treat the case  $(a, k) = 1$  throughout this paper, and we shall prove the following theorem.

**Theorem 1.2.** *For every positive integer  $k$ , there is a positive integer  $l_0(k)$  such that for all integer  $n \geq l_0(k)$ , there exists an arithmetic sequence  $A = \langle a, a+k, a+2k, \dots, a+(n-1)k \rangle$  which has no 1-near relprimes.*

In other words, every element in this arithmetic sequence  $A$  with length  $n$  (large enough) contains at least 2 non-coprimes. This result also proves Theorem 1.1.

## 2. THE MAIN RESULT

Since we are mainly looking for a sequence which contains mostly the composite integers, the estimation of the number of primes between two distinct integers is crucial. The following result plays an important role in counting the number of primes.

**Lemma 2.1.** ([1] cf. Erdős 1949). *For every  $\lambda > 1$ , there exist real numbers  $C(\lambda) > 0$  and  $x_0 = x_0(\lambda)$  such that*

$$C(\lambda) \frac{x}{\log x} < \pi(\lambda x) - \pi(x)$$

for  $x \geq x_0(\lambda)$ , where  $\pi(x)$  denotes the number of primes  $\leq x$ .

In counting, we also need an inequality which is a basic result in Calculus.

**Lemma 2.2.** *Let  $n$  be a positive integer and  $\varepsilon$  and  $c$  be positive numbers. Then, there exists a positive number  $N_0 = N_0(\varepsilon, n, c)$  such that  $(\log cx)^n \leq \varepsilon x$  for  $x \geq N_0$ .*

The plan of our proof is to form a sequence of  $n$  (sufficiently large) positive integers  $A = \langle a, a + k, a + 2k, \dots, a + (n - 1)k \rangle$  such that for each  $x \in A$  there exist at least 2 distinct integers in  $A$  which are not relatively prime with  $x$ . Since  $k$  is given, it suffices to find  $a$  and  $n$ . For brevity, we prove only for the case when  $n = 4m + 1$ ,  $m \geq 2$ , and the other cases  $4m, 4m + 2, 4m + 3$  are similar. Therefore, we may let all sequences we consider throughout this paper are of the following form  $A = \{a_{-2m}, a_{-2m+1}, \dots, a_{-1}, a_0, a_1, \dots, a_{2m-1}, a_{2m}\}$  in which  $a_i - a_{i-1} = k \geq 2$ ,  $i = -2m + 1, -2m + 2, \dots, 2m$ . In  $A$ , we also let  $J_A(a_0, m) = \{j | (a_0, a_j) = 1, 1 \leq j \leq 2m\}$  and  $|J_A(a_0, m)| = \mu_A(a_0, m)$ . In case that  $\mu_A(a_0, m)$  is fixed, we denote  $\mu_A(a_0, m)$  by  $\mu$  for brevity. The following lemma is essential to the proof of our main theorem.

**Lemma 2.3.** *Given  $k$ . Then, there exist  $a_0$  and  $m$  such that*

$$(1) \quad 4\mu(a_0, m) \leq \pi(2m) - \pi(m) \text{ and}$$

$$(2) \quad 16\mu(a_0, m) \leq \pi(3m) - \pi(2m).$$

*Proof.* For each positive integer  $m$ , let  $K = \{k_1, k_2, \dots, k_\alpha\}$  be the set of prime factors of  $k$  and let  $Q = \{q_1, q_2, \dots, q_\beta\}$ ,  $R = \{r_1, r_2, \dots, r_\gamma\}$  and  $T = \{t_1, t_2, \dots, t_\delta\}$  be the set of primes in  $(0, 2m)$ ,  $(m, 2m)$  and  $(2m, 3m)$  respectively. For convenience, the primes in  $K$ ,  $Q$ ,  $R$  and  $T$  are listed in increasing order. Now, let  $a_0$  be a positive integer such that  $q|a_0$  for each  $q \in Q \setminus K$ . Then it is easily checked that  $(a_0, a_j) = 1$  if and only if  $j = \prod_{i=1}^\alpha k_i^{f_i}$  where  $f_i$ 's are non-negative integers. Let  $l_i$ ,  $i = 1, \dots, \alpha$ , be the maximum integers satisfying  $k_i^{l_i} \leq 2m$ . Then, we have  $\mu(a_0, m) \leq (l_1 + 1) \cdots (l_\alpha + 1)$ . Therefore,

$$\begin{aligned} \mu(a_0, m) &\leq \prod_{i=1}^\alpha \left( \frac{\log 2m}{\log k_i} + 1 \right) \\ &\leq \left( \frac{\log 2m}{\log k_1} + 1 \right)^\alpha \\ &\leq \left( \frac{1}{\log k_1} \right)^\alpha \frac{(\log 2k_1 m)^{\alpha+1}}{\log 2m} \\ &\leq \left( \frac{1}{\log k_1} \right)^\alpha \frac{(\log 2k_1 m)^{\alpha+1}}{\log m} \end{aligned}$$

holds. Let  $C(2)$  and  $x_0(2)$  be positive numbers defined in Lemma 2.1 such that

$$C(2) \frac{m}{\log m} \leq \pi(2m) - \pi(m)$$

for  $m \geq x_0(2)$ . Let  $\varepsilon = \frac{1}{4}(\log k_1)^\alpha C(2)$ . By Lemma 2.2, there exists an  $N_0 = N_0(\varepsilon, \alpha + 1, 2k_1)$  such that

$$(\log 2k_1 m)^{\alpha+1} \leq \frac{1}{4}(\log k_1)^\alpha C(2)m$$

for  $m \geq N_0$ . Then we have

$$\mu(a_0, m) \leq \frac{1}{4}C(2) \frac{m}{\log m}$$

for  $m \geq N = \max(N_0, x_0(2))$ . Therefore, we have

$$4\mu(a_0, m) \leq \pi(2m) - \pi(m).$$

satisfying the condition (2.1).

By the same process, let  $C(\frac{3}{2})$  and  $x_0(\frac{3}{2})$  be positive numbers defined in Lemma 2.1 such that

$$C\left(\frac{3}{2}\right) \frac{2m}{\log 2m} < \pi(3m) - \pi(2m)$$

for  $2m \geq x_0(\frac{3}{2})$ . Let  $\varepsilon' = \frac{1}{16}(\log k_1)^\alpha C(\frac{3}{2})$ ,  $x'_0 = \frac{1}{2}x_0(\frac{3}{2})$  and  $N_1 = N(\varepsilon', \alpha + 1, k_1)$ , where  $N(\varepsilon', \alpha + 1, k_1)$  is the positive number described in Lemma 2.2. Then, from Lemma 2.1 and Lemma 2.2,

$$16\mu(a_0, m) \leq \pi(3m) - \pi(2m)$$

which is the inequality (2.2). Now, by choosing  $m \geq \max(N_0, x_0(2), N_1, \frac{1}{2}x_0(\frac{3}{2}))$ , we have the desired  $a_0$  and  $m$ . This concludes the proof. ■

By observation, the candidates of 1-near relprimes in  $A$  are  $a_j$  and  $a_{-j}$  where  $j \in J_A(a_0, m)$  (Lemma 2.3) and the total number of candidates is  $2\mu_A(a_0, m)$ . All the others are not 1-near relprimes, since for each  $i \notin J_A(a_0, m)$ ,  $a_0$ ,  $a_i$  and  $a_{-i}$  do have common divisors larger than 1. Now, we are ready to prove the main result. Review that we shall arrange the prime factors less than the length of the sequence  $4m + 1$  such that each prime factor occurs at least three times. Therefore, at the beginning stage,  $a_i \in A$  represents the position of an integer and the exact solution for  $A$  will be obtained later by using the Chinese Remainder Theorem.

*Proof of Theorem 1.2.* By Lemma 2.3, there exist  $a_0$  and  $m$  such that equations 1 and 2 hold. It follows that there are at most  $2\mu_A(a_0, m) = 2\mu$  1-near relprimes in  $A$ .

If these  $2\mu$  integers are indeed not 1-near relprimes in  $A$ , then we are done. Otherwise, we have to arrange prime factors such that these  $2\mu$  integers do have common prime factors with at least two other integers in  $A$ . In order to do that we rename  $A$  as  $A' = \langle a'_{-2m}, \dots, a'_0, \dots, a'_{2m} \rangle$  where the following three conditions hold.

- (1)  $q|a'_0$  for each  $q \in Q \setminus (K \cup \{r_1, r_2, \dots, r_{4\mu}\})$  where  $r_1, \dots, r_{4\mu}$  are  $4\mu$  primes in  $(m, 2m)$ . (By Lemma 2.3.)
- (2)  $r_i|a'_{j_i}, r_{i+\mu}|a'_{j_i}$  for  $j_i \in J_A(a_0, m), 1 \leq i \leq \mu$ .
- (3)  $r_{i+2\mu}|a'_{-j_i}, r_{i+3\mu}|a'_{-j_i}$ , for  $j_i \in J_A(a_0, m), 1 \leq i \leq \mu$ .

By the above adjustment, we conclude that  $a'_{j_i}$  and  $a'_{-j_i}$  are not 1-near relprimes any more for each  $j_i \in J_A(a_0, m)$ . But, now, we have to make sure the  $8\mu$  integers  $a'_i, i \in \{\pm r_1, \pm r_2, \dots, \pm r_{4\mu}\}$ , where  $|i| > m$ , are not 1-near relprimes. So, again, we use the trick to arrange the  $16\mu$  primes,  $t_1, t_2, \dots, t_{16\mu}$  in  $(2m, 3m)$ , obtained from Lemma 2.3 to the above  $8\mu$  integers as their prime factors (two for each one of them). This can be done by letting

- (4)  $t_i|a'_{r_i}, t_{i+4\mu}|a'_{r_i}$  for  $i = 1, 2, \dots, 4\mu$ ; and
- (5)  $t_{i+8\mu}|a'_{-r_i}, t_{i+12\mu}|a'_{-r_i}$ , for  $i = 1, 2, \dots, 4\mu$ .

Now since  $m < r_i < 2m$  and  $2m < t_i < t_{i+4\mu} < 3m$ , we have  $-2m < r_i - t_i < 0$  and  $-2m < r_i - t_{i+4\mu} < 0$ . Then  $(a'_{r_i}, a'_{r_i - t_i}) = t_i > 0$  and  $(a'_{r_i}, a'_{r_i - t_{i+4\mu}}) = t_{i+4\mu} > 0$ , where  $i = 1, 2, \dots, 4\mu$ . By the same process,  $(a'_{-r_i}, a'_{-r_i + t_{i+8\mu}}) = t_{i+8\mu} > 0$  and  $(a'_{-r_i}, a'_{-r_i + t_{i+12\mu}}) = t_{i+12\mu} > 0$ , where  $i = 1, 2, \dots, 4\mu$ . Therefore, all integers in  $A'$  are not 1-near relprimes in their corresponding positions. By the Chinese Remainder Theorem, there exists a sequence satisfying the equations (1) to (5). This completes the proof.  $\blacksquare$

### 3. CONCLUDING REMARKS

For the case  $s > 1$  or  $k > 1$ , the following problem is an interesting one. For which  $s$  and  $k$ , there is a sequence  $A = (a, a + k, a + 2k, \dots, a + (n - 1)k)$  containing no  $s$ -near relprimes. We believe that for fixed integers  $s$  and  $k$  such a sequence does exist, but we are not able to obtain a proof at this moment.

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