TAIWANESE JOURNAL OF MATHEMATICS
Vol. 14, No. 1, pp. 35-45, February 2010
This paper is available online at http://www.tjm.nsysu.edu.tw/

SOME CONSEQUENCES OF A THEOREM ON FANS

David Cariolaro

Abstract. Using a fundamental identity concerning fans called "Fan Theorem" we give new proofs of classical edge colouring theorems. We also derive from the same identity a formula for the chromatic index of Class 2 multigraphs (i.e. multigraphs G such that $\chi'(G) > \Delta(G)$) and a new generalization of Vizing's Adjacency Lemma to multigraphs, which is more general than the one given by the author in [D. Cariolaro, *On fans in multigraphs*, J. Graph Theory, 51 (4), 2006, 301–318].

1. Introduction

All graphs considered in this paper are finite and without loops but may have multiple edges. The term "multigraph" will be used as a synonym of "graph". We shall use the term "simple graph" to denote graphs with no multiple edges. Let G be a graph. We denote the *degree* of a vertex v in G, i.e. the number of edges incident with v, by $d_G(v)$. The maximum degree will be denoted by $\Delta(G)$. If the edge e joins the vertices u and v, we denote this by $e \in uv$ (or $e = uv$, if G is a simple graph). The number of edges joining two given distinct vertices u and v in G is called the *multiplicity of the edge* uv and denoted by $\mu_G(uv)$. We denote the *maximum multiplicity* of G (i.e. the maximum of the multiplicities of the edges of G) by $\mu(G)$. Notation and terminology, not explicity introduced here, will follow Bollobás [2]. For an introduction to edge colouring we refer the reader to Fiorini and Wilson [4].

An edge colouring of a graph G is a map $\varphi : E(G) \to \mathcal{C}$, where $\mathcal{C}$ is a set called the *colour set*, and φ has the property that $\varphi(e_1) \neq \varphi(e_2)$ for any pair e_1, e_2 of adjacent edges. If $\mathcal{C}$ is chosen to have the minimum cardinality, then φ is called an *optimal colouring* and $|\mathcal{C}|$ is called the *chromatic index* of G, denoted by $\chi'(G)$.

Received February 9, 2006, accepted April 11, 2008.
Communicated by Hung-Lin Fu.
2000 *Mathematics Subject Classification*: 05C15.
Key words and phrases: Critical graph, Chromatic index, Fan, Fan Digraph, Fan Theorem.

It is immediate to see that $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of G, as all the edges incident with a vertex of maximum degree must receive a different colour. We call G *Class 1* if $\chi'(G) = \Delta(G)$ and *Class 2* otherwise. To establish reasonable upper bounds on the chromatic index seems to be far less trivial than to establish lower bounds. The two main theorems in this respect are the inequalities $\chi'(G) \leq \lfloor \frac{3}{2}\Delta(G) \rfloor$ for any graph G (Shannon's Theorem [8]) and $\chi'(G) \leq \Delta(G) + \mu(G)$ for any graph G (Vizing's Theorem [9]), which are both cornerstones in the theory of edge colouring.

Notice that Vizing's theorem gives the sharp bound $\chi'(G) \leq \Delta(G) + 1$ in the case that G is a simple graph. Notwithstanding this excellent approximation, to decide whether a simple graph G is Class 1 is NP-complete (see [6]). This fact seems to suggest that, if we are given the information that the graph G is Class 2, then it would be somehow possible to determine $\chi'(G)$ exactly (this fact holds trivially for simple graphs in view of Vizing's Theorem). This is exactly what we will achieve here by deriving an exact expression for the chromatic index of a Class 2 multigraph (unfortunately it will be seen that the computation of some of the objects appearing in the formula itself is NP-hard). Nonetheless the formula is useful, not only in itself, but also to derive non-trivial upper bounds on the chromatic index of a multigraph. Our result lends credit to (but unfortunately, does not seem to imply in any obvious way) the following well known conjecture.

Goldberg-Seymour Conjecture. Let G be a Class 2 multigraph. Then

$$\chi'(G) = \max\{\Delta(G) + 1, \max\lceil |E(H)|/\lfloor |V(H)|/2 \rfloor \rceil\},$$

where the maximum is extended to all submultigraphs H of G of order at least two.

We begin with a simultaneous proof of Vizing's theorem (for simple graphs) [9] and Vizing's Adjacency Lemma [10, 11]. We then obtain, with a method slightly different than the one we used in [3], a short proof of Ore's Theorem [7] (which implies the multigraph version of Vizing's Theorem [9]). We then prove the exact formula for the chromatic index mentioned above and, using this, we give a proof of the Andersen-Goldberg upper bound on the chromatic index of a multigraph [1, 5], which implies Shannon's Theorem [8]. We conclude with a new generalization of Vizing's Adjacency Lemma to multigraphs, which extends the one given by the author in [3], and hence is (to the best of our knowledge) more general than all the other existing ones.

All the above results are deduced as corollaries of the same identity. We believe that this identity is so fundamental in edge colouring that it deserves to have a name, and we have christened it in [3] "Fan Theorem" (it appears to have been used implicitly by other authors, e.g. Andersen [1], Goldberg [5], but has never been stated as a theorem on its own). For details concerning the definitions and proof of the Fan Theorem, we refer the reader to [3].

2. Preliminary Results and Definitions

An edge e of a multigraph G is called *critical* if $\chi'(G-e) < \chi'(G)$. The multigraph G itself is called *critical* if it is Class 2, has no isolated vertices, and all its edges are critical. It is well known (and easy to prove) that every Class 2 multigraph contains a subgraph with the same chromatic index and such that, not just one, but all the edges are critical. An *e-tense colouring* ϕ of G is a partial edge colouring of G which assigns no colour to e and whose restriction to $E(G-e)$ is an optimal colouring of $G-e$. The colour set of ϕ is defined to be the colour set of its restriction to $G-e$. Given an e-tense colouring ϕ of G with colour set $\mathcal{C}$ and a vertex $w \in V(G)$, we say that a colour $\alpha \in \mathcal{C}$ is *missing* at w (or that w is *missing* the colour α) if there is no edge, having w as an endpoint, which is assigned the colour α by ϕ. The set of colours missing at w is denoted by $\mathcal{C}_w$ and its cardinality is called the *colour-deficiency* of w and denoted by $cdef(w)$, i.e.

$$cdef(w) = |\mathcal{C}_w|.$$

It is easily seen that

$$cdef(w) = |\mathcal{C}| - d_{G-e}(w)$$

for any $w \in V(G)$, and, if e is critical,

$$cdef(w) = \begin{cases} \chi'(G) - d_G(w) & \text{if } w = u, v \\ \chi'(G) - 1 - d_G(w) & \text{if } w \neq u, v \end{cases} \tag{1}$$

for any $w \in V(G)$.

Let $u \in V(G)$. A *fan* at u with respect to ϕ is a sequence of edges of the form

$$F = [e_0, e_1, e_2, \ldots, e_{k-1}, e_k],$$

where $e_0 = e$, $e_i \in uv_i$, and where the vertex v_i is missing the colour of the edge e_{i+1}, for every $i = 0, 1, \ldots, k-1$. An edge f is called a *fan edge* at u if it appears in at least one fan at u. A vertex w is called a *fan vertex* at u if it is joined to u by at least one fan edge. The set of fan vertices is denoted by $V(\mathcal{F})$. A colour $\alpha \in \mathcal{C}$ is called a *fan colour* if it is the colour of a fan edge. The set of fan colours is denoted by $\mathcal{C}_{\mathcal{F}}$. If w is a fan vertex at u, we denote by $\mu^*(uw)$ the number of *fan edges* joining u and w, and call $\mu^*(uw)$ the *fan multiplicity of the edge uw*. The main contribution of [3] was the introduction of a new concept in edge colouring, the *Fan Digraph*, which we now define.

Let G be a Class 2 multigraph, let $e \in uv$ be a critical edge and let ϕ be an e-tense colouring of G. The *e-Fan Digraph* at u with respect to ϕ is the directed multigraph $\mathcal{F} = (V(\mathcal{F}), A(\mathcal{F}), \psi_{\mathcal{F}})$, where

1. $V(\mathcal{F}) = \{w \mid w$ is a fan vertex at $u\}$;
2. $A(\mathcal{F}) = \mathcal{C}_{\mathcal{F}} = \{\alpha \mid \alpha$ is a fan colour at $u\}$;
3. $\psi_{\mathcal{F}} : A(\mathcal{F}) \to V(\mathcal{F}) \times V(\mathcal{F})$

$$\alpha \longmapsto \quad (w_\alpha, z_\alpha),$$

where w_α is the unique fan vertex at u missing colour α and z_α is the unique fan vertex at u joined to u by an edge coloured α.

The Fan Digraph is rooted at the vertex v, endpoint of the uncoloured edge e. Notice that the fact that w_α and z_α exist and are unique is not trivial at all and follows from an important lemma due to Andersen [1] and, independently, Goldberg [5] (see [3, Lemma 2]).

The definition of the Fan Digraph may at first look cumbersome. However, if we postulate the existence of the Fan Digraph, it is extremely easy to define and handle fans based at a given vertex. Indeed it was shown in [3] that we can simply define fans based at a vertex u as *directed trails* in the corresponding Fan Digraph $\mathcal{F}$ having as initial vertex the root v of $\mathcal{F}$. Moreover, every property of the ordinary fans is reflected in the property of these directed trails and it is much more easily understood in the framework of the Fan Digraph. Technically speaking, for the purposes of this paper we shall only use the following double identity concerning the Fan Digraph (proved in [3]).

Theorem 1. (The Fan Theorem). *Let ϕ be a tense colouring of a Class 2 graph having a critical edge $e \in uv$. Let $V(\mathcal{F})$ be the set of fan vertices at u. Then*

$$\sum_{w \in V(\mathcal{F})} cdef(w) = \sum_{w \in V(\mathcal{F})} \mu^*_G(uw) - 1 = |\mathcal{C}_{\mathcal{F}}|.$$

If G is simple, the fan multiplicities $\mu^*_G(uw)$ which appear in Theorem 1 are all equal to 1 and hence we have the following, from which it is easily seen that $\mathcal{F}$ is a tree in this case.

Corollary 1. (Fan Theorem, simple graphs). *Under the hypotheses of Theorem 1, let G be a simple graph. Then*

$$\sum_{w \in V(\mathcal{F})} cdef(w) = |V(\mathcal{F})| - 1 = |\mathcal{C}_{\mathcal{F}}|.$$

We shall need also the following lemma, whose easy proof can be found in [5].

Lemma 1. *Let $n_1 \geq n_2 \geq \ldots \geq n_k$ be positive integers, with $k \geq 2$. Then the following inequality holds:*

$$\lfloor \frac{n_1 + n_2 + \ldots + n_k}{k} + \frac{k-2}{k} \rfloor \leq \lfloor \frac{n_1 + n_2}{2} \rfloor. \tag{2}$$

3. Some Corollaries of the Fan Theorem

Our first objective is to give a short and simultaneous proof of Vizing's Theorem and Vizing's Adjacency Lemma for simple graphs (from which it will be clear that these are related, and yet independent, results).

Unified proof of Vizing's Theorem (simple graphs) and Vizing's Adjacency Lemma. Let G be a simple graph. Without loss of generality we may assume that G is Class 2 (i.e. $\chi'(G) > \Delta(G)$). Furthermore, by possibly removing edges from G successively without decreasing the chromatic index, we may assume that G is critical. Let ϕ be a tense colouring with respect to an edge $e = uv$. It follows from Corollary 1 that

$$\sum_{w \in V(\mathcal{F}) \setminus \{v\}} cdef(w) = |V(\mathcal{F})| - 1 - cdef(v). \tag{3}$$

As a consequence of (3) and the fact that $cdef(w)$ is a nonnegative integer for any w, there are at least $\boldsymbol{cdef(v)}$ vertices $\boldsymbol{w}$ in $\boldsymbol{V(\mathcal{F}) \setminus \{v\}}$ such that $\boldsymbol{cdef(w) = 0}$. (Notice that, by (1) and the assumption on G, $cdef(v) = \chi'(G) - d_G(v) \geq \Delta(G) + 1 - d_G(v)$). For any such vertex w we have

$$\begin{aligned} 0 &= cdef(w) = \chi'(G) - d_G(w) - 1 \\ &\geq (\Delta(G) + 1) - d_G(w) - 1 = \Delta(G) - d_G(w) \geq 0. \end{aligned} \tag{4}$$

Hence all the inequality signs can be replaced by equality signs in (4) and we deduce that

- $\chi'(G) = \Delta(G) + 1$, i.e. Vizing's Theorem;
- $d_G(w) = \Delta(G)$, i.e. Vizing's Adjacency Lemma. ∎

In the multigraph case there are some additional complications, but arguing as above we can do as follows. First we introduce the following notation. For any fan vertex w at u, we call *fan degree* the quantity

$$d^*_G(w) = d_G(w) + \mu^*_G(uw).$$

Let $\Delta^*(\mathcal{F})$ be defined by

$$\Delta^*(\mathcal{F}) = \max_{w \in V(\mathcal{F})} \{d^*_G(w)\}.$$

Assume $\chi'(G) \geq \Delta^*(\mathcal{F})$. Write, for each fan vertex w,

$$s(w) = cdef(w) - \mu_G^*(uw) + 1. \tag{5}$$

With this notation we can express the Fan Theorem (Theorem 1) as

$$\sum_{w \in V(\mathcal{F})} s(w) = |V(\mathcal{F})| - 1,$$

or

$$\sum_{w \in V(\mathcal{F}) \setminus \{v\}} s(w) = |V(\mathcal{F})| - 1 - s(v). \tag{6}$$

Notice that, by (5) and (1), we have

$$s(w) = \begin{cases} \chi'(G) - d_G^*(w) & \text{if } w \neq v \\ \chi'(G) - d_G^*(v) + 1 & \text{if } w = v. \end{cases} \tag{7}$$

By our initial assumption, $s(w) \geq 0$ for all $w \in V(\mathcal{F})$ and $s(v) > 0$. Using (6) and the fact that $s(w)$ is a nonnegative integer for each $w \in V(\mathcal{F})$, there exist at least $s(v) = \chi'(G) - d_G^*(v) + 1$ fan vertices $w \neq v$ such that $s(w) = 0$. For each of these we have

$$0 = s(w) = \chi'(G) - d_G^*(w) \geq \Delta^*(\mathcal{F}) - d_G^*(w) \geq 0, \tag{8}$$

where we have used the initial assumption and the definition of $\Delta^*(\mathcal{F})$. But then all the inequalities in (8) are equalities and in particular we have

1. $\chi'(G) = \Delta^*(\mathcal{F})$;
2. $d_G^*(w) = \Delta^*(\mathcal{F})$.

Notice that (1.) above is a slightly stronger version than Ore's Theorem [7] (and in particular is stronger than Vizing's Theorem) and (2.) is essentially the multigraph version of Vizing's Adjacency Lemma given by Andersen [1].

In the above argument the initial hypothesis that $\chi'(G) \geq \Delta^*(\mathcal{F})$ is quite strong. Without using this assumption we can nonetheless obtain a formula for $\chi'(G)$ as follows. Using (1) we write

$$\sum_{w \in V(\mathcal{F})} cdef(w) = \chi'(G)|V(\mathcal{F})| + 1 - \sum_{w \in V(\mathcal{F})} (d_G(w) + 1). \tag{9}$$

Using (9) and Theorem 1 we have

$$\chi'(G)|V(\mathcal{F})| + 1 - \sum_{w \in V(\mathcal{F})} (d_G(w) + 1) = \sum_{w \in V(\mathcal{F})} \mu_G^*(uw) - 1.$$

Hence

$$\chi'(G)|V(\mathcal{F})| = \sum_{w\in V(\mathcal{F})} (d_G(w) + 1 + \mu^*_G(uw)) - 2. \tag{10}$$

Using the notation $d^*_G(w) = d_G(w) + \mu^*_G(uw)$ introduced earlier and rearranging the terms at the right-hand side of (10), we can rewrite (10) as

$$\chi'(G)|V(\mathcal{F})| = \sum_{w\in V(\mathcal{F})} d^*_G(w) + |V(\mathcal{F})| - 2. \tag{11}$$

The above identity yields the following exact formula for the chromatic index of G:

$$\chi'(G) = \frac{1}{|V(\mathcal{F})|} \sum_{w\in V(\mathcal{F})} d^*_G(w) + \frac{|V(\mathcal{F})| - 2}{|V(\mathcal{F})|}\,. \tag{12}$$

The expression $\frac{1}{|V(\mathcal{F})|}\sum_{w\in V(\mathcal{F})} d^*_G(w)$ represents the "average fan degree" and, accordingly, we denote it by $d^*(\mathcal{F})$. With this notation we can express (12) as

$$\chi'(G) = d^*(\mathcal{F}) + \frac{|V(\mathcal{F})| - 2}{|V(\mathcal{F})|}\,. \tag{13}$$

We have thus proved the following.

Theorem 2. *Let G be a Class 2 graph. Let ϕ be an e-tense colouring of G, where $e \in uv$ is a critical edge of G. Let $V(\mathcal{F})$ be the set of fan vertices at u. Let $d^*(\mathcal{F}) = \frac{1}{|V(\mathcal{F})|}\sum_{w\in V(\mathcal{F})}(d_G(w) + \mu^*_G(uw))$ be the average fan degree. Then*

$$\chi'(G) = d^*(\mathcal{F}) + \frac{|V(\mathcal{F})| - 2}{|V(\mathcal{F})|}. \tag{14}$$

Corollary 2. *With the notation and assumptions of Theorem 2 we have*

$$\chi'(G) = \lceil d^*(\mathcal{F})\rceil; \tag{15}$$

$$\sum_{w\in V(\mathcal{F})} d^*_G(w) \equiv 2 \pmod{|V(\mathcal{F})|}; \tag{16}$$

$$|V(\mathcal{F})| = \frac{2}{d^*(\mathcal{F}) + 1 - \lceil d^*(\mathcal{F})\rceil}. \tag{17}$$

Proof. Since $|V(\mathcal{F})| \geq 2$ (see [3, Lemma 4]), the quantity $\frac{|V(\mathcal{F})|-2}{|V(\mathcal{F})|}$ satisfies

$$0 \leq \frac{|V(\mathcal{F})| - 2}{|V(\mathcal{F})|} < 1 \tag{18}$$

and, since $\chi'(G)$ is an integer, we have from (14) and (18) that

$$\chi'(G) = \lceil d^*(\mathcal{F}) \rceil,$$

which proves (15). From (14), or equivalently, from (11) we deduce (16). Finally the expression (17) can be obtained using (14) and (15). ∎

Another important corollary of Theorem 2 is the following theorem of Andersen [1] and Goldberg [5] .

Corollary 3. *Let w_1 maximize $d^*_G(w)$ over $V(\mathcal{F})$ and let w_2 maximize $d^*_G(w)$ over $V(\mathcal{F}) \setminus \{w_1\}$. Then*

$$\chi'(G) \leq \lfloor \frac{1}{2}(d^*_G(w_1) + d^*_G(w_2)) \rfloor. \tag{19}$$

Proof. By (14) we have

$$\chi'(G) = \frac{1}{|V(\mathcal{F})|} \sum_{w \in V(\mathcal{F})} d^*_G(w) + \frac{|V(\mathcal{F})| - 2}{|V(\mathcal{F})|}. \tag{20}$$

Using (20) and Lemma 1, with $V(\mathcal{F}) = \{w_1, w_2, \ldots, w_k\}$ and $d^*_G(w_1) \geq d^*_G(w_2) \geq \ldots \geq d^*_G(w_k)$, we have

$$\chi'(G) \leq \lfloor \frac{1}{2}(d^*_G(w_1) + d^*_G(w_2)) \rfloor,$$

which proves the corollary.

From (19), using

$$\begin{aligned} d^*_G(w_1) + d^*_G(w_2) &\leq \mu_G(uw_1) + d_G(w_1) + \mu_G(uw_2) + d_G(w_2) \\ &\leq d_G(u) + d_G(w_1) + d_G(w_2) \leq 3\Delta(G) \end{aligned}$$

we deduce immediately Shannon's Theorem [8].

To obtain a stronger version of the Adjacency Lemma we write

$$\sigma(w) = cdef(w) - \mu^*(uw).$$

We can then express the Fan Theorem as

$$\sum_{w \in V(\mathcal{F})} \sigma(w) = -1$$

or, substracting $\sigma(v)$ from both sides,

$$\sum_{w \neq v} \sigma(w) = -1 - \sigma(v). \tag{21}$$

Notice that, by (1), if $w \in V(\mathcal{F})$ then

$$\sigma(w) = \begin{cases} \chi'(G) - d^*_G(w) - 1 & \text{if } w \neq v \\ \chi'(G) - d^*_G(v) & \text{if } w = v. \end{cases}$$

We distinguish two cases.

Case 1. $\sigma(v) \geq 0$, i.e. $\chi'(G) \geq d^*_G(v)$.
In this case the right-hand side of (2) is negative. Let

$$X = \{w \in V(\mathcal{F}) \mid w \neq v, \sigma(w) < 0\}.$$

Then $X \neq \emptyset$ and

$$\sum_{w \in X} \sigma(w) \leq -1 - \sigma(v),$$

or, reversing the sign of inequality,

$$\sum_{w \in X} (-\sigma(w)) \geq \sigma(v) + 1. \tag{22}$$

Using formula (22) we can write

$$X = \{w \in V(\mathcal{F}) \mid w \neq v, \chi'(G) \leq d^*_G(w)\}$$

and we can write (23) as

$$\sum_{w \in X} (d^*_G(w) - \chi'(G) + 1) \geq \chi'(G) - d^*_G(v) + 1,$$

which is precisely the generalization of Vizing's Adjacency Lemma that we obtained in [3] (by different means).

Case 2. $\sigma(v) < 0$, i.e. $\chi'(G) \leq d^*_G(v) - 1$.

In this case the quantity on the right-hand side of (21) is nonnegative and, by letting

$$Y = \{w \in V(\mathcal{F}) \mid w \neq v, \sigma(w) \geq 0\}$$

we have $Y \neq \emptyset$ and

$$\sum_{w \in Y} \sigma(w) \geq -1 - \sigma(v). \tag{23}$$

Notice that $Y = \{w \in V(\mathcal{F}) \mid w \neq v, \chi'(G) \geq d^*_G(w) + 1\}$ and (24) can be rewritten as

$$\sum_{w \in Y} (\chi'(G) - d^*_G(w) - 1) \geq d^*_G(v) - \chi'(G) - 1.$$

We have thus proved the following.

Theorem 3. *Let G be a Class 2 graph and let $e \in uv$ be a critical edge. Let ϕ be an e-tense colouring and let $\mathcal{F}$ be the corresponding Fan Digraph. Let, for any $w \in V(\mathcal{F})$, $d^*_G(w) = d_G(w) + \mu^*_G(uw)$. Then one of the two following cases occurs.*

1. $\chi'(G) \geq d^*_G(v)$.
   Let $X = \{w \in V(\mathcal{F}) \mid w \neq v, \chi'(G) \leq d^*_G(w)\}$. Then $X \neq \emptyset$ and

$$\sum_{w \in X} (d^*_G(w) - \chi'(G) + 1) \geq \chi'(G) - d^*_G(v) + 1.$$

2. $\chi'(G) < d^*_G(v)$.
   Let $Y = \{w \in V(\mathcal{F}) \mid w \neq v, \chi'(G) > d^*_G(w)\}$. Then $Y \neq \emptyset$ and

$$\sum_{w \in Y} (\chi'(G) - d^*_G(w) - 1) \geq d^*_G(v) - \chi'(G) - 1.$$

Theorem 3 extends [3, Theorem 8] and hence, as proved in [3], it is more general than all the other known generalizations of Vizing's Adjacency Lemma to multigraphs.

References

1. L. D. Andersen, On edge-colourings of graphs, *Math. Scand.*, **40** (1977), 161-175.
2. B. Bollobás, *Modern Graph Theory*, Springer-Verlag, New York, 1998.
3. D. Cariolaro, On fans in multigraphs, *J. Graph Theory*, **51(4)** (2006), 301-318.

4. S. Fiorini and R. J. Wilson, *Edge-colourings of graphs,* Research notes in Mathematics, Pitman, 1977.

5. M. K. Goldberg, Edge-colouring of multigraphs: recolouring technique, *J. Graph Theory*, **8** (1984), 123-137.

6. I. Holyer, The NP-completeness of edge colouring, *SIAM J. Comput.*, **10** (1981), 718-720.

7. O. Ore, *The Four-colour Problem,* Academic Press, New York, 1967.

8. C. E. Shannon, A theorem on colouring the lines of a network, *J. Math. Phys.*, **28** (1949), 148-151.

9. V. G. Vizing, On an estimate of the chromatic class of a p-graph, *Diskret. Analiz.*, **3** (1964), 25-30 (in Russian).

10. V. G. Vizing, Critical graphs with a given chromatic class, *Diskret. Analiz.,* **5** (1965), 6-17 (in Russian).

11. V. G. Vizing, The chromatic class of a multigraph, *Cybernetics*, **3** (1965), 32-41.

David Cariolaro
Institute of Mathematics,
Academia Sinica,
Nankang, Taipei 11529,
Taiwan
E-mail: cariolaro@math.sinica.edu.tw